Bases in Orlik–Solomon Type Algebras

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Let $M$ be a matroid on $[n]$ and $E$ be the graded algebra generated over a field $k$ generated by the elements $1, e_1, \ldots, e_n$. Let $\mathcal{Z}(M)$ be the ideal of $E$ generated by the squares $e_1^2, \ldots, e_n^2$, elements of the form $e_i e_j + a_{ij} e_j e_i$ and ‘boundaries of circuits’, i.e., elements of the form $\sum \chi_j e_1 \ldots e_{j-1} e_{j+1} \ldots e_m$ with $\chi_j \in k$ and $e_1, \ldots, e_m$ a circuit of the matroid with some special coefficients. The $\chi$-algebra $A(M)$ is defined as the quotient of $E$ by $\mathcal{Z}(M)$. Recall that the class of $\chi$-algebras contains several studied algebras and in first place the Orlik–Solomon algebra of a matroid. We will essentially construct the reduced Gröbner basis of $\mathcal{Z}(M)$ for any term order and give some consequences.

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1. INTRODUCTION

In a vector space, a (central) hyperplane arrangement is a finite collection of codimension 1 subspaces. The matroid of an hyperplane arrangement can be defined by saying that a subset of the arrangement is independent if and only if the codimension of its intersection is equal to its cardinality. Manifolds defined as complements of complex hyperplane arrangements are important in the Aomoto–Gelfand theory of hypergeometric functions. In [5] the cohomology algebra of a manifold of this form is shown to be isomorphic to the Orlik–Solomon (OS) algebra of the matroid of the arrangement. This result has motivated further research on OS algebras. It is known that for OS algebras of matroids the set of ‘no broken circuits’ (NBC) gives a basis. We refer the reader to [6, 9] for more details on OS-algebras and to [2, 8] for good sources of matroid and oriented matroid theory.

In Section 2, we recall the construction of $\chi$-algebras [4] as the quotient of an algebra $E$ by an ideal $\mathcal{Z}(M)$. This is a generalization of OS algebras for which the set of NBC gives also a basis. We also recall two commutative examples of $\chi$-algebras: an algebra defined for an arrangement of hyperplane [7] and an algebra defined for an oriented matroid [3]. A $\chi$-algebra is defined by the quotient of an algebra $E$ by an ideal $\mathcal{Z}(M)$ defined from the circuits of $M$.

In Section 3, we construct the reduced Gröbner basis of the ideal $\mathcal{Z}(M)$ for any term order (Theorem 3.5). This gives as a corollary a universal Gröbner basis which is shown to be minimal. Finally we remark that the bases given by the NBC are also the bases corresponding to the reduced Gröbner bases for the different term orders.

2. $\chi$-ALGEBRAS

Let $M$ be a simple matroid of rank $r$ on ground set $[n] := \{1, 2, \ldots, n\}$. We say that a subset $U \subseteq [n]$ is unidependent if it contains exactly one circuit, denoted by $C(U)$. For any $i \in C(U)$ the subset $U \setminus i$ is independent. This property characterizes unidependents among dependents: a dependent $D$ is unidependent if and only if there is $i \in D$ such that $D \setminus i$ is independent.

Let $I$ be an independent of $M$. We say that an element $i \in [n]$ is active with respect to $I$ if $I \cup i$ contains a circuit with smallest element $i$. An independent set with at least one active element is said to be active, and inactive otherwise. We denote by $a(I)$ the smallest active element with respect to an active independent $I$. Inactive independents are often called NBC in the literature, since a subset of $[n]$ is an inactive independent if and only if it contains NBC, where a broken circuits are the sets obtained by removing the smallest element from a circuit.
Fix a set $E = \{e_1, \ldots, e_n\}$. Let $E$ be the graded algebra over a field $k$ generated by the elements $1, e_1, \ldots, e_n$ and satisfying the relations $e_i^2 = 0$ for all $e_i \in E$ and $e_i \cdot e_j = a_{i,j} e_i \cdot e_j$ with $a_{i,j} \in k \setminus \{0\}$ for all $i < j$. Both the free exterior algebra and the free commutative algebra

with squares zero generated by the elements of $E$ are such algebras (take $a_{i,j} = -1$ resp. $a_{i,j} = 1$ for all $i < j$) and will be the only ones to be used in the examples. When writing a set in the form $X = \{i_1, i_2, \ldots, i_m\}$ we always suppose w.l.o.g. that we have $i_1 < i_2 < \cdots < i_m$. Given a subset $X = \{i_1, i_2, \ldots, i_m\} \subset [n]$ we will denote by $e_X$ the corresponding (pure) element $e_1 \cdot e_{i_2} \cdot \ldots \cdot e_m$. Fix a mapping $\chi : 2^n \to k$. We define the $\chi$-boundary of an element $e_X$ by

$$
\partial e_X = \sum_{\ell=1}^{\ell=m} (-1)^{\ell} \chi(X \setminus i_\ell) e_{X \setminus i_\ell}.
$$

We extend $\partial$ to $E$ by linearity.

Let $\mathcal{A}_\chi(M)$ be the (right) ideal of $E$ generated by the $\chi$-boundaries $\partial e_U : C$ circuit. We say that

$$
\mathcal{A}_\chi(M) = E/\mathcal{A}_\chi(M)
$$

is a $\chi$-algebra if $\chi$ satisfies the following two properties:

(UC1) $\chi(I) \neq 0$ if $I$ is independent,

(UC2) for any unidependent $U$ of $M$ there is a $e \in k \setminus \{0\}$, such that

$$
\partial e_U = a(\partial e_{U \setminus C}) e_{U \setminus C(U)}.
$$

It can be observed that (UC2) implies that $\chi(U) = 0$ for a unidependent $U$ containing no basis of $M$. Values of $\chi$ on other dependents are irrelevant and can always be chosen null. For convenience, we will also note $e_X$ for the residue class of $e_X$ in $\mathcal{A}_\chi(M)$. Note that a $\chi$-algebra is defined by the matroid $M$, the algebra $E$ and the function $\chi$.

**Example 2.1.** The OS algebra of a matroid [6]. Let $M$ be a matroid on $[n]$. The OS algebra $\mathcal{O}M$ is the quotient of $E$, the graded exterior algebra of the vector space $\sum_{i=1}^n k e_i$, by the ideal generated by boundaries of circuits of $M$.

The OS algebra of $M$, $\mathcal{O}M$, is the $\chi$-algebra obtained for $M$, the algebra $E$ as above and $\chi$ defined for $X \subseteq [n]$ by $\chi(X) = 1$ for every independent.

**Example 2.2.** The Orlik–Terao algebra of a set of vectors [7]. Let $V = \{v_1, v_2, \ldots, v_n\}$ be a set of vectors in a vector space over $k$. The Orlik–Terao algebra $\mathcal{O}V$ is the quotient of $E$, the commutative graded algebra over the field $k$ generated by the elements $1, e_1, \ldots, e_n$, with squares zero, by the ideal generated by the elements of $E$ of the form $\sum_{j=1}^n \lambda_{i,j} e_{i_1} e_{i_2} \cdots e_{i_{j-1}} e_{i_{j+1}} \cdots e_{i_n}$ for any minimal non-trivial linear dependency $\sum_{i=1}^n \lambda_{i,j} v_{i,j} = 0$ among the vectors of $V$.

The Orlik–Terao algebra, $\mathcal{O}V$, is the $\chi$-algebra obtained as follows. Let $M$ be the matroid of linear dependencies of the vectors in $V$ and $E$ be the algebra as above. We fix a basis $B_F$ for any flat $F$ of the matroid $M$. Then for $I = \{i_1, i_2, \ldots, i_k\}$ independent in $M$ we define $\chi(I)$ as the determinant $\det(v_{i_1}, v_{i_2}, \ldots, v_{i_k})$ with respect to $B_d(I)$.

**Example 2.3.** A commutative algebra defined for an oriented matroid [3]. Let $OM$ be an oriented matroid on $[n]$. The commutative algebra $A(OM)$ is the quotient of $E$, the commutative graded algebra over the field $k$ generated by the elements $1, e_1, \ldots, e_n$, with squares zero, by the ideal generated by the elements of $E$ of the form $\sum_{i \in C} s g_C(i) e_{C \setminus i}$ for any signed circuit $C$ of $OM$ with signature $sg_C$. 
The algebra \( A(OM) \) is the \( \chi \)-algebra obtained as follows. Let \( M \) be the underlying matroid of \( OM \) and \( E \) the algebra as above. To define \( \chi \), we fix a basis signature independently in all restrictions of \( OM \) to a flat \( F \) of \( M \) (we recall that a basis signature of an oriented matroid is determined up to a factor \( \pm 1 \)). Then for \( I \) independent in \( M \) we define \( \chi(I) \) as the sign of \( I \) in standard form for the chosen basis signature of the submatroid of \( OM \) on the geometric closure of \( I \) in \( M \).

We say that a unidependent \( U \) is inactive if there is a (necessarily unique) active independent \( I \) such that \( U = I \cup \alpha(I) \). Let \( D \) be the right ideal of \( E \) generated by the elements \([e_C : C \text{ circuit}]\). We will note \( \mathcal{E}_I \) and \( \mathcal{Y} \), the algebra quotient \( E/D \) and its ideal quotient \( \mathcal{Y}/D \) respectively. We now rephrase the principal result of [4].

**Theorem 2.4** ([4]). Let \( M \) be a matroid on \([n]\) and \( A_\chi(M) \) be a \( \chi \)-algebra. Then the set \([e_I : I \text{ inactive independent of } M]\) is a basis of \( A_\chi(M) \) and the set \([\partial e_U : U \text{ inactive unidependent of } M]\) is a basis of \( \mathcal{Y}_I \).

### 3. Reduced and Universal Gröbner Basis

For general definitions on Gröbner bases, see [1]. We begin by adapting some of them to our context. Let \( M \) be a matroid, \( \mathcal{E} \) be an algebra and \( A_\chi(M) \) a \( \chi \)-algebra as defined in the previous section. A total order \( \prec \) of the set of monomials (which is a standard basis of \( \mathcal{E} \)):

\[
\mathcal{T} := \{ e_X : X = \{i_1, \ldots, i_m\} \subseteq [n], i_1 < \cdots < i_m \},
\]

is said to be a term order of \( \mathcal{T} \) if \( e_\emptyset = 1 \) is the minimal element and

\[
\forall e_X, e_Y, e_Z \in \mathcal{E}, \quad (e_X \prec e_Y) \cdot (e_X \cdot e_Z \neq 0) \cdot (e_Y \cdot e_Z \neq 0) \implies e_X \cup z \prec e_Y \cup z.
\]

**Example 3.1.** A permutation \( \pi \in S_n \) defines a linear re-ordering of the elements of \([n]\), \( \pi^{-1}(1) \prec_\pi \pi^{-1}(2) \prec_\pi \cdots \prec_\pi \pi^{-1}(n) \). Consider the ordering \( e_{\pi^{-1}(1)} \prec_\pi e_{\pi^{-1}(2)} \prec_\pi \cdots \prec_\pi e_{\pi^{-1}(n)} \). The corresponding degree lexicographic ordering in \( \mathcal{T} \) is a term order, denoted here by \( \prec_\pi \).

Given a term order \( \prec \), and a non-zero element \( f \in \mathcal{E} \), we may write

\[
f = a_1e_{X_1} + a_2e_{X_2} + \cdots + a_m e_{X_m},
\]

where \( a_i \in \mathbb{K} \setminus \{0\} \) and \( e_{X_m} < \cdots < e_{X_1} \). We say that the \( a_i e_{X_i} \), [resp. \( e_{X_i} \)] are the terms [resp. powers] of \( f \). We say that \( \text{lp}_\prec(f) := e_{X_1} \) [resp. \( \text{lt}_\prec(f) := a_1e_{X_1} \)] is the leading power [resp. leading term] of \( f \) (with respect to \( \prec \)). Note that we can have \( \text{lp}_\prec(hg) \neq \text{lp}_\prec(h)\text{lp}_\prec(g) \) when \( \text{lp}_\prec(h)\text{lp}_\prec(g) = 0 \). Let \( \mathcal{H} \) be an ideal of \( \mathcal{E} \) and let \( \prec \) be a term order of \( \mathcal{T} \). A subset of non-zero elements \( \mathcal{G} \subseteq \mathcal{H} \) is a Gröbner basis of the ideal \( \mathcal{H} \) with respect to \( \prec \), iff, for all non-zero element \( f \in \mathcal{H} \), there exists \( g \in \mathcal{G} \) such that \( \text{lp}_\prec(g) = e_Y \text{ divides } \text{lp}_\prec(f) = e_X \Leftrightarrow Y \subseteq X \).

For any subset \( S \) of \( \mathcal{E} \), we define the leading power ideal of \( S \) with respect to \( \prec \), \( \text{lp}_\prec(S) \), to be the ideal of \( \mathcal{E} \) spanned by the elements \( \{\text{lp}_\prec(s) : s \in S\} \). Consider the subset of powers

\[
\mathcal{T}_I := \{ e_I : I \text{ independent} \} \quad \text{and} \quad \mathcal{T}_D := \{ e_D : D \text{ dependent} \}.
\]

Let \( k[\mathcal{T}_I] \) and \( k[\mathcal{T}_D] \) be the \( k \)-vector subspace of \( \mathcal{E} \) generated by the bases \( \mathcal{T}_I \) and \( \mathcal{T}_D \), respectively. So \( \mathcal{E} = k[\mathcal{T}_I] \oplus k[\mathcal{T}_D] \). With the notation of Section 2, we have that \( k[\mathcal{T}_D] = D \) and \( k[\mathcal{T}_I] \cong \mathcal{E}_I \). Let \( p_I : \mathcal{E} \to k[\mathcal{T}_I] \) be the first projection. We define the term orders of \( \mathcal{T}_I \) in a similar way to term orders of \( \mathcal{T} \). It is clear that the restriction of every term order of \( \mathcal{T} \) to the
subset $\mathbb{T}_l$ is also a term order of $\mathbb{T}_l$. We can also add to $k[\mathbb{T}_l]$ a structure of $k$-algebra with the product $\star : k[\mathbb{T}_l] \times k[\mathbb{T}_l] \to k[\mathbb{T}_l]$, determined by the equalities $e_I \star e_I' = p_I(e_I e_I')$ for all $I, I'$ independents. Note that if $e_I \star e_I' \neq 0$, then $e_I \star e_I' = e_I e_I' (\Leftrightarrow e_I e_I' \neq 0$ if $I \cap I' = \emptyset$ and $I \cup I'$ is an independent set of $M$). So $\mathcal{G}_i(M) := p_I(\mathcal{S}(M))$ is an ideal of $k[\mathbb{T}_l]$.

**Proposition 3.2.** Let $<$ be a term order of $\mathbb{T}$. A Gröbner basis of $\mathcal{G}_i(M)$ with respect to $<$ is also a Gröbner basis of $\mathcal{S}(M)$ with respect to $<$. 

**Proof.** Let $G_i$ be a Gröbner basis of $\mathcal{G}_i(M)$ with respect to the term order $<$. Pick a non-null element $f \in \mathcal{S}(M)$. If we see $\mathcal{S}(M)$ as a $k$-vector space it is clear that $\mathcal{S}(M) = \mathcal{G}_i(M) \oplus k[\mathbb{T}_d]$. So $e_X := l_p(f) \in \mathcal{S}(M)$ if $X$ is an independent set of $M$ or $e_X \in k[\mathbb{T}_d]$ \(\emptyset\) if $X$ is a dependent set of $M$. If $X$ is independent there is an element $g \in G_i$ such that $l_p(g) = e_X$ such that $I \subset X$, so $l_p(g)$ divides $l_p(f)$ in $\mathcal{S}(M)$. Suppose now that $X$ is a dependent set of $M$. Then there is a circuit $C \subset X$. We know that $\delta e_C \in \mathcal{S}(M)$ and if $l_p(\delta e_C) = e_Y$ then $Y \subset C \subset X$. So, $l_p(\delta e_C)$ divides $l_p(f)$ in $\mathcal{S}(M)$ and $G_i$ is also a Gröbner basis of $\mathcal{S}(M)$.

A Gröbner basis $G$ of an ideal $\mathcal{I}$ is called reduced (with respect to the term order $<$) if for every element $g \in G$ we have $l_p(g) \neq l_p(f)$, and for every two distinct elements $g, g' \in G$, no term of $g'$ is divisible by $l_p(g)$. A (finite) subset $U \subset \mathcal{I}$ is called a universal Gröbner basis if $U$ is a Gröbner basis of $\mathcal{I}$ with respect to all term orders simultaneously.

**Proposition 3.3.** Let $G$ be a Gröbner basis of the ideal $\mathcal{S}(M)$ with respect to the term order $<$ of $\mathbb{T}$. Then

$$B_G := \{ e_X : X \subset [n], e_X \notin l_p(G) = l_p(\mathcal{S}(M)) \}$$

is a basis of $A_x(M)$.

We say that $B_G$ is the canonical basis of the $\chi$-algebra $A_x(M)$ for the Gröbner basis $G$ of the ideal $\mathcal{S}(M)$.

**Remark 3.4.** From the preceding proposition we see that, for every term order $<$ of $\mathbb{T}$, there is a unique monomial basis of $A_x(M)$ denoted by $B_\prec$. We say that $B_\prec$ is the canonical basis of $A_x(M)$. On the other hand it is well known that the term order $<$ determines a unique reduced Gröbner basis of $\mathcal{S}(M)$ denoted $(G_{\prec})_\prec$. From the definitions we can also deduce that $B_\prec = B_\prec \Rightarrow (G_{\prec})_\prec = (G_\prec)_\prec \Rightarrow l_p(G_\prec) = l_p(G_\prec)$. 

For a term order $<$ of $\mathbb{T}$ we say that $\pi_\prec \in S_n$ is the permutation compatible with $<$ if, for every pair $i, j \in [n]$, we have $e_i < e_j$ if $i \prec_\pi j (\Rightarrow \pi_\prec^{-1}(i) < \pi_\prec^{-1}(j))$. Let $\mathcal{G}_\pi(M)$ be the subset of circuits of $M$ such that $\inf_{\pi_\prec} (C) = \alpha_\pi(C)$ and $C \backslash \alpha_\pi(C)$ is inclusion minimal with this property. $(\alpha_\pi(C))$ is the minimum active element of $C \backslash \inf_{\pi_\prec} (C)$ where the order used for activity and taking inf is $\prec_\pi$. In the following we may replace ‘$\pi_\prec$’ by ‘$\pi$’ when no mistake can result.

**Theorem 3.5.** Let $<$ be a term order $\mathcal{T}$ compatible with the permutation $\pi \in S_n$. Then the family $\mathcal{G}_{\text{red}} := \{ \delta e_C : C \in \mathcal{G}_\pi(M) \}$ form a reduced Gröbner basis of $\mathcal{S}(M)$ with respect to the term order $\prec$.

**Proof.** From Proposition 3.2 it is enough to prove that $(G_{\prec})_\prec$ is a reduced Gröbner of $\mathcal{G}(M)$. Let $f$ be any element of $\mathcal{G}_\pi(M)$, we have from Theorem 2.4 (we note $\mathcal{U}_\prec$ the set of inactive unidependent for the order $\prec_\pi$) that $f = \sum_{U \in \mathcal{U}_\prec} \xi_U \delta e_U; \xi_U \in k$. Let now remark that $l_p(\delta e_U) = e_{U \backslash \alpha_\pi(U)}$ and that these terms are all different. We have then clearly that
\[ \text{lp}_{<}(f) = \sup_{\alpha} [\text{lp}_{>}(\partial \varepsilon_{U})]. \]

Given \( U \in \mathfrak{U}_{\pi}(M) \) it is clear that \( \alpha_{\pi}(C(U)) = \alpha_{\pi}(U) \). So, \( C(U) \setminus \alpha_{\pi}(C(U)) \) is a circuit of \( \mathfrak{C}_{\pi} \) such that \( C' \setminus \alpha_{\pi}(C') \subset C(U) \setminus \alpha_{\pi}(C(U)) \). Let \( C' \) be a circuit of \( \mathfrak{C}_{\pi} \) such that \( C' \setminus \alpha_{\pi}(C') \subset C(U) \setminus \alpha_{\pi}(C(U)) \). So we have that \( \text{lp}_{<}(\partial \varepsilon_{C'}) \) divides \( \text{lp}_{<}(\partial \varepsilon_{U}) \), and \( (G_{r})_{<} \) is a Gröbner basis of \( \mathfrak{Z}_{\pi}(M) \).

Suppose for a contradiction that \( (G_{r})_{<} \) is not a reduced Gröbner basis; i.e., there exist two circuits \( C \) and \( C' \) in \( \mathfrak{C}_{\pi} \) and an element \( c \in C \) such that \( e_{C \setminus \alpha_{\pi}(C')} \) divides \( e_{C \setminus \alpha_{\pi}(C')} \). First we can say that \( c \neq \alpha_{\pi}(C) \) because the sets \( C' \setminus \alpha_{\pi}(C') \) and \( C \setminus \alpha_{\pi}(C) \) are incomparable. This, in particular, implies that \( \alpha_{\pi}(C) \in C' \setminus \alpha_{\pi}(C') \) and \( \alpha_{\pi}(C') \in C \setminus \alpha_{\pi}(C) \), a contradiction.

COROLLARY 3.6. \( \mathcal{G}_{\pi} := \{ \partial \varepsilon_{C} : C \in \mathfrak{C}(M) \} \) form a minimal universal Gröbner basis of \( \mathfrak{Z}(M) \).

PROOF. From Theorem 3.5, the reduced Gröbner basis constructed for the different orders \( \prec \) are all contained in \( \mathcal{G}_{\pi} \) which proves the universality. We prove the minimality by contradiction. Let \( C_{0} = \{ i_{1}, \ldots, i_{m} \} \) be a circuit of \( M \) and let \( \pi \in S_{m} \) be a permutation such that \( \pi^{-1}(i_{j}) = j, j = 1, \ldots, m \). Then \( C_{\pi} := \{ \partial \varepsilon_{C} : C \in \mathfrak{C} \setminus C_{0} \} \) is not a Gröbner basis since \( \text{lp}_{<\pi}(\partial \varepsilon_{C_{0}}) = e_{C_{0} \setminus i_{1}} \) is not in \( \text{lp}_{<\pi}(C_{\pi}) \). \( \square \)

To finish we give a characterization of the NBC bases of the \( \chi \)-algebras in terms of the Gröbner bases of their ideals. Consider a permutation \( \pi \in S_{m} \) and the associated re-ordering \( <_{\pi} \) of \([n]\). When the \( <_{\pi} \)-smallest element \( \inf_{<_{\pi}}(C) \) of a circuit \( C \in \mathfrak{C}(M) \), \( |C| > 1 \), is deleted, the remaining set, \( C \setminus \inf_{<_{\pi}}(C) \), is called a \( \pi \)-broken circuit of \( M \). We set

\[ \text{nbc}_{<_{\pi}}(M) := \{ e_{X} : X \subset [n] \text{ contains no } \pi \text{-broken circuit of } M \}. \]

As the algebra \( \mathcal{A}_{\pi}(M) \) does not depend on the ordering of the elements of \( M \) it is clear that \( \pi \text{-nbc}(M) \) is a NBC basis of \( \mathcal{A}_{\pi}(M) \).

COROLLARY 3.7. Let \( B \) be a basis of \( \mathcal{A}_{\pi}(M) \). Then are equivalent:

\begin{align*}
(3.7.1) & \text{ } B \text{ is the canonical basis } B_{<_{\pi}} \text{ for some term order } <_{\pi} \text{ of } \mathbb{T}. \\
(3.7.2) & \text{ } B \text{ is the } \pi \text{-NBC basis } \pi \text{-nbc}(M), \text{ for some permutation } \pi \in S_{m}. \\
(3.7.3) & \text{ } B \text{ is the canonical basis } B_{<_{\pi}}, \text{ for some reduced Gröbner basis } \mathcal{G}_{\pi}, \text{ of the ideal } \mathfrak{Z}(M). 
\end{align*}

PROOF. (3.7.1) \( \Rightarrow \) (3.7.2). Let \( <_{\pi} \) be a term order of \( \mathbb{T} \). Since from Corollary 3.6 \( \mathcal{G}_{\pi} \) is a universal Gröbner basis of \( \mathfrak{Z}(M) \) it is trivially a Gröbner basis relatively to \( <_{\pi} \). We have already remarked that the leading term of \( \partial \varepsilon_{C} \) is \( e_{C \setminus \inf_{<_{\pi}}(C)} \). From Proposition 3.3 we conclude that \( B_{<_{\pi}} = \pi \text{-nbc}(M) \).

(3.7.2) \( \Rightarrow \) (3.7.3). Suppose that \( B = \pi \text{-nbc}(M) \). Let \( <_{\pi} \) be the degree lexicographic order of \( \mathbb{T} \) determined by the permutation \( \pi \in S_{m} \). Note that \( <_{\pi} \leq \pi \). From Theorem 3.5 we know that \( (G_{w})_{<_{\pi}} = \{ \partial \varepsilon_{C} : C \in \mathfrak{C}_{<_{\pi}} \} \) is the reduced Gröbner basis of \( \mathfrak{Z}(M) \) with respect to the term order \( <_{\pi} \). Then \( B \) is the canonical basis for the reduced Gröbner basis \( (G_{r})_{<_{\pi}} \).

(3.7.3) \( \Rightarrow \) (3.7.1). This is a consequence of Proposition 3.3 and Remark 3.4. \( \square \)

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