ON CHARACTERIZING TERRAIN VISIBILITY GRAPHS*

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Abstract. A terrain is an $x$-monotone polygonal line in the $xy$-plane. Two vertices of a terrain are mutually visible if and only if there is no terrain vertex on or above the open line segment connecting them. A graph whose vertices represent terrain vertices and whose edges represent mutually visible pairs of terrain vertices is called a terrain visibility graph. We would like to find properties that are both necessary and sufficient for a graph to be a terrain visibility graph; that is, we would like to characterize terrain visibility graphs.

Abello et al. [Discrete and Computational Geometry, 14(3):331–358, 1995] showed that all terrain visibility graphs are “persistent”. They showed that the visibility information of a terrain point set implies some ordering requirements on the slopes of the lines connecting pairs of points in any realization, and as a step towards showing sufficiency, they proved that for any persistent graph $M$ there is a total order on the slopes of the (pseudo) lines in a generalized configuration of points whose visibility graph is $M$.

We give a much simpler proof of this result by establishing an orientation to every triple of vertices, reflecting some slope ordering requirements that are consistent with $M$ being the visibility graph, and prove that these requirements form a partial order. We give a faster algorithm to construct a total order on the slopes. Our approach attempts to clarify the implications of the graph theoretic properties on the ordering of the slopes, and may be interpreted as defining properties on an underlying oriented matroid that we show is a restricted type of 3-signotope.

1 Introduction

Problems related to geometric visibility have arisen from applications in graphics and motion planning in robotics. In graphics, for example, the hidden line problem for computer-drawn polyhedra is to determine which edges, or parts of edges, of a polyhedron are visible from a given vantage point. In robotics or more specifically motion planning, we would like to find the shortest path between two positions for a robot without hitting the objects around. Visibility problems include the well-known art-gallery problems: planning a path for a guard so that it can see the entire art gallery, or determining the minimum number of the guards which together can see the whole art gallery.

A deeper understanding of the combinatorial structure of visibility between geometric objects may help to address problems involving visibility in computational geometry.

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A visibility graph is a fundamental combinatorial structure which has proven useful for addressing such problems. The vertices of a visibility graph correspond to geometric components such as points or line segments. There is an edge between two vertices of the graph if the components are visible to each other. There are variations in the definition of visibility depending on the underlying application. For the class of art gallery problems, the vertex visibility graph of polygons is more commonly used. Here, the geometric components are the vertices of the polygon, and two vertices are visible if the line segment connecting them is in the interior or along the boundary of the polygon. Throughout this paper, by the term visibility graph, we mean the vertex visibility graph of the geometric object unless otherwise stated.

Ideally, we would like to fully understand the combinatorial properties of visibility in polygons. We would like to find properties that are both necessary and sufficient for a graph to be a visibility graph; that is, we would like to characterize, graph theoretically, visibility graphs. If a graph satisfies the sufficient graph theoretic properties, it is realizable as a visibility graph of a polygon. Determining whether a graph with certain properties is the visibility graph of a simple polygon is known as visibility graph recognition or realizability. The actual drawing of the point set whose visibility graph is the desired graph is called reconstruction. Studying the recognition and characterization of visibility graphs may help us find more efficient algorithms for problems related to geometric visibility.

Although there are some partial results for restricted polygons, no characterization of visibility graphs of simple polygons is known. The trouble appears to be proving the sufficiency of certain properties, for which a two step approach might work: First, prove combinatorial conditions on point sets that can represent graphs with these properties; and second, show that such point sets can be geometrically realized to form the required visibility graph. The first step yields a characterization of visibility graphs but only in a generalized, non-geometric setting. Here the combinatorial conditions may be interpreted in the language of oriented matroids. The second step is then to realize a class of oriented matroids. Not all oriented matroids are realizable (the realizability problem is NP-hard), however, it may be possible to show that all matroids in this class are realizable.

### 1.1 Problem statement

A one-dimensional terrain is an x-monotone polygonal line in the xy-plane. The endpoints of the terrain line segments mark terrain vertices. Two vertices of a terrain are mutually visible if and only if there is no terrain vertex on or above the open line segment connecting them. The visibility graph of a terrain with n vertices is the undirected graph with vertex set \{1, 2, ..., n\} and edges \{\{a, b\} | terrain vertices a and b are mutually visible\}. We assume that the terrain visibility graph is ordered meaning that the ordering of the vertices along the terrain agrees with the vertex numbering.

It is relatively simple to show that all terrain visibility graphs satisfy three properties described in Section 3. It remains challenging to show that all graphs with these three properties, which are called persistent graphs (see Definition 3), are terrain visibility graphs. In fact, though Abello et al. [4] claimed this result, a complete proof has never been published.
Abello et al. studied core induced subgraphs of the visibility graphs of staircase polygons (definitions at the start of Section 2.2), which are equivalent to ordered terrain visibility graphs. They showed that the visibility information of a given terrain vertex implies some ordering requirements on the slopes of the lines connecting this vertex to the others in any realization. A “maximal chain” in the “weak Bruhat order” (or equivalently, a balanced tableau, which is defined in Section 4.1) may be used to express these slope ordering requirements. As a step towards showing sufficiency, they provided a $\Theta(n^5)$-time algorithm that finds one maximal chain (of perhaps many) in the weak Bruhat order consistent with a given persistent graph. Their algorithm starts with a representative maximal chain of a clique graph and repeatedly performs complicated operations on the chain, in order to generate a maximal chain incrementally closer to that of the desired persistent graph. The idea was to use this maximal chain to construct a point set realizing it. However, the additional step turned out to be tricky.

1.2 Generalized Configuration

A balanced tableau may be viewed as representing a total order on the slopes of pseudolines that connect pairs of points in a generalized configuration (definitions follow). A pseudoline is a simple curve that separates the plane. An arrangement of pseudolines is a collection of pseudolines, such that each pair of them meet in exactly one point, where they cross. Let $P$ be a set of points in the Euclidean plane, and let $L$ be an arrangement of pseudolines such that every pair of points in $P$ lie on exactly one pseudoline, and each pseudoline in $L$ contains exactly two points of $P$. Then the pair $(P, L)$ is a generalized configuration of points in general position (in general position indicates that no three points of $P$ lie on the same pseudoline of $L$). The concept of the slope order may also be generalized to pseudolines. If a pseudoline arrangement is intersected with a vertical line such that all intersection points of the arrangement lie to its right, then the order in which the pseudolines cross the vertical line (decreasing by the y-coordinates of the crossings) is the (increasing) slope order of the pseudolines [52].

1.3 Description of results

We give a streamlined proof of Abello et al.’s result [4] (that there is a representative balanced tableau for every persistent graph). Our approach is to translate the constraints on slope ordering imposed by the persistent graph into orientations on triples of vertices and to come up with an orientation on all vertex triples that agrees with these constraints (Section 4.3.1).

We interpret our orientation as defining properties on an underlying oriented matroid, and show that our orientation is a restricted type of a 3-signotope. This, together with the work of Felsner and Weil [28] on acyclicity of signotopes, gives an immediate proof of the result (Section 4.3.3). In fact, this is a slightly stronger result than that of Abello et al.’s.

We also give an alternate self-contained proof (Section 4.3.4) that clarifies the implications of the graph theoretic properties on the slope orders in more detail. This has the
potential to determine additional constraints on the slopes that may aid in realization.

Felsner and Weil give an abstract combinatorial proof that the graphs associated with signotopes are acyclic in general. Our direct proof may be viewed as an alternative and perhaps more intuitive proof for the acyclicity of such graphs when restricted to 3-signotopes. While the substantial ideas of both proofs are the same, our proof explains in more detail the manner in which certain slope orderings entail others.

Our proofs imply a $\Theta(n^3)$-time algorithm to construct a representative balanced tableau (Section 4.3.5), which improves on the $\Theta(n^5)$ algorithm of Abello et al. [4]. Moreover, the slope orders imposed by our orientation can be preserved when extending the graph by additional vertices. Our orientation also forbids certain substructures in the point set realizing it. These properties may help prove realizability (Section 4.3.6). Lastly, we put our work and the previous results in the context of oriented matroids, and give a thorough comparison of these studies (Section 5).

2 Related work

Visibility graphs have been extensively studied. Visibility may be defined among the vertices of a polygon, line segments in the plane, or various other geometric objects in two or higher dimensions. O’Rourke [41], and Ghosh and Goswami [33] give an excellent review of research on visibility graphs. Ghosh’s book [32] is also a good source of information on visibility graphs.

Characterizing visibility graphs of simple polygons and finding algorithms to recognize them seem challenging. There is no polynomial-time algorithm known to recognize visibility graphs. Nor is the problem known to be NP-hard, or even in NP. Everett [25] shows that the problem can be solved in PSPACE.

The class of visibility graphs does not lie in any of the well-known classes of graphs such as planar graphs, chordal, circle or perfect graphs [25, 31]. Everett [25] shows that there is no finite set of forbidden induced subgraphs that characterizes visibility graphs.

Results on visibility graphs so far have involved restricting the class of graphs, or restricting the class of polygons, or adding extra information to the graph.

2.1 Restricting the class of graphs

ElGindy [24] shows that any “maximal outerplanar graph” is a visibility graph, and provides a linear time embedding algorithm to construct a uni-monotone polygon whose visibility graph is the desired maximal outerplanar graph. A monotone polygonal chain is a set of vertices in the plane, ordered along some direction, with consecutive vertices joined. A polygon is monotone if it can be broken into two monotone polygonal chains (both ordered along the same direction). A uni-monotone polygon is a monotone polygon where one chain consists of a single edge. A terrain, whose visibility properties we study here, is the same as a monotone polygonal chain (monotone in the horizontal direction).

Colley [20] extends the range of graphs that can be identified as visibility graphs. He
defines a new class of graphs, “tree of cliques graphs”, and extends ElGindy’s algorithm to recognize these graphs by embedding a uni-monotone polygon with the required visibility. Both ElGindy’s and Colley’s algorithms can be modified to create a uniform uni-monotone polygon (that is, a uni-monotone polygon whose vertices are uniformly spaced in the direction of monotonicity). Not every uniform uni-monotone polygon has a tree of cliques visibility graph.

2.2 Restricting the class of polygons

A staircase polygon is a polygon consisting of alternate horizontal and vertical edges forming a polygonal chain leading to the right and down, and a path of one horizontal and one vertical edge connecting its endpoints. These polygons are also known as orthogonal convex fans. A convex fan is a polygon that has a convex vertex, called the kernel, visible to all other vertices. An orthogonal convex fan is a convex fan consisting of only horizontal and vertical edges. The core of a staircase polygon is the set of reflex vertices plus the vertices adjacent to the kernel vertex. A uniform step-length staircase polygon is a staircase polygon whose vertices are evenly spaced with respect to the horizontal direction.

Abello and Egecioglu [2] give a polynomial time recognition algorithm, using linear programming, to recognize visibility graphs of uniform step-length staircase polygons. Using this method, they show that there exist graphs that are visibility graphs of staircase polygons but not realizable as uniform step-length staircase polygons.

Colley [20, 21] proves a strong relationship between the visibility graphs of uni-monotone polygons and staircase polygons. He uses the result to show that recognizing the visibility graph of a staircase polygon is equivalent, under linear-time reduction, to the problem of recognizing the visibility graph of a monotone polygonal chain, where the additional information of the order of the vertices on the chain is given. The vertex ordering of the core of a staircase polygon is determined by the visibility graph of the staircase polygon. A crucial part of his argument is that the core induced subgraph of the visibility graph of a staircase polygon is identical to the visibility graph of a monotone polygonal chain (or equivalently a terrain visibility graph), and vice versa. Using this and the result from Abello and Egecioglu [2], Colley shows that the visibility graphs of uniform uni-monotone polygons are a strict subset of the visibility graphs of general uni-monotone polygons, if the outside face of the polygon is fixed.

There have been few results that give a graph theoretic characterization of visibility graphs for restricted classes of polygons. Everett and Corneil [26] characterize the visibility graphs of “1-spiral” polygons and give a linear time algorithm for recognizing them.

Colley et al. [22] characterize the visibility graphs of “towers” and present a linear time algorithm to recognize them. Choi et al. [19] also characterize these graphs, where they use the term “funnel” instead of tower. They give a linear time algorithm to reconstruct the funnel from its visibility graph.
2.3 Adding extra information

Many results on visibility graphs involve adding extra information to the graphs. Ghosh [30] conjectures three necessary conditions for recognizing visibility graphs of simple polygons, when the Hamiltonian cycle forming the polygon boundary is specified. Everett [25] gives a counterexample to this conjecture and suggests a stronger version of the third necessary condition, which Srinivasaraghavan and Mukhopadhyay [48] prove is indeed necessary. Abello et al. [9] show that, even with the stronger version of the third condition, the necessary conditions are insufficient. Ghosh [31] identifies another necessary condition to circumvent the new counterexample and conjectures that the four necessary conditions are sufficient, but Streinu [51] later proves Ghosh’s conjecture is false.

Coullard and Lubiw [23] introduce further necessary conditions for a graph to be a visibility graph. They develop a new structural property of visibility graphs: Each 3-connected component of a visibility graph has a vertex ordering in which every vertex is adjacent to a previous 3-clique; that is, each 3-connected component of a visibility graph has a 3-clique ordering. The weaker result that each vertex is adjacent to a previous 2-clique is a consequence of polygon triangulation. The 3-clique ordering property is not sufficient. The property can be tested in polynomial time, and is used to give an algorithm for the distance visibility graph problem, which is the problem of whether an edge-weighted graph is the visibility graph of a simple polygon with the weights as Euclidean distances.

ElGindy [24] conjectures a characterization of visibility graphs of convex fans. He suggests a “decomposition strategy” and gives an algorithm to check whether a graph is the visibility graph of a convex fan, when the Hamiltonian cycle forming the boundary is known. However, the reconstruction appears tricky and the correctness of the algorithm is not clear.

Abello et al. [3, 4] claim to have characterized visibility graphs of staircase polygons (or equivalently orthogonal convex fans) in a series of two papers, only one of which has appeared in the literature. The preliminary results by Abello et al.[8] are precursors to these two papers. The difficult part of such a characterization is to characterize the core induced subgraphs of these visibility graphs. The non-core vertices are easy to determine from the graph. They identify the Hamiltonian cycle and as a result the ordering of the core vertices in the graph. Abello et al.[4] and Abello [1] present a necessary property for such core induced subgraphs, which they call “persistent” property. They show that the visibility between core vertices (in a staircase polygon) implies some ordering requirements on the slopes of the lines that connect pairs of these vertices in any realization. They approach the problem of whether the persistent property is sufficient, given the ordering of the core vertices, by constructing a total order on the slopes of (pseudo) lines that connect pairs of core vertices in a generalized configuration such that the slope order is consistent with the desired visibility graph. The non-core vertices would be easy to add to any core realization. Abello et al. [4, 8] claim that the proposed slope order is realizable by a point set but a complete proof has not been published. We describe their work in more detail in Section 4.2.

Following Ghosh’s result [30], Abello and Kumar [5, 7] study visibility graphs of simple polygons, when the Hamiltonian cycle forming the boundary of the polygon is given.
They define a new class of graphs called “quasi-persistent graphs”, which they show is equivalent to the class of graphs satisfying the first two necessary conditions of Ghosh [30]. Using a geometric interpretation of a polygon realizing a quasi-persistent graph, they determine vertices that block the line of sight between pairs that are not mutually visible. This gives a blocking vertex assignment. (Different polygons with the same visibility graph may have different blocking vertex assignments.) They show that a blocking vertex assignment, if determined by a polygon realization, satisfies four necessary conditions. The last three conditions are based on properties of Euclidean shortest paths (as the shortest path between a pair that is not mutually visible is determined by the blocking vertex assignment of the pairs on the path). Ghosh [31] shows all four necessary conditions of Abello and Kumar follow from his third and fourth conditions. To address the realizability problem, Abello and Kumar introduce an oriented matroid approach: first find the combinatorial properties on the point set corresponding to the vertices of the visibility graph (that is represented by an oriented matroid); then decide whether such a point set is realizable. In particular, they show how to construct a “uniform rank 3 oriented matroid” for every quasi-persistent graph satisfying the four conditions, which if affinely realizable yields a simple polygon with the desired visibility graph.

Abello and Kumar [6] show that these conditions are sufficient to reconstruct a polygon from the graph when restricted to “2-spiral” polygons.

O'Rourke and Streinu [43] introduce a new polygon visibility graph, the vertex-edge visibility graph, that represents visibility between vertices and edges. They suggest that the additional geometric information such graphs provide may simplify the problem of characterizing them. They show that a vertex-edge visibility graph determines the convexity of vertices, the vertex visibility graph, and, for each vertex, the partial local sequence (definition below) and the “shortest path tree”.

The local sequence for a vertex \( v \) is the circular sequence of all other vertices as they are encountered by a rotating line through \( v \). The partial local sequence contains only the visible vertices. The collection of all local sequences is a version of what Goodman and Pollack [34] called a cluster of stars [49], which forms an affine (or acyclic) uniform rank 3 oriented matroid, whose topological representation is a generalized configuration of points [34]. The set of all local sequences of a generalized configuration of points determines the chirotope information, or equivalently the set of all triple orientations, of the point set. The orientation of a triple \((i, j, k)\) shows whether \( k \) is to the right or to the left of the pseudoline through \( i \) to \( j \). The same is true when pseudolines are straight-lines. The set of all triple orientations determines the order type.

O'Rourke and Streinu [42] generalize the notion of straight-line visibility to visibility along pseudolines, which they call “pseudo-visibility”. The idea of pseudo-visibility comes from the concept of duality between pseudoline arrangements and generalized configurations of points. They give a complete characterization of vertex-edge (pseudo) visibility graphs of “pseudo-polygons”. They define a predicate on any triple of vertices (the predicates match the chirotope definition of Abello and Kumar [7]). They show the predicates satisfy Knuth’s “CC system” axioms [38]. CC systems are equivalent to uniform rank 3 acyclic oriented matroids [38], which in turn are equivalent to Goodman and Pollack’s generalized config-
urations of points in general position [12]. As a consequence of the relationship between vertex-edge visibility graphs and vertex visibility graphs, they show that the recognition problem for vertex visibility graphs of pseudo-polygons is in NP (the same problem with straight-line visibility is only known to be in PSPACE).

Streinu [49] gives a characterization of clusters of stars of generalized configurations of points, and provides efficient algorithms for recognizing them. Her characterization conditions indicate that an orientation (clockwise or counterclockwise) of every triple that is consistent over a set of local sequences (that is, obeys a generalized transitivity law) is realizable as a generalized configuration of points, for some ordering of the point set. Knuth’s CC systems [38] can also be interpreted as characterizing the local sequences of generalized configurations of points.

The concept of pseudo-visibility detaches the stretchability question from the combinatorial aspects of the problem. A pseudoline arrangement (or equivalently an acyclic uniform rank 3 oriented matroid) is stretchable or realizable if it is isomorphic to a straight-line arrangement. The main difficulty in fully characterizing vertex-edge visibility graphs of (straight-line) polygons is to decide whether a certain class of acyclic uniform rank 3 oriented matroids is stretchable. It is well-known that stretchability of pseudoline arrangements, in general, is NP-hard [39, 47]. However, there exist various techniques to prove stretchability for particular instances [16, 17, 13, 44, 14, 15, 45]. O’Rourke and Streinu [42] remark that the class of acyclic uniform rank 3 oriented matroids generated by the vertex-edge pseudo-visibility graphs is a strict subclass of all acyclic uniform rank 3 oriented matroids, so it may be possible to characterize or recognize them efficiently.

Streinu [50] shows that the class of vertex-edge (straight-line) visibility graphs is properly contained in the class of vertex-edge pseudo-visibility graphs. She introduces “star-like” pseudo-polygons and shows that a star-like vertex-edge pseudo-visibility graph may not be realizable. However, since vertex-edge pseudo-visibility graphs contain more geometric information than pseudo-visibility graphs, it is possible that a pseudo-visibility graph is associated with several (possibly exponentially many) vertex-edge pseudo-visibility graphs, some of which are stretchable and some not [51]. For instance, all star-like pseudo-visibility graphs are realizable [50]. Later, Streinu [51] shows an infinite family of pseudo-visibility graphs that are not realizable. Since the graphs in this family satisfy the necessary conditions of O’Rourke and Streinu [42], Abello and Kumar [7], and Ghosh [31], this implies that these necessary conditions are not sufficient to characterize (straight-line) visibility graphs.

Everett et al. [27] study other combinatorial objects (containing more information than visibility graphs) that describe the structure of simple polygons, which they call the “stabbing information”. The stabbing information stores certain information about the intersections that each line connecting two vertices makes with the other edges of the polygon. O’Rourke [40], and Jackson and Wismath [36] study weaker variants of such objects for orthogonal polygons that involve the horizontal and vertical visibility information. Jackson and Wismath [36] show how to reconstruct an orthogonal polygon from both the internal and external horizontal and vertical visibility information.
3 Necessary properties for terrain visibility graphs

We first define the graph properties that we use, and then present the set of conditions satisfied by all terrain visibility graphs.

For integers $a$ and $b$ we use the notation $[a..b]$ to indicate the interval of all integers between and including them. The two numbers $a$ and $b$ are called the endpoints of the interval. To exclude an endpoint from the interval we replace the associated bracket with a parenthesis. For example $(i..j]$ is $\{x \in \mathbb{N} \mid i < x \leq j\}$. We use $[n]$ for $[1..n]$ for succinctness.

**Definition 1.** An ordered graph $G = ([n], E)$ has the X-property if for every four vertices $a < b < c < d$, if $\{a, c\} \in E$ and $\{b, d\} \in E$ then $\{a, d\} \in E$.

This property is called “inversion complete” by Abello [1].

**Definition 2.** An ordered graph $G = ([n], E)$ has the bar-property if for every edge $\{a, c\} \in E$ where $a + 1 < c$, there exists a vertex $b \in (a..c)$ such that $\{a, b\} \in E$ and $\{b, c\} \in E$.

In an ordered graph with Hamiltonian path $1, 2, \ldots, n$, this property is an ordered version of chordality [1, 31]: every ordered cycle of length at least four has a chord. 

**Definition 3.** An ordered graph $G = ([n], E)$ that has both the X-property and the bar-property and contains the Hamiltonian path $1, 2, \ldots, n$ is called persistent.

Abello et al. [4] initially defined persistent graphs in 1995 in a different and slightly incorrect manner so that the X-property was not guaranteed (incorrect in the sense that some properties that they assume from their definition are not true). Abello modified the definition in 2004 [1] to include this property (which he called inversion completeness). His subsequent definition gives the same class of graphs as our definition does.

We prove that ordered terrain visibility graphs are persistent. Colley [20] proves that the core induced subgraph of the visibility graph of a staircase polygon is identical to a terrain visibility graph, and vice versa. Abello et al. [4] show that the core induced subgraph of the visibility graph of a staircase polygon (and as a result a terrain visibility graph), with respect to its ordering, is persistent. Here, for completeness of results, we re-prove the necessary conditions for terrain visibility graphs by a simpler and more geometric approach. Our proof relies on two lemmas. The first is called the “Order Claim” in the literature [11, 18, 37], and the second is called the “Midpoint Claim” by King [37].

**Lemma 1.** Ordered terrain visibility graphs have the X-property.

*Proof.* Consider four vertices $a < b < c < d$ in an ordered terrain visibility graph $G = ([n], E)$, such that $\{a, c\} \in E$ and $\{b, d\} \in E$. Since $\{a, c\} \in E$, we know the terrain does not intersect the half-strip above the segment $\overline{ac}$. Denote the half-strip above the segment $\overline{ac}$ by $H^+(\overline{ac})$. Similarly, we know the terrain does not intersect $H^+(\overline{bd})$. Hence $b$ is below the segment $\overline{ac}$ and $c$ is below the segment $\overline{bd}$, which means the two segments $\overline{ac}$ and $\overline{bd}$ intersect. Thus the line segment $\overline{ad}$ lies in the region $H^+(\overline{ac}) \cup H^+(\overline{bd})$. Therefore, the terrain does not intersect $H^+(\overline{ad})$, which implies $\{a, d\} \in E$. See Figure 1. \qed
Lemma 2. Ordered terrain visibility graphs have the bar-property.

Proof. Consider an ordered terrain visibility graph $G = ([n], E)$. For any edge $\{a, c\} \in E$ where $a + 1 < c$, the terrain induced on the vertices in $[a..c]$, together with the edge $\{a, c\}$, creates a simple polygon. The fact that every simple polygon admits a triangulation implies the existence of a vertex $b \in (a..c)$ such that $\{a, b\} \in E$ and $\{b, c\} \in E$. See Figure 2.

Figure 2: Bar-property in ordered terrain visibility graphs.

We know that ordered terrain visibility graphs contain the Hamiltonian path $1, 2, \ldots, n$. Moreover, by Lemma 1 and Lemma 2, they also satisfy both the X-property and the bar-property. Thus we conclude the following theorem (which is equivalent to Theorem 3.5 by Abello et al. [4]).

Theorem 1. Ordered terrain visibility graphs are persistent.

4 On the sufficiency of the persistent property

Ideally, we would like to show that the persistent property is sufficient to imply that the graph is a terrain visibility graph, and to be able to recover a terrain from a given persistent graph; but this seems challenging. Abello et al. [4] showed that the visibility information of a terrain point set implies some ordering requirements on the slopes of the lines connecting pairs of points in any realization, and constructed a total order on the slopes of the lines in a generalized configuration of points with the desired visibility. It is unknown whether
there is a point set that realizes the resulting slope order. It is worth mentioning that a slope ordering consistent with the desired terrain visibility is not unique.

We give a much simpler proof that the slope ordering requirements obtained from any persistent visibility graph form a partial order. Our approach is to establish an orientation on every triple of vertices, reflecting some slope ordering requirements, such that it is consistent with the desired visibility information. Our proof also gives a faster algorithm for constructing a total order on the slopes.

Here, we first introduce the terminology that we use for the representation of the slope ordering. We next describe the overall approach of Abello et al. \[4\] briefly. We then show that our orientation is a restricted type of a 3-signotope (Section 4.3.3), which together with the work of Felsner and Weil \[28\], gives an immediate proof of the result. We also give an alternate self-contained proof (Section 4.3.4) that clarifies the implications of the graph theoretic properties on the slope orders, which may help in approaching realizability. Lastly, we present a $\Theta(n^3)$-time algorithm for constructing a total order on the slopes (Section 4.3.5) and conclude by discussing properties of our orientation.

4.1 Representation of the slope order

We use terminology similar to that used by Abello et al. \[4\] for the representation of the slope order of the (pseudo) lines connecting pairs of points in a (generalized) configuration.

A tableau of size $n$ is a two-dimensional array of $n - 1$ rows (indexed from 2 to $n$) where row $r$ contains $r - 1$ entries (indexed from 1 to $r - 1$), and whose entries are the integers $1, 2, \ldots, \binom{n}{2}$. For a tableau $T$, we refer to the entry in row $r$ and column $c$ as $T[r, c]$.

Note that $r > c$.

Consider a non-degenerate point set $\{1, 2, \ldots, n\}$. We may represent the slope ordering of the lines connecting all pairs of the points by a tableau $T$ of size $n$, such that $T[j, i]$ is the rank of the slope of the line through $i$ and $j$. (In other words, $T[j, i] = s$ if and only if the slope of the line through $i$ and $j$ is the $s$-th smallest of all such slopes.) We know that every three points $a < b < c$ have either a positive orientation (that is, $b$ lies below the segment $ac$) or a negative orientation (that is, $b$ lies above the segment $ac$). This implies that, in the tableau $T$ representing the slope ordering, we have either $T[b, a] < T[c, a] < T[c, b]$ or $T[b, a] > T[c, a] > T[c, b]$. (This establishes Lemma 3.1 of Abello et al. \[4\].)

For three integers $a < b < c$, we say the triple $T[b, a]$, $T[c, a]$, and $T[c, b]$ is oriented positively if $T[b, a] < T[c, a] < T[c, b]$. Similarly, the triple is oriented negatively if $T[b, a] > T[c, a] > T[c, b]$. The triple is balanced if either $T[b, a] < T[c, a] < T[c, b]$ or $T[b, a] > T[c, a] > T[c, b]$. A balanced tableau is a tableau whose triples are all balanced.

The skeleton $S_T$ of a tableau $T$ is a two-dimensional array of the same dimensions as $T$ where

$$S_T[c, a] = \begin{cases} 1 & \text{if } c = a + 1 \text{ or } T[c, a] > T[b, a] \text{ for all } b \in (a..c), \\ 0 & \text{otherwise}. \end{cases}$$

Figure 3 shows a balanced tableau and its skeleton.
An \( n \)-triangle is the strict lower triangle of an \( n \times n \) \((0,1)\)-matrix. For an \( n \)-triangle \( M \), \( M[r,c] \) is the entry in row \( r \) and column \( c \), and \( M[[c..d];[a..b]] \) is the matrix formed by the rows \([c..d]\) and the columns \([a..b]\). A persistent \( n \)-triangle is the \( n \)-triangle of the adjacency matrix of a persistent graph.

A skeleton \( S_T \) of a balanced tableau \( T \) of size \( n \) can be interpreted as the \( n \)-triangle of an adjacency matrix for an undirected graph with vertices \( 1, 2, \ldots, n \) and edges \( \{\{a,c\} | S_T[c,a] = 1\} \). This graph is the skeleton graph of the balanced tableau. It is easy to show that the skeleton graph of a balanced tableau representing the slope order of a terrain point set is identical to the terrain visibility graph. We know \( S_T[c,a] = 1 \) if and only if \( c = a + 1 \) or the slope of line \( ac \) is greater than the slopes of all lines \( ab \), where \( b \in (a..c) \). This means that \( S_T[c,a] = 1 \) if and only if the terrain vertices \( a \) and \( c \) are adjacent in the terrain, or all terrain vertices in \( (a..c) \) are below the line \( ac \); that is, terrain vertices \( a \) and \( c \) are mutually visible. (This argument is Lemma 3.2 in Abello et al. [4].)

A tableau representing the slope order of a simple configuration of points is balanced (See Lemma 3.1 of Abello et al. [4]). This and Theorem 1 imply that the skeleton graph of a balanced tableau representing a terrain is persistent.

Ideally, we would like to show that persistent graphs are visibility graphs of terrains. This is equivalent to showing that every persistent graph is the skeleton of a balanced tableau whose entries represent the ranks of the slopes between pairs of points in a terrain. We have not proved this but here we show that every persistent graph is the skeleton graph of a balanced tableau (Lemma 5.12 of Abello et al. [4]).

**Theorem 2.** If \( M \) is a persistent \( n \)-triangle, then there exists a balanced tableau whose skeleton is identical to \( M \).

We will prove this theorem in Section 4.3.

### 4.2 Abello et al.’s approach

Abello et al. [4] prove that a graph is persistent if and only if it is the skeleton graph of a balanced tableau. They argue that a skeleton graph of a balanced tableau is persistent directly by using the definitions. Here we only sketch their approach for the reverse direction (that is, every persistent graph is the skeleton graph of a balanced tableau).
For a persistent graph $G = (V, E)$, they define an edge $e$ to be reversible if the graph $G = (V, E \setminus e)$ remains persistent. They use the idea of reversible edges to partially order persistent graphs so that they can generate any of them in a canonical manner. Given a persistent graph with $n$ vertices, they give an algorithm that starts from a clique of $n$ vertices and successively removes a reversible edge until the desired persistent graph is generated. They use the basis of this algorithm to reconstruct a balanced tableau from a given persistent graph. Namely, they start from a balanced tableau whose skeleton represents a clique, and perform operations on the tableau entries so that the underlying skeleton becomes incrementally closer to the desired graph. They define “flush” and “augmentation” operations, which are compositions of “Coxeter” type I and type II transformations, on a balanced tableau. Each flipping of a 1 entry in the skeleton (that is, removing a reversible edge) is done through a sequence of flush and augmentation operations on the tableau and is complicated. They establish a loop invariant to show the correctness of their proposed algorithm. However, the loop invariant is not intuitive and requires a complex proof. The overall complexity of their algorithm is $\Theta(n^5)$.

### 4.3 Our main result

We reprove that every persistent graph is the skeleton graph of a balanced tableau but in a much simpler way. By the definition of a tableau’s skeleton, we know that the skeleton entries (whether 0 or 1) imply certain inequality relations amongst the tableau entries. The following lemma summarizes these relations.

**Lemma 3.** For a tableau $T$ and an $n$-triangle $M$, $S_T = M$ if and only if

1. If $M[c, a] = 1$, then $c = a + 1$ or $T[c, a] > T[b, a]$ for all $b \in (a..c)$, and
2. If $M[c, a] = 0$, then there exists $b \in (a..c)$ such that $T[b, a] > T[c, a]$.

**Proof.** Lemma 3 follows directly from the definition of the skeleton of a tableau. □

Using the combinatorial properties of a persistent graph $G$, we would like to derive a balanced tableau satisfying the conditions in Lemma 3, where $M$ is the $n$-triangle of the adjacency matrix of $G$. We show that the inequality relations required by Lemma 3 and by the balanced property give a partial order on the tableau entries; and as a result any tableau that realizes this partial order is balanced, with a skeleton graph identical to the given persistent graph.

Our approach is to orient all tableau triples so that they are consistent with Lemma 3. Since our orientation is guaranteed to be balanced, to show the existence of such a balanced tableau, we only need to show that our orientation forms a partial order on the tableau entries.

#### 4.3.1 Orienting the triples

We infer combinatorial properties on the structure of a persistent $n$-triangle, and use these structural properties in determining the orientation of a tableau triple.
Definition 4. Let $M$ be an $n$-triangle. We define $hook(abc)$ to be the substructure $M[[b..c];a] \cup M[c;[a..b]]$, where $a < b < c$. The corner of $hook(abc)$ is $M[c,a]$. The column-arm of $hook(abc)$ is $M[[b..c];a]$, and the row-arm of $hook(abc)$ is $M[c;[a..b]]$.

The half-strict $abc$-rectangle is the substructure $M[[b..c];[a..b]]$, and is denoted by $rect(abc)$. $rect(abc) \subset rect(xyz)$ means $rect(abc)$ is contained in $rect(xyz)$. (Figure 4 illustrates the half-strict $abc$-rectangle and $hook(abc)$.)

![Figure 4: The half-strict $abc$-rectangle (shaded) and $hook(abc)$ (bold outline).](image)

The following lemma identifies the structure of $hook(abc)$ in a persistent $n$-triangle $M$.

Lemma 4. If $M$ is a persistent $n$-triangle then each $hook(abc)$ is of one of the following forms (See Figure 5):

- The corner is 1, or
- Either the row-arm or the column-arm contains only zeros, or
- $M[b,a] = M[c,b] = 1$ and all other entries in $hook(abc)$ are zeros.

![Figure 5: Possible structures of $hook(abc)$ in a persistent $n$-triangle.](image)

Proof. Suppose to the contrary that there exists a $hook(abc)$ that does not have any of the above properties. Thus, we have

1. $M[c,a] = 0$;
2. there exists $i \in (a..b]$ such that $M[c,i] = 1$; and
3. there exists $j \in [b..c)$ such that $M[j,a] = 1$;
such that either \( i \neq b \) or \( j \neq b \). But this contradicts \( M \) having the X-property on the four points \( a < i < j < c \), which is impossible.

It is easy to deduce the possible structures of a half-strict \( abc \)-rectangle in a persistent \( n \)-triangle, from Lemma 4 and the X-property. Figure 6 illustrates these possible structures with regards to \( \text{hook}(abc) \). As shown in the picture, in the top four cases, each half-strict \( abc \)-rectangle contains a 1 entry whereas in the bottom four cases, each half-strict \( abc \)-rectangle contains only zeros.

\[
\begin{array}{c}
\begin{array}{c|c}
 b & \star \\
 \hline
 c & 1 \\
 a & \end{array} &
\begin{array}{c|c}
 b & 1 \\
 \hline
 c & 0 \cdots 0 \\
 a & \end{array} &
\begin{array}{c|c}
 b & 0 \\
 \hline
 c & 0 \hline
 a & 1 \\
\end{array} &
\begin{array}{c|c}
 b & 0 \\
 \hline
 c & 0 \cdots 0 \\
 a & \end{array}
\end{array}
\]

(a) \( \text{rect}(abc) \) contains a 1.

\[
\begin{array}{c}
\begin{array}{c|c}
 b & \text{all 0} \\
 \hline
 c & 0 \cdots 0 \\
 a & \end{array} &
\begin{array}{c|c}
 b & \text{all 0} \\
 \hline
 c & 0 \cdots 0 \\
 a & \end{array} &
\begin{array}{c|c}
 b & \text{all 0} \\
 \hline
 c & 0 \cdots 1 \\
 a & \end{array} &
\begin{array}{c|c}
 b & \text{all 0} \\
 \hline
 c & 0 \cdots 0 \\
 a & \end{array}
\end{array}
\]

(b) \( \text{rect}(abc) \) contains only zeros.

Figure 6: Possible structures of a half-strict \( abc \)-rectangle with regards to \( \text{hook}(abc) \) in a persistent \( n \)-triangle.

From the bar-property we conclude the following lemma.

**Lemma 5.** Let \( M \) be a persistent \( n \)-triangle. For every \( a < b < c < d \) in \([n]\), if both \( \text{rect}(abc) \) and \( \text{rect}(bcd) \) contain only zeros then \( M[(c..d); [a..b)] \) is a zero matrix.

**Proof.** The \( k \)-upper triangle of a matrix is the collection of the entries above the \( k \)-th diagonal. We prove the contrapositive. Suppose \( M[(c..d); [a..b)] \) contains a 1 entry. Let \( M[c', a'] \) be a 1 entry lying on the \( k \)-th diagonal in the matrix \( M[(c..d); [a..b)] \) such that the \( k \)-upper triangle of the matrix contains only zeros. It is easy to see that such an entry exists. We have \( M[c', a'] \) with \( a \leq a' < b < c < c' \). Since \( M \) is persistent, by the bar-property we know there exists \( b' \in (a'..c') \) such that \( M[b', a''] = 1 \) and \( M[c', b'] = 1 \). Since the \( k \)-upper triangle in the matrix \( M[(c..d); [a..b)] \) contains only zeros, we infer \( b' \leq c' \). See Figure 7(a). If \( \text{rect}(bcd) \) contains only 0 entries, then \( b' = c \). Hence \( M[b', a'] = M[c, a'] = 1 \), which implies that there is a 1 entry in \( M[c; [a..b)] \). See Figure 7(b). Therefore either \( \text{rect}(abc) \) or \( \text{rect}(bcd) \) contains a 1 entry, which concludes the proof.

Let \( M \) be a persistent \( n \)-triangle. Given \( M \), we define the **orientation function** \( \alpha_M \) from the 3-element subsets of \([n]\) to \{+, –\} as follows:
(a) The grey region contains only zeros, and hence $b \leq b' \leq c$.

(b) $\text{rect}(bcd)$ containing only zeros implies that $b' = c$, and $M[c, a'] = 1$.

Figure 7: For a persistent $M$, if $M[(c..d); [a..b)]$ contains a 1 then, by the bar-property, either $\text{rect}(abc)$ or $\text{rect}(bcd)$ contains a 1 as well.

For every $a < b < c$ in $[n]$, let

$$
\alpha_M(abc) = \begin{cases} 
+ & \text{if } \text{rect}(abc) \text{ contains a 1 entry}, \\
- & \text{if } \text{rect}(abc) \text{ contains only 0 entries}.
\end{cases}
$$

**Observation 1** (rectangle containment). For $\text{rect}(abc)$ and $\text{rect}(xyz)$ such that $\text{rect}(abc) \subset \text{rect}(xyz)$, we have

1.1. if $\alpha_M(abc) = +$, then $\alpha_M(xyz) = +$, and

1.2. if $\alpha_M(xyz) = -$, then $\alpha_M(abc) = -$.

**Observation 2.** Let $M$ be a persistent $n$-triangle. For every $a < b < c < d$ in $[n]$,

2.1. if $\alpha_M(abc) = \alpha_M(bcd) = -$, then $\alpha_M(abd) = \alpha_M(acd) = -$;

2.2. if $\alpha_M(abc) = -$ and $\alpha_M(acd) = +$, then $\alpha_M(bcd) = +$; and

2.3. if $\alpha_M(bcd) = -$ and $\alpha_M(acd) = +$, then $\alpha_M(abc) = +$.

**Proof.** Observation 2.1 is a restatement of Lemma 5. Observations 2.2 and 2.3 logically follow from Observation 2.1.

A tableau $T$ agrees with orientation function $\alpha_M$ if for all $a < b < c$,

$$
T[b, a] < T[c, a] < T[c, b] \quad \text{if } \alpha_M(abc) = +, \quad \text{and}
$$

$$
T[b, a] > T[c, a] > T[c, b] \quad \text{if } \alpha_M(abc) = -.
$$

Figure 8 illustrates the inequality relations required in a tableau that agrees with $\alpha_M$.

Suppose there exists a tableau $T$ that agrees with $\alpha_M$. The following lemma shows that our orientation captures both the balanced property and the properties required to satisfy $S_T = M$. 

---

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Lemma 6. Let $M$ be a persistent $n$-triangle. If there is a tableau $T$ whose entries agree with orientation function $\alpha_M$, then

1. $T$ is balanced, and
2. $S_T = M$.

Proof. We have each tableau triple oriented either positively or negatively, and hence all triples are balanced. Therefore $T$ is a balanced tableau. As illustrated in Figure 9, it is easy to observe that having each tableau triple oriented by $\alpha_M$ implies that:

1. if $M[c,a] = 1$, then $c = a + 1$ or $T[c,a] > T[b,a]$ for all $b \in (a..c)$, and
2. if $M[c,a] = 0$, then there exists $b \in (a..c)$ such that $T[b,a] > T[c,a]$. (A particular $b$ is $b \in (a..c)$ such that $M[b,a] = 1$ and $M[x,a] = 0$ for all $x \in (b..c)$. Notice that such $b$ always exists because $M[a+1,a] = 1$ due to the Hamiltonian path $1, 2, \ldots, n$, which is guaranteed by the persistent property.)

Thus the triple orientations induce the conditions in Lemma 3, which implies $S_T = M$. □

Figure 9: Possible triple orientations and their implied inequality relations over a tableau that agrees with the orientation are illustrated with regards to the associated half-strict rectangles.
If a tableau $T$ agrees with $\alpha_M$ then the entries of $T$ have to obey a set of inequalities (as shown in Figure 9). From the above lemma, we conclude that if the set of all these inequalities gives a partial order on the tableau entries, then any tableau realizing this partial order would be a balanced tableau with skeleton $M$. In the following, we prove that our orientation gives a partial order.

### 4.3.2 The M-tableau relation digraph $D_M$

We introduce a directed graph to represent the required inequality relations between the tableau entries so that the tableau agrees with $\alpha_M$. We show that the resulting digraph is acyclic if $M$ is persistent, which concludes the proof of Theorem 2.

**Definition 5.** Let $M$ be a persistent $n$-triangle. The M-tableau relation digraph is a directed graph $D_M = (V_M, E_M)$ such that

- $V_M = \{rc | r, c \in [n] \text{ and } r > c\}$ (the vertex $rc$ represents the tableau entry in row $r$ and column $c$), and
- $E_M = \{(ba, ca) \text{ and } (ca, cb) | \alpha_M(abc) = -\} \cup \{(cb, ca) \text{ and } (ca, ba) | \alpha_M(abc) = +\}$ (the edges represent the inequality relations over the entries of a tableau that agrees with $\alpha_M$; for a tableau $T$, the edge $(ba, ca)$ represents that $T[b,a] > T[c,a]$).

In the next section, we show that orientation function $\alpha_M$ is a signotope of a particular type. This, coupled with a result from Felsner and Weil [28], proves the acyclicity of $D_M$.

### 4.3.3 Orientation function $\alpha_M$ is a restricted type of signotope

Felsner and Weil introduced signotopes in their study of sweeps of pseudoline arrangements. We first describe their terminology.

The notation $\binom{[n]}{k}$ represents the $k$-element subsets of $[n]$.

**Definition 6.** Let $\prec$ be $\prec +$. For an integer $r$ such that $1 \leq r \leq n$, an $r$-signotope on $[n]$ is a function $\alpha : \binom{[n]}{r} \to \{-, +\}$ such that for every $(r+1)$-element subset $P$ of $[n]$ and all $i$, $j$, and $k$ such that $1 \leq i < j < k \leq r+1$ either $\alpha(P[i]) \preceq \alpha(P[j]) \preceq \alpha(P[k])$ or $\alpha(P[i]) \succeq \alpha(P[j]) \succeq \alpha(P[k])$, where $P[x]$ denotes the set $P$ minus the $x$-th largest element of $P$. We refer to this property as monotonicity.

It is easy to restate Definition 6 for $r = 3$ as follows.

**Definition 7.** A 3-signotope on $[n]$ is a function $\alpha : \binom{[n]}{3} \to \{-, +\}$ such that for every 4-element subset $\{a, b, c, d\}$ of $[n]$ with $a < b < c < d$, the orientation sequence $(\alpha(abc), \alpha(abd), \alpha(acd), \alpha(bcd))$ is monotone (that is, it has at most one change of sign). In other words, it is one of the eight columns of the table in Figure 10.

We use this definition in proving the following Theorem.
\[\begin{array}{c|cccccccc}
\alpha(abc) & + & - & - & - & + & + & - \\
\alpha(abd) & + & + & - & - & + & + & - \\
\alpha(acd) & + & + & + & - & + & - & - \\
\alpha(bcd) & + & + & + & - & - & - & - \\
\end{array}\]

Figure 10: Monotone orientation sequences.

**Theorem 3.** Let \(M\) be a persistent \(n\)-triangle. The orientation function \(\alpha_M : \binom{[n]}{3} \to \{-, +\}\) where

\[
\alpha_M(abc) = \begin{cases} 
+ & \text{if rect}(abc) \text{ contains a 1 entry} \\
- & \text{otherwise}, 
\end{cases}
\]

is a 3-signotope. Moreover, for every 4-tuple, the resulting orientation sequence excludes \((-,-,-,+)\) and \((+,-,-,-)\).

**Proof.** We need to show that for any 4-element subset \(P = \{a, b, c, d\}\) of \([n]\) with \(a < b < c < d\), the orientation sequence \((\alpha_M(abc), \alpha_M(abd), \alpha_M(acd), \alpha_M(bcd))\) is monotone. Figure 11 illustrates the relationships between \(\text{rect}(abc)\), \(\text{rect}(abd)\), \(\text{rect}(acd)\), and \(\text{rect}(bcd)\). Let \(P^{[x]}\) denote the set \(P\) minus the \(x\)-th largest element of \(P\). The position \(i\) of the orientation sequence contains \(\alpha_M(P^{[i]}).

![Figure 11: The relationships between rect(abc) (lightly shaded red), rect(abd) (antidiagonally striped red), rect(acd) (diagonally striped blue), and rect(bcd) (boldly shaded blue).](image)

We know either \(\alpha_M(bcd) = +\) or \(\alpha_M(bcd) = -\). Consider the following cases.

**Case 1.** \(\alpha_M(bcd) = +\): Since \(\text{rect}(bcd) \subseteq \text{rect}(acd)\), we conclude \(\alpha_M(acd) = +\). Thus, we may get a non-monotone orientation sequence only if \(\alpha_M(abd) = -\) and \(\alpha_M(abc) = +\). But this is impossible because \(\text{rect}(abc) \subseteq \text{rect}(abd)\).

**Case 2.** \(\alpha_M(bcd) = -\): Suppose we have a non-monotone orientation sequence. Thus the sequence contains at least one + sign. We show that the rightmost + sign in the orientation sequence propagates all the way to the left, which implies that the sequence is monotone. Note that, because \(\text{rect}(abc) \subseteq \text{rect}(abd)\), if \(\alpha_M(abc) = +\) then \(\alpha_M(abd) = +\). Therefore, the rightmost + sign is either at position 2 or at position 3. We consider each case below.
Case 2.1. \( \alpha_M(abd) = + \) and \( \alpha_M(acd) = - \): We know \( \text{rect}(abd) \) contains a 1 entry whereas \( \text{rect}(acd) \) contains only zeros. This implies that \( \text{rect}(abc) \) contains a 1 entry. Thus \( \alpha_M(abc) = + \), which concludes this case.

Case 2.2. \( \alpha_M(acd) = + \): By Observation 2.3 we infer \( \alpha_M(abc) = + \). Since \( \text{rect}(abc) \subset \text{rect}(abd) \), we conclude \( \alpha_M(abd) = + \), which concludes this case.

Therefore, \( \alpha_M \) gives a monotone orientation sequence for each 4-tuple, and hence is a 3-signotope. Moreover, by the argument above, it is easy to see that it never gives the monotone sequence \((-,-,-,+)\) or \((+,-,-,-)\), and thus it is a restricted subclass of 3-signotopes.

Felsner and Weil [28] associate an \( r \)-signotope \( \alpha \) on \([n]\) with a directed graph whose vertices are the \((r-1)\)-element subsets of \([n]\), and whose edges are

\[
\rightarrow_\alpha \equiv \{P^{[i]} \rightarrow_\alpha P^{[j]} \mid P \in \binom{[n]}{r}, 1 \leq i < j \leq r, \alpha(P) = +\}
\]

\[
\cup \{P^{[j]} \rightarrow_\alpha P^{[i]} \mid P \in \binom{[n]}{r}, 1 \leq i < j \leq r, \alpha(P) = -\}.
\]

They prove the following lemma.

**Lemma 7** (Felsner and Weil [28]). For an \( r \)-signotope \( \alpha \) on \([n]\) the graph with vertices \( \binom{[n]}{r-1} \) and edges \( \rightarrow_\alpha \) is acyclic.

For the 3-signotope \( \alpha_M \), the edge set \( \rightarrow_{\alpha_M} \) of this associated directed graph consists of the edges of \( D_M \) with their directions reversed. Therefore, \( \alpha_M \) being a 3-signotope (Theorem 3), together with Lemma 7, implies that \( D_M \) is acyclic. Hence our orientation forms a partial order on the tableau entries. Lemma 6 guarantees that if a tableau agrees with our orientation, then it is balanced and has the desired skeleton. As a result, any tableau that realizes this partial order would be a balanced tableau with the desired skeleton. This concludes Theorem 2.

A 3-signotope is realizable as an ordered generalized configuration of points (by Theorem 7 of Felsner and Weil [28] and the concept of duality between pseudoline arrangements and generalized configurations of points). By ordered here we mean that in some direction the projection of the points (of the generalized configuration) are ordered from 1 to \( n \). However, not all 3-signotopes are realizable as ordered point sets. Our orientation is a strict subclass of 3-signotopes. It excludes the monotone orientation sequences \((-,-,-,+)\) and \((+,-,-,-)\), which puts more constraints on the point set realizing it (that is, it forbids certain substructures in any realization). This may help prove realizability.

Felsner and Weil give an abstract combinatorial proof that the graphs associated with signotopes are acyclic in general. A deeper understanding of the implications of the persistent property on the slope ordering has the potential to determine additional constraints that may aid in realization. In the next section, we give an alternate self-contained proof to clarify these implications and explain in more detail how the persistent property prevents cycles.
4.3.4 An alternate self-contained proof that $D_M$ is acyclic

Since the vertices of the digraph $D_M$ are in one-to-one correspondence with the entries of the $n$-triangle $M$, and to simplify referring to the graph edges in the following lemmas, we consider a drawing of the graph such that each vertex $yx$ is placed at the position of the entry $M[y,x]$ and each edge is a straight-line segment. Throughout this section, by an edge we mean an edge of the drawing of the graph. The row $b$ of the graph is the subgraph induced on the vertices $\{bx \mid x \in [1..b)\}$. The column $a$ of the graph is the subgraph induced on the vertices $\{ya \mid y \in (a..n]\}$.

The following lemma shows that each row or column of the graph is acyclic.

**Lemma 8.** Let $D_M$ be an $M$-tableau relation digraph where $M$ is a persistent $n$-triangle. Each row or column of $D_M$ is acyclic.

**Proof.** For an edge $e$ that has both its endpoints in the same row or column, we define $Rect(e)$ as follows: For every $a < b < c$, let $Rect((ba,ca)) = Rect((ca,ba)) = Rect((ca,cb)) = Rect((cb,ca)) = rect(abc)$. We say $Rect(e)$ is the half-strict rectangle associated with the edge $e$.

Suppose there is a cycle that has all its vertices in the same row. Let $e_{out}$ be the out-going cycle edge incident on the rightmost point of the cycle. Since $e_{out}$ directs to the left, we know $Rect(e_{out})$ contains a 1 entry. It is easy to see that $Rect(e_{out})$ is contained in the union of all half-strict rectangles associated with cycle edges directed to the right. These half-strict rectangles contain only zeros. This contradicts $Rect(e_{out})$ having a 1 entry. See Figure 12.

A similar contradiction arises if a cycle occurs in a column (we consider the in-going cycle edge incident on the topmost point in this case).

![Figure 12](image)

Figure 12: $Rect(e_{out})$ is contained in the union of the half-strict rectangles associated with edges directed to the right.

**Corollary 1** (Transitivity on a row or a column). Let $D_M = (V_M,E_M)$ be an $M$-tableau relation digraph where $M$ is a persistent $n$-triangle. We have:

- if $(ba_1,ba_2)$ and $(ba_2,ba_3)$ are in $E_M$, then $(ba_1,ba_3)$ is in $E_M$, and
- if $(b_1a,b_2a)$ and $(b_2a,b_3a)$ are in $E_M$, then $(b_1a,b_3a)$ is in $E_M$. 
Proof. Since there is a directed edge between every pair of vertices in the same row or column, if the desired edge, for example \((ba_1, ba_3)\), is not in \(E_M\), then the edge with the opposite direction is in the graph, for example \((ba_3, ba_1)\). But that creates a cycle contradicting Lemma 8.

The following two lemmas establish that the existence of certain paths in \(D_M\) implies the existence of other edges in \(D_M\). If \(D_M\) contains a cycle, the edges implied by the cycle lead to a contradiction.

**Lemma 9.** Let \(D_M = (V_M, E_M)\) be the \(M\)-tableau relation digraph defined on a persistent \(n\)-triangle \(M\). For every \(a < b < c < d\) in \([n]\) the digraph \(D_M\) satisfies the four properties below (see Figure 13):

**Property 1:** if \((db, cb) \in E_M\), then \((da, ca) \in E_M\).

**Property 2:** if \((ca, da) \in E_M\), then \((cb, db) \in E_M\).

**Property 3:** if \((da, ca) \in E_M\) and \((ca, cb) \in E_M\), then \((db, cb) \in E_M\).

**Property 4:** if \((ca, cb) \in E_M\) and \((cb, db) \in E_M\), then \((ca, da) \in E_M\).

![Figure 13: The properties of \(D_M\). The solid edges imply the existence of the dashed edges.](image)

**Proof.** We prove each property as follows:
Property 1. If \((db, cb) \in E_M\) then \(\alpha_M(bcd) = +\) (by Definition 5). Since \(\text{rect}(bcd) \subset \text{rect}(acd)\), we conclude \(\alpha_M(acd) = +\), which implies \((da, ca) \in E_M\).

Property 2. If \((ca, da) \in E_M\) then \(\alpha_M(acd) = -\). Since \(\text{rect}(bcd) \subset \text{rect}(acd)\), we conclude \(\alpha_M(bcd) = -\), which implies \((cb, db) \in E_M\).

Property 3. If \((da, ca) \in E_M\) and \((ca, cb) \in E_M\), then \(\alpha_M(acd) = +\) and \(\alpha_M(abc) = -\). Thus by Observation 2.2 we have \(\alpha_M(bcd) = +\), which implies \((db, cb) \in E_M\).

Property 4. If \((ca, cb) \in E_M\) and \((cb, db) \in E_M\), then \(\alpha_M(abc) = -\) and \(\alpha_M(bcd) = -\). Thus by Observation 2.1 we have \(\alpha_M(acd) = -\), which implies \((ca, da) \in E_M\).

Lemma 10. Let \(D_M = (V_M, E_M)\) be an \(M\)-tableau relation digraph where \(M\) is a persistent \(n\)-triangle. Let \(P\) be a path \(b_0a_0, b_1a_1, \ldots, b_ka_k\) in \(D_M\) such that the vertex-induced subgraph of \(D_M\) on the vertex set \(\{b_0a_0, b_1a_1, \ldots, b_ka_k\}\) is acyclic. Let \(m\) be the index such that \(b_m\) is a greatest element in \(\{b_0, b_1, \ldots, b_k\}\). If \(b_k < b_0\) then \((b_ma_k, b_ka_k)\) is in \(E_M\). If \(b_0 < b_k\) then \((b_0a_0, b_ma_0)\) is in \(E_M\). (See Figure 14.)

![Diagram](image)

(a) If \(b_k < b_0\), then \((b_ma_k, b_ka_k)\) (dashed) is in \(E_M\).

(b) If \(b_0 < b_k\), then \((b_0a_0, b_ma_0)\) (dashed) is in \(E_M\).

Figure 14: The solid path \(b_0a_0, b_1a_1, \ldots, b_ka_k\) implies the existence of the dashed edge.

Proof. First observe that, for every \(i\) such that \(0 \leq i < k\), either \(a_i = a_{i+1}\) or \(b_i = b_{i+1}\). We proceed by induction on the number of edges in \(P\). For succinctness, we occasionally use \(v_i\) for \(b_ia_i\). Clearly the statement holds when the length of \(P\) equals one. We show that if the statement holds for all paths of length less than \(k\), then the statement holds for all paths of length \(k\).
Suppose that $b_m$ is greater than both $b_0$ and $b_k$. If $b_k < b_0$, let $P_1$ be the path from $v_m$ to $v_k$. If $b_0 < b_k$, let $P_1$ be the path from $v_0$ to $v_m$. We know $P_1$ is a path of length less than $k$ whose vertex-induced subgraph in $D_M$ is acyclic. Thus, by the induction hypothesis, we have $P_1$ implies the existence of the desired edge in $E_M$. See Figure 15.

![Figure 15](image)

Figure 15: The case when $b_m > b_0$ and $b_m > b_k$. The thick part of the solid path, $P$, is $P_1$. By the induction hypothesis, $P_1$ implies the existence of the dashed edge.

Now suppose that $b_m = b_0$. Let $P_1$ be the path from $v_0$ to $v_{k-1}$. Note that $b_{k-1} \leq b_0$. If $b_{k-1} = b_0$, then the edge $(v_{k-1}, v_k)$ is already the desired edge. This is because if $b_{k-1} = b_0$ then $b_k > b_k$ (by the conditions of the lemma, $b_k$ is not $b_0$), and hence $a_{k-1} = a_k$. Thus $(v_{k-1}, v_k) = (b_{k-1}a_{k-1}, b_ka_k) = (b_m a_k, b_ka_k)$. So we assume $b_{k-1} < b_0$. By the induction hypothesis on $P_1$, we infer that $(b_0 a_{k-1}, b_{k-1} a_{k-1})$ is in $E_M$. By Corollary 1 and Lemma 9 (properties 1 and 3), it is easy to observe that this and $(v_{k-1}, v_k) \in E_M$ imply that $(b_0 a_k, b_ka_k)$ is in $E_M$. See Figure 16(a).

Finally assume that $b_m = b_k$. The argument is similar to the previous case. Let $P_1$ be the path from $v_1$ to $v_k$. Note that $b_1 \leq b_k$. If $b_1 = b_k$ then the edge $(v_0, v_1)$ is already the desired edge. This is because $b_1 = b_k$ implies $b_1 > b_0$ (because $b_k$ is not $b_0$ by the conditions of the lemma), and hence $a_1 = a_0$. Thus $(v_0, v_1) = (b_0 a_0, b_ka_0)$. So we assume $b_1 < b_k$. By the induction hypothesis on $P_1$, we infer that $(b_1 a_1, b_k a_1)$ is in $E_M$. By Corollary 1 and Lemma 9 (properties 2 and 4), it is easy to observe that this and $(v_0, v_1) \in E_M$ imply that $(b_0 a_0, b_ka_0)$ is in $E_M$. See Figure 16(b). This concludes the proof.

**Lemma 11.** Let $D_M = (V_M, E_M)$ be an $M$-tableau relation digraph where $M$ is a persistent $n$-triangle. The digraph $D_M$ is acyclic.

**Proof.** Suppose to the contrary that $D_M$ is cyclic. Let $C = b_0 a_0, b_1 a_1, \ldots, b_{k-1} a_{k-1}, b_0 a_0$ be a shortest cycle in $D_M$. Observe that, for every $i$ such that $0 \leq i \leq k-1$, either $a_i = a_{i+1 \mod k}$ or $b_i = b_{i+1 \mod k}$. Again, for succinctness, we occasionally use $v_i$ for $b_i a_i$.

From Corollary 1, we know that $C$ is composed of edges that alternate horizontal and vertical alignment. Assume that $(v_0, v_1)$ is a topmost edge in $C$ (we have $b_0 = b_1$ and no element in $\{b_0, b_1, \ldots, b_{k-1}\}$ is smaller than $b_0$). Let $(v_m, v_{m+1})$ be a bottommost edge in $C$ (so we have $b_m = b_{m+1}$ and no element in $\{b_0, b_1, \ldots, b_{k-1}\}$ is greater than $b_m$). By Lemma 8, we know that $b_m > b_0$. Now, let $P_1 = v_1, v_2, \ldots, v_m$ and $P_2 = v_m+1, v_{m+2}, \ldots, v_{k-1}, v_0$. Both $P_1$ and $P_2$ are of length at least one (because $(v_0, v_1)$ and $(v_m, v_{m+1})$ are in different rows). Since the length of the paths $P_1$ and $P_2$ are less than the length of $C$, we know
(a) The case when \( b_m = b_0 \) and \( b_k \neq b_0 \). We have \( a_k < a_{k-1} \), \( b_k = b_{k-1} \) or \( a_k < a_{k-1} \), \( b_k > b_{k-1} \) or \( a_k = a_{k-1} \), \( b_k < b_{k-1} < b_0 \). In the first and second subcases, we use the properties 1 and 3 in Lemma 9, respectively. In the third and fourth subcases we use Corollary 1.

(b) The case when \( b_m = b_k \) and \( b_1 \neq b_k \). We have \( a_0 < a_1 \), \( b_0 = b_1 \) or \( a_0 > a_1 \), \( b_0 = b_1 \) or \( a_0 = a_1 \), \( b_0 > b_1 \) or \( a_0 = a_1 \), \( b_0 < b_1 < b_k \). In the first and second subcases, we use the properties 4 and 2 in Lemma 9, respectively. In the third and fourth subcases we use Corollary 1. Note that the third subcase is impossible because \((v_0, v_1)\) contradicts the existence of the dotted edge.

Figure 16: The cases \( b_m = b_0 \) and \( b_m = b_k \). The thick black path is \( P_1 \). The path \( P \) is composed of \( P_1 \) and the thin solid (blue) edge. From \( P_1 \), by the induction hypothesis, we conclude the existence of the dotted (blue) edge. The thin solid and dotted (blue) edges imply the existence of the very thick (red) edge.
that their vertex-induced subgraphs in $D_M$ are acyclic. Thus, by Lemma 10, we conclude $(b_1a_1, b_ma_1)$ and $(b_{m+1}a_0, b_0a_0)$ are in $E_M$. Therefore, we have the three edges $(b_{m+1}a_0, v_0)$, $(v_0, v_1)$, and $(v_1, b_ma_1)$ in $E_M$, which contradicts Properties 3 and 4 of Lemma 9. See Figure 17.

Figure 17: The contradiction if $D_M$ were cyclic. The bold path $P_1$ (in blue) implies the downward dotted (blue) edge. The bold path $P_2$ (in red) implies the upward dotted (red) edge. The path outlined with thick lines (in green) is a contradiction.

This Lemma gives an alternate proof of Theorem 2.

Lemma 11 may be viewed as an alternative and perhaps more intuitive proof for the acyclicity of graphs associated with signotopes, when restricted to 3-signotopes. While the substantial ideas of both proofs are the same, our proof explains in more detail the manner in which sets of edges imply the existence of other edges. In the case of a cycle, this results in a contradiction. In fact, exploring these implications is what led us to establishing our orientation over the tableau entries. Our proof has the potential to determine additional slope ordering constraints that may aid in realization. Certain slope orders on $n-1$ vertex terrain visibility graphs can be preserved when extending the graph by an additional vertex; others cannot. Our proof helps to understand the slope order constraints that would exclude the non-extensible slope orders.

4.3.5 Time complexity of constructing a total order

We may orient all triples in $\Theta(n^3)$ time, by computing $\alpha_M$ for all triples in order of the distance of the corner of the associated half-strict rectangle from the diagonal. From the triple orientation $\alpha_M$, we construct the digraph $D_M$ representing the required inequality relations over a tableau that agrees with $\alpha_M$ (by Definition 5). This takes $\Theta(n^3)$ time because the digraph has $\Theta(n^2)$ vertices and $\Theta(n^3)$ edges. The graph $D_M$ forms a partial order on the tableau entries (by Theorem 2). Then, we do a topological sort on the digraph, which also takes $\Theta(n^3)$ time. As a result, we give an $\Theta(n^3)$ time algorithm for constructing a total order. See Algorithm 1.
Algorithm 1: Algorithm to construct a total order over the entries of a balanced tableau whose skeleton is $M$.

**Input:** A persistent $n$-triangle $M$.

**Output:** A total order on the entries of a balanced tableau with skeleton $M$.

```
for row ← 2 to n do
    LeftmostOne[row] ← row - 1;
for diag ← 2 to n - 1 do
    for col ← 1 to n - diag do
        row ← col + diag;
        if $M[row, col] = 1$ then
            LeftmostOne[row] ← col;
            foreach $x$ in (col..row) do
                $\alpha_M(col, x, row) ← +$;
        else
            foreach $x$ in (col..row) do
                if $\alpha_M(col, x, row - 1) = +$ or $LeftmostOne[row] < x$ then
                    $\alpha_M(col, x, row) ← +$;
                else
                    $\alpha_M(col, x, row) ← -$;

Construct digraph $D_M$ by Definition 5
return the topological sort of $D_M$
```

4.3.6 Remarks on the partial order resulting from orientation $\alpha_M$

We would like to mention that the partial order resulting from our orientation may not encode all possible slope orderings. This is because we orient all triples, which may put more constraints on the tableau entries than are required in order to guarantee a balanced tableau with the desired skeleton. In general, a triple orientation of a point set realizing an induced subgraph may not be extensible to a triple orientation of a point set realizing the entire graph. For instance, in Figure 18, the induced subgraph on the vertex set $\{1, 2, 3, 4\}$ admits either orientation for the points 1, 3, 4 for a realization. But the triple needs to be oriented positively in any realization of the entire graph.

![Figure 18](image.png)

Figure 18: A skeleton graph in which the points 1, 3, 4 must be oriented positively in any realization of the entire graph, but may be oriented arbitrarily in the subgraph outlined in dashed red.
It is easy to see that the additional constraints that our orientation defines make the triple orientation of vertices \{1, 2, \ldots, i\} extensible to the triple orientation of vertices \{1, 2, \ldots, i + 1\}, for all \(i\) such that \(3 \leq i < n\). This suggests that, using our orientation, it may be possible to reconstruct the terrain by incrementally (from left to right) realizing the point set. In addition, our orientation excludes \((-,-,-,+)\) and \((+,-,-,-)\) from the possible orientation sequences on the 4-tuples, hence forbids certain substructures in any realization. This puts more constraints on the point set realizing it, which may help to prove realizability and perhaps to reconstruct the terrain.

5 Relationship to oriented matroids and related work

Our approach may be interpreted in the context of oriented matroids. The same context has been used by Abello and Kumar [5, 7] and later by O’Rourke and Streinu [42] in addressing visibility graphs of pseudo simple polygons. Both these results require more geometric information than the mere vertex visibility graphs in order to establish the triple orientations. In fact, they use additional combinatorial concepts to impute unique blocking vertices (articulation points in the terminology of O’Rourke and Streinu) to pairs of vertices that are not mutually visible, in order to define a unique shortest path between two distinct vertices. The orientation of a triple is then based on whether all three vertices lie on the same shortest path or not. In our work, we restrict ourselves to terrain visibility graphs. We determine a triple orientation based on whether the associated half-strict rectangle contains a 1 entry or not. Our orientation is established solely from the vertex visibility graph (rather than requiring a blocking vertex assignment). For terrains, a shortest ordered path in the visibility graph is the Euclidean shortest path in any realization of the graph. The term shortest path, in what follows, refers to the shortest ordered path in the visibility graph. It is easy to see that, for \(a < b < c\), if the half-strict \(abc\)-rectangle contains a 1 entry, then \(b\) is not on any shortest path containing both \(a\) and \(c\). This is because a 1 entry in the half-strict \(abc\)-rectangle indicates the existence of an edge \(\{u, v\}\) that passes over \(b\) (where \(a \leq u < b < v \leq c\)). Let \(\{u, v\}\) be the longest such edge. If \(b\) lies on a shortest path from \(a\) to \(c\) then \(\{u, v\}\) is not \(\{a, c\}\). Also, the shortest path must cross \(\{u, v\}\), but then the X-property contradicts \(\{u, v\}\) being the longest such edge. Similarly, if the half-strict \(abc\)-rectangle does not contain a 1 entry, then all shortest paths containing \(a\) and \(c\), contain \(b\) as well. This implies that our orientation happens to be the same as the orientation defined in the previous results. However, we use specific graph theoretic properties of terrain visibility graphs, and consequently we show more about the structure of the underlying oriented matroid, which may help to prove realizability.

Since oriented matroids are closely related to Knuth’s axioms [38] on three-point orientation predicates, we give a succinct description of these axioms and the relevant results here. Later we use Knuth’s terminology, which may help us better distinguish between the results from Abello and Kumar [7], O’Rourke and Streinu [42], and our work.

The clockwise relation \(pqr\) states that the circle through points \((p, q, r)\) is traversed clockwise when we encounter the points in cyclic order \(p, q, r, p\). The following axioms hold for the clockwise relation between sets of up to five points in the Euclidean plane:
Axiom 1 (cyclic symmetry) \( pqr \implies qrp. \)

Axiom 2 (antisymmetry) \( pqr \implies \neg prq. \)

Axiom 3 (nondegeneracy) \( pqr \lor \neg prq. \)

Axiom 4 (interiority) \( tqr \land ptr \land pqt \implies pqr. \)

Axiom 5 (transitivity) \( tsr \land tsp \land tsq \land tpr \land tqr \implies tpr. \)

The ternary relations that satisfy Axioms 1 – 5 are called CC systems (short for “counterclockwise systems”). The ternary relations that satisfy Axioms 1 – 3 and Axiom 5 are pre-CC systems. Uniform oriented matroids correspond to Axiom 5. Axiom 4 is equivalent to a special class of uniform oriented matroids, called acyclic. In fact, Axiom 5 captures almost all the important properties of Axiom 4, and thus pre-CC systems are not much different from full CC systems. More precisely, a set of triples is a pre-CC system if and only if it can be obtained from a CC system by negating a subset of its points, where negating a point complements the value of all triples that contain that point.

Abello and Kumar [7] show that their orientation forms a simplicial chirotope of rank 3, which equivalently represents a uniform rank 3 oriented matroid. It has been shown that simplicial rank 3 chirotopes are equivalent to Knuth’s pre-CC systems [10, 38]. Abello and Kumar show that if the resulting chirotope is realizable then their necessary conditions would characterize visibility graphs of simple polygons (when the additional information of the blocking vertex assignment is given). However, since some pre-CC systems that satisfy their conditions are not realizable (that is, they do not arise from actual points in the plane), such conditions fail to characterize visibility graphs of straight-line simple polygons.

O’Rourke and Streinu [42] show that their orientation forms a CC system. CC systems are equivalent to uniform acyclic rank 3 oriented matroids (or identically affine rank 3 simplicial chirotopes). Uniform acyclic rank 3 oriented matroids are in turn equivalent to arrangements of pseudolines, by the Folkman-Lawrence representability theorem [29]. Shor [47] and Mnëv [39] show that not all arrangements of pseudolines are stretchable. In fact, Goodman and Pollack [35] show that almost all CC systems on \( n \) points are unrealizable, in the limit as \( n \to \infty \). Even though O’Rourke and Streinu [42] remark that their orientation forms a strict subclass of all CC systems, Streinu [50, 51] proves that their characterization (of a variant of pseudo-visibility graphs containing more information than vertex pseudo-visibility graphs) allows graphs that are not visibility graphs of straight-line polygons.

We study terrain visibility graphs, which induce stronger conditions. Namely, we know that terrain visibility graphs satisfy the X-property, which not all simple polygons possess. Using the specific graph properties of terrain visibility graphs, we prove stronger results: our orientation is a strict subclass of 3-signotopes. By case analysis, it is easy to see that every 3-signotope is a CC system (and consequently a pre-CC system). However, we may have CC systems that are not 3-signotopes. For instance, let \( f \) be a cyclic symmetric and antisymmetric ternary relation on the set \( \{a, b, c, d, e\} \) such that 
\[
\begin{align*}
f(abc) &= -, \quad f(abd) = +, \quad f(abe) = -, \quad f(acd) = -, \quad f(ace) = -, \quad f(ade) = +, \quad f(bcd) = -, \quad f(bce) = -, \quad f(bde) = +, \quad \text{and} \quad f(cde) &= +, \quad \text{where} \quad a < b < c < d < e.
\end{align*}
\]
Then \( f \) satisfies all the five axioms of
CC systems, but does not satisfy the monotonicity property on the 4-tuples \((a,b,c,d)\) and \((a,b,d,e)\), and hence is not a 3-signotope. Figure 19 illustrates the relation between pre-CC systems, CC systems, and 3-signotopes.

Figure 19: The relation between pre-CC systems, CC systems, and 3-signotopes.

Knuth’s CC systems may be interpreted as characterizing the local sequences of generalized configurations of points. Likewise, Streinu [49] gives the necessary and sufficient properties of local sequences that arise from generalized configurations of points (where the resulting allowable sequence\(^1\) may not include the identity permutation). These properties indicate that an orientation (clockwise or counterclockwise) of every triple that obeys a “generalized transitivity” law, is realizable as a generalized configuration of points for some ordering of the point set (or equivalently, it forms a uniform rank 3 acyclic oriented matroid).

In this work, we study ordered graphs (that is, a graph with a total order \(1,\ldots,n\) over its vertices). We orient all triples such that they are balanced, and prove that our orientation is a 3-signotope on the ordered set \([n]\). A 3-signotope on \([n]\) is realizable as a generalized configuration of points with ordering \(1,\ldots,n\). We prove that our orientation obeys the transitivity law; and hence forms a partial order on the slope order requirements. Felsner and Weil [28] show that the transitivity of the balanced orientations in an ordered set is identical to the monotonicity of all the 4-tuples; or in other words, to the orientation function being a 3-signotope. This suggests that our self-contained proof for the acyclicity of the M-tableau relation digraph implies that our orientation function is a 3-signotope. We prove this fact directly in Theorem 3.

6 Summary

We give a streamlined proof of Abello et al.’s result [4]. Our approach is to establish an orientation on every triple of vertices, reflecting some slope ordering requirements, such that it is consistent with the desired visibility information. We prove that the slope ordering

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\(^1\) Suppose \(P\) is the set of numbered points \(\{p_1,\ldots,p_n\}\) in the Euclidean plane. Let \(L\) be a directed line not orthogonal to any line determined by two members of \(P\), and project \(P\) orthogonally onto \(L\). The projected points define a permutation of the indices \(1,\ldots,n\). If \(L\) rotates counterclockwise, the permutation changes whenever \(L\) passes through a direction perpendicular to a line connecting two points of \(P\). (When \(L\) has turned through an angle \(\pi\), the resulting permutation of the indices will be the reverse of the initial permutation.) Such a periodic sequence of permutations is called an allowable sequence.
requirements obtained from our orientation form a partial order. Our proof gives a \( \Theta(n^3) \)-time algorithm for constructing a total order on the slopes, which improves on the \( \Theta(n^5) \)-time algorithm of Abello et al. [4]. We show that our orientation is a restricted type of a 3-signotope, which together with the work of Felsner and Weil [28], gives an immediate proof of the result. We also give an alternate self-contained proof that clarifies the implications of the X-property and the bar-property on the slope orders, which has the potential to introduce new slope order constraints that exclude certain non-extensible slope orders. Our orientation is preserved when extending the graph by additional vertices. It also forbids certain substructures in any point set realizing it. These properties may be helpful in approaching realizability.

References


