REFERENCES


stable equilibria or periodic orbits, which are usually referred to
as multistability or multiperiodicity, respectively. The stability of multiple equilibria of recurrent neural networks has been
studied in [2], [3], [10], [11], [19], [21].

As models of human brains, neural networks have memory
function; i.e., the state of the present time is related to the
state of the past. Time delay provides information of history.
As in many other dynamical systems, it is well known that
delays may result in instability. So, to study delayed neural
networks, one must address the problem of how to remove
this destabilizing effect.

In the past decades, the studies of recurrent neural networks
with their various generalizations have attracted considerable
research interest (see [3], [6], [8], [10], [18] and the refer-
ences therein). It is shown that the n-neuron recurrent neural
networks with one step piecewise linear activation function
have 2^n locally exponentially stable equilibrium points located
in saturation regions (see [14], [15]). In order to increase
storage capacity, for example, a star-style activation function
can be defined with k steps. As a result, it is desirable to
develop effective and efficient design procedures such that the
resulting neural network can have k^n (k ≥ 2) locally stable
equilibria or locally attractive periodic states.

Motivated by the above discussions, our aim in this brief is
to explore the multistability of recurrent neural networks with
time-varying delays evoked by the piecewise linear activation
function [recurrent neural networks evoked by the piecewise
linear activation function (PLRNNs)]. Some sufficient condi-
tions are obtained to ensure that an n-neuron recurrent neural
network with k-step activation function can have (4k - 1)^n
equilibrium points and (2k)^p of them are locally exponentially
stable.

The remainder of this brief consists of the following
sections. Section II describes some preliminaries. The main
results are stated in Sections III. Simulation results of an
illustrative example are given in Section IV. Finally, our
cclusion is given in Section V.

II. PRELIMINARIES

A. Notations

Let C([t_0 - τ, t_0), D) be the Banach space of functions
mapping [t_0 - τ, t_0) into D ⊆ ℝ^n with norm defined by
||f|| = max_{t ∈ [t_0 - τ, t_0)} ||f(t)||, where φ(x) =
(φ_1(x), φ_2(x), ..., φ_n(x)) ∈ C([t_0 - τ, t_0), D). Denote ||x|| =
max_{t ∈ [t_0 - τ, t_0)} ||x(t)|| as the vector norm of the vector x =
(x_1, x_2, ..., x_n). For any given integer k ≥ 1, ℝ = (−∞, +∞) can be divided
into 4k - 1 intervals

(−∞, −(4k - 3)] ∪ [−(4k - 3), −(4k - 5)]
∪ ... ∪ [−(3, −1)] ∪ [−1, 1) ∪ [1, 3)
∪ ... ∪ [4k - 5, 4k - 3) ∪ [4k - 3, +∞). (1)

For example, when k = 2, (−∞, +∞) can be divided into 7
intervals, and all intervals of (−∞, +∞) are depicted in Fig. 1.

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Let

\[ D_1 = \left\{ \left( -\infty, -(4k - 3) \right], \left[ 4k - 3, +\infty \right) \right\} \]

\[ D_2 = \left\{ \left( -1, 1 \right] \right\} \]

\[ D_3 = \left\{ \left[ 3 - 4q, 5 - 4q \right], \left[ 4q - 5, 4q - 3 \right], q = 2, 3, \ldots, k \right\} \]

Then, D_1 is made up of 2k intervals; D_2 is made up of one
interval; D_3 is made up of 2k - 2 intervals, and (−∞, +∞) =
D_1 ∪ D_2 ∪ D_3. Let ε be a very small constant such that 0 <
ε < 1/(4k - 3). Denote

\[ D_{\Omega_1} = \left\{ \left. \left[ -1/e, -(4k - 3) \right], \left[ 4k - 3, 1/e \right) \right\} \right\} \]

\[ D_{\Omega_2} = \left\{ \left[ -1, 1 - \epsilon \right] \right\} \]

\[ D_{\Omega_3} = \left\{ \left[ 3 - 4q + \epsilon, 5 - 4q \right], \left[ 4q - 5, 4q - 3 - \epsilon \right] \right\} \]

\[ q = 2, 3, \ldots, k \]

\[ \Omega_1 = \left\{ \prod_{i=1}^{n} \ell_i \mid \ell_i = \left\{ \begin{array}{ll}
(4k - 3), & (4k - 5) \mid \cdots \mid \{-3, -1\} \mid \{-1, 1\} \mid \{1, 3\} \mid \cdots \mid \{4k - 5, 4k - 3\} \mid \{4k - 3, +\infty\} \end{array} \right. \right\} \]

\[ \Omega_2 = \left\{ \prod_{i=1}^{n} \ell_i \mid \ell_i = \left\{ \begin{array}{ll}
(4k - 3), & (4k - 5) \mid \cdots \mid \{-3, -1\} \mid \{-1, 1\} \mid \{1, 3\} \mid \cdots \mid \{4k - 7, 4k - 5\} \mid \{4k - 3, +\infty\} \end{array} \right. \right\} \]

It is obvious that Ω_1 is made up of (2k)^n parts and Ω_2 ⊂ Ω_1,
where Ω_2 is defined in (2). For example, when n = 2, Ω_2 is
made up of 7^2 parts, and all parts of Ω_2 are depicted in
Fig. 2.

Meanwhile, when n = 2, Ω_2 is made up of 4^2 parts, and all
parts of Ω_2 are depicted in Fig. 3.
Recall that $\varepsilon$ is a very small constant such that $0 < \varepsilon \ll 1/(4k - 3)$. Denote

$$
\Omega_b = \left\{ \prod_{i=1}^n (\ell_i^0, \ell_i^0) = \left[ -1/\varepsilon, -(4k - 3) \right] \text{ or } [e - (4k - 3), -(4k - 5)] \text{ or } \cdots \text{ or } [e - 3, -1] \text{ or } [e - 1, 1 - \varepsilon] \text{ or } [1, 3 - \varepsilon] \text{ or } \cdots \right\}
$$

$$
\Omega_w = \left\{ \prod_{i=1}^n (\ell_i^0, \ell_i^0) = \left[ -1/\varepsilon, -(4k - 3) \right] \text{ or } [e - (4k - 5), -(4k - 7)] \text{ or } \cdots \text{ or } [e - 5, -7] \text{ or } [e - 3, -1] \text{ or } [1, 3 - \varepsilon] \text{ or } [5, 7 - \varepsilon] \text{ or } \cdots \right\}
$$

It is obvious that if $\Omega \in \Omega_b$ (or $\Omega_w$), then $\Omega$ is a bounded and closed set.

**B. Model**

Consider a class of PLRNNs described by the following equation:

$$
\frac{dx_i(t)}{dt} = -x_i(t) + \sum_{j=1}^n \omega_{ij} f(x_j(t)) + \xi_i,
$$

(6)

where $x = (x_1, \ldots, x_n)^T \in \mathbb{R}^n$ is the state vector; $A = (a_{ij})$ and $B = (b_{ij})$ are connection weight matrices that are not assumed to be symmetric; $u = (u_1, \ldots, u_n)^T \in \mathbb{R}^n$ is an input vector; $\forall t \geq t_0$, $\forall i, j \in \{1, 2, \ldots, n\}$, $\tau_{ij}(t)$ is the time-varying delay that satisfies $0 \leq \tau_{ij}(t) \leq \tau = \max_{i,j \in \mathbb{N}_n} [\sup_{t \geq t_0} \tau_{ij}(t)]$ with $\tau$ being a constant number; and $f$ is a neuron activation function satisfying

$$
f(r) = \left\{ \begin{array}{ll}
4k - 3, & r \in [4k - 3, +\infty) \\
2r - (4k - 3), & r \in [4k - 5, 4k - 3) \\
\vdots & \\
2r - 5, & r \in [3, 5) \\
1, & r \in [1, 3) \\
-1, & r \in (-1, 1) \\
2r + 5, & r \in [-5, -3) \\
\vdots & \\
2r + 4k - 3, & r \in [3 - 4k, 5 - 4k] \\
3 - 4k, & r \in (-\infty, 3 - 4k]
\end{array} \right.
$$

(7)

Such PLRNN represents a general class of neural networks with or without delays. In particular, when $k = 2$, $f(r)$ is depicted in Fig. 4.

**C. Properties**

The initial condition of PLRNN (6) is assumed to be

$$
\phi(\theta) = (\phi_1(\theta), \phi_2(\theta), \ldots, \phi_n(\theta))^T
$$

(8)

where $\phi(\theta) \in C([t_0 - \tau, t_0], D)$, $D \in \mathbb{R}^n$. Let $x(t; t_0, \phi)$ be the solution of PLRNN (6) with initial condition (8). It means that $x(t; t_0, \phi)$ is continuous and satisfies PLRNN (6) and $x(t; t_0, \phi) = \phi(u)$, for $u \in [t_0 - \tau, t_0]$. Also let $x(t)$ be the solution of PLRNN (6).
Definition 2: The equilibrium point $x^*$ of PLRNN (6) is said to be locally exponentially stable in region $D$, if there exist constants $\alpha > 0$, $\beta > 0$ such that $\forall t \geq t_0$

$$\| x(t; x_0, \phi) - x^* \| \leq \beta \| \phi - x^* \| \exp(-\alpha t - \tau_0)$$

where $x(t; x_0, \phi)$ is the solution of PLRNN (6) with any initial condition $\phi(0) \in C([t_0, t] \times D)$, and $D$ is said to be a locally exponentially attractive set of the equilibrium point $x^*$. When $D = \mathbb{R}^n$, $x^*$ is said to be globally exponentially stable.

Lemma 5 (13)): Let $D$ be a bounded and closed set in $\mathbb{R}^n$, and $H$ be a mapping on complete metric space $(D, || \cdot ||)$, where $x, \ y \in D, ||x - y|| = \max_{i \in \{1, \ldots, n\}} |x_i - y_i|$ is measurement in $D$. If $H(D) \subseteq D$ and there exists a constant $\alpha < 1$ such that $\forall x, y \in D, ||H(x) - H(y)|| \leq \alpha ||x - y||$, then there exists a unique $x^* \in D$ such that $H(x^*) = x^*$.

III. MULTIPLE EQUILIBRIA OF PLRNNs

A. Equilibria in $\Omega_1$:

In this section, we show that an $n$-neuron neural network can have $(4k - 1)^p$ equilibria located in $\Omega_1$.

Theorem 3: For the given integer $k \geq 1$, if $\forall i \in \{1, 2, \ldots, n\}$

$$a_{ii} + b_{ii} - (4k - 3) \sum_{j \neq i} (a_{ij} + b_{ij}) - |a_{ii}| > 1$$

then PLRNN (6) has $(4k - 1)^p$ equilibria located in $\Omega_1$.

Proof: (From (9) and (10), there exists a small enough $\varepsilon_i > 0$, such that)

$$(1 - \varepsilon_i)(a_{ii} + b_{ii} - 1) >$$

$$\frac{(4k - 3) \sum_{j \neq i} (a_{ij} + b_{ij}) + |a_{ii}|}{1 + 2 - \varepsilon_i}$$

$$a_{ii} + b_{ii} - (4k - 3) \sum_{j \neq i} (a_{ij} + b_{ij}) - |a_{ii}| >$$

$$\frac{1 + 2 - \varepsilon_i}{(4k - 3)}$$

$$\forall \varepsilon > 0, \ \varepsilon \leq \min\{1/(4k - 3), \varepsilon_i\}, \ \forall \Omega \in \Omega_{\kappa}, \ \text{where} \ \Omega_{\kappa} \text{ is defined in (4)}. \ \Omega \text{ is a bounded and closed set for PLRNN (6), there exists an equilibrium point } x^* = (x^*_1, \ldots, x^*_n)^T \in \Omega \text{ if and only if for } i = 1, 2, \ldots, n$$

$$-x^*_i + \sum_{j \neq i} (a_{ij} + b_{ij}) f(x^*_j) + u_i = 0.$$
From (9), (12), and (20)

\[ H(k) = (4q - 3)x(k) + \sum_{j=1}^{m} (a_{ij} + b_{ij})f(x_j) + u_i \]

\[ \geq (4q - 3)(a_{ii} + b_{ii}) + \sum_{j=1}^{m} (a_{ij} + b_{ij})f(x_j) + u_i \]

\[ \leq (4q - 3)(a_{ii} + b_{ii}) + \sum_{j=1}^{m} (a_{ij} + b_{ij})f(x_j) + u_i \]

When \( i = m \), from (11) and (21)

\[ H_m(k) = \sum_{j=1}^{n} (a_{mj} + b_{mj})f(x_j) + u_m \]

\[ \leq \frac{(4q - 3)\|a\|_m + \|b\|_m}{(\|a\|_m + \|b\|_m) - 1} \]

\[ \leq 1 - \epsilon \]  

\[ H_m(k) \leq (4q - 3)\|a\|_m + \|b\|_m - |u_m| \]

\[ \geq \frac{(4q - 3)\|a\|_m + \|b\|_m - |u_m|}{(\|a\|_m + \|b\|_m) - 1} \]

\[ \geq \epsilon - 1. \]

When \( i \in \{m+1, m+2, \ldots, n\} \), from (17), assume that \( f(x_i) = 2x_i, -(4q - 3) \) (the case of \( f(x_i) = 2x_i, +(4q - 3) \) can be similarly proved). From (10), (13), and (22)

\[ H_i(k) = \left( \sum_{j=1}^{m} (a_{ij} + b_{ij})f(x_j) + u_i \right) / \left( 1 - 2(a_{ii} + b_{ii}) \right) \]

\[ = (4q - 3)(a_{ii} + b_{ii}) + u_i \]

\[ \leq (4q - 3)(a_{ii} + b_{ii}) + u_i \]

\[ \leq (4q - 3)(a_{ii} + b_{ii}) + u_i \]

\[ \leq \left( \sum_{j=1}^{m} (a_{ij} + b_{ij})f(x_j) + u_i \right) / \left( 2(a_{ii} + b_{ii}) - 1 \right) + 4q - 3 \]

\[ \leq 2 - \epsilon + 4q - 3 = 4q - 1 - \epsilon \]

\[ \leq 4q - 3 < 4q - 3 - \epsilon \]

\[ H_i(k) = \left( \sum_{j=1}^{m} (a_{ij} + b_{ij})f(x_j) + u_i \right) / \left( 2(a_{ii} + b_{ii}) - 1 \right) + 4q - 3 \]

\[ \leq \left( \sum_{j=1}^{m} (a_{ij} + b_{ij})f(x_j) + u_i \right) / \left( 2(a_{ii} + b_{ii}) - 1 \right) + 4q - 3 \]

\[ \geq 2 - \epsilon + 4q - 3 = 4q - 1 - \epsilon \]

\[ \leq 4q - 3 < 4q - 3 - \epsilon \]

\[ (27) \]

\[ H_i(k) = \left( \sum_{j=1}^{m} (a_{ij} + b_{ij})f(x_j) + u_i \right) / \left( 1 - 2(a_{ii} + b_{ii}) \right) \]

\[ = (4q - 3)(a_{ii} + b_{ii}) + u_i \]

\[ \leq (4q - 3)(a_{ii} + b_{ii}) + u_i \]

\[ \leq (4q - 3)(a_{ii} + b_{ii}) + u_i \]

\[ \leq \left( \sum_{j=1}^{m} (a_{ij} + b_{ij})f(x_j) + u_i \right) / \left( 1 - 2(a_{ii} + b_{ii}) \right) \]

\[ \leq \left( \sum_{j=1}^{m} (a_{ij} + b_{ij})f(x_j) + u_i \right) / \left( 1 - 2(a_{ii} + b_{ii}) \right) \]

\[ \leq 2 - \epsilon + 4q - 3 = 4q - 1 - \epsilon \]

\[ \leq 4q - 3 < 4q - 3 - \epsilon \]

\[ (28) \]

Hence, from (23)-(28)

\[ H(\Omega) \subset \Omega. \]
In addition, $\forall x, y \in \bar{\Omega}$, when $i \in \{1, 2, \ldots, m - 1\}$, from (20)

$$
\|H(x) - H(y)\| = \|(a_{in} + b_{io} + k_{in} - y_a) + \sum_{j \neq m} (a_{ij} + b_{ij})(f(x_i) - f(y_i))\| \\
\leq 2 \sum_{j=1,j \neq m} |a_{ij} + b_{ij}| |x_j - y_j| 
$$

(30)

when $i = m$, from (21)

$$
\|H(x) - H(y)\| = \sum_{j=1,j \neq m} \frac{(a_{ij} + b_{ij})}{(a_{ii} + b_{ii} - 1)} (f(x_i) - f(y_i)) \\
\leq 2 \sum_{j=1,j \neq m} |a_{ij} + b_{ij}| |x_j - y_j| 
$$

(31)

when $i \in \{m + 1, m + 2, \ldots, n\}$, from (22)

$$
\|H(x) - H(y)\| = \sum_{j=1,j \neq m} \frac{(a_{ij} + b_{ij})}{(a_{ii} + b_{ii} - 1)} (f(x_i) - f(y_i)) \\
\leq 2 \sum_{j=1,j \neq m} |a_{ij} + b_{ij}| |x_j - y_j| 
$$

(32)

When $k \geq 2$, from (9) and (10)

$$(a_{ii} + b_{ii}) > 1 + 5 \sum_{j \neq i, j \neq m} (a_{ij} + b_{ij})$$

$$\frac{2}{5} \geq \sum_{j \neq i, j \neq m} |a_{ij} + b_{ij}|.$$

Let

$$a = \max \left\{ 2 \sum_{j=1,j \neq m} |a_{ij} + b_{ij}|, \frac{2}{5} \sum_{j=1,j \neq m} |a_{ij} + b_{ij}| / (a_{ii} + b_{ii} - 1) \right\}$$

then $a < 1$. From (30)-(32)

$$\|H(x) - H(y)\| \leq a \|x - y\|.$$

From (29) and $a < 1$, according to Lemma 5, there exists $x^* \in \bar{\Omega}$ such that (19) has a unique solution located in $\bar{\Omega}$, i.e., $x^*$ is an isolated equilibrium point located in $\bar{\Omega}$ of (6).

When $k = 1$, in (30) and (31), $2 \sum_{j=1,j \neq m} (a_{ij} + b_{ij})$ can be replaced by $\sum_{j \neq i, j \neq m} (a_{ij} + b_{ij})$. Hence, it follows from (9), (10), (20), and Lemma 5 that there exists $x^* \in \bar{\Omega}$ such that $x^*$ is an isolated equilibrium point located in $\bar{\Omega}$ of (6).

Since $\bar{\Omega}$ is divided into $(4k - 1)^p$ parts, by arbitrariness of $\varepsilon$, the PLRNN (6) has $(4k - 1)^p$ isolated equilibrium points.

**B. Equilibria in $\bar{\Omega}$**

In this section, we show that an n-neuron neural network can have $(2k)^p$ equilibrium points located in $\bar{\Omega}$ and these equilibrium points are locally stable.

**Theorem 4:** For any given integer $k \geq 1$, if $\forall \ v \in \{1, 2, \ldots, m\}$, (9) and (10) hold, then PLRNN (6) has $(2k)^p$ stable equilibrium points located in $\bar{\Omega}$.

**Proof:** Let $\varepsilon > 0$, $x \in (4k - 3), \forall \Omega, \bar{\Omega} \in \bar{\Omega}_k$, where $\bar{\Omega}_k$ is defined in (5), according to Theorem 3, for PLRNN (6), there exists an equilibrium point $x^* = (x^*_1, \ldots, x^*_n) \in \bar{\Omega}$.

Let $\gamma(t) = x(t) - x^*$. From (7), when $x(t) \in \bar{\Omega}$, $\sum_{j=1}^n f(x_j(t))$ is a constant, i.e., $\sum_{j=1}^n f(x_j(t)) = \sum_{j=1}^n f(x^*_j)$. So when $x(t), x(t - \tau(t))$, and $x^* \in \bar{\Omega}$

$$
\frac{dx_j(t)}{dt} = -\gamma_j(t). 
$$

(33)

Obviously, the equilibrium point of (33) is exponentially stable, so $x^*$ is a locally exponentially stable equilibrium point of PLRNN (6) in $\bar{\Omega}$.

Obviously, the number of such region $\bar{\Omega}$ is $(2k)^p$ in $\mathbb{R}^n$. Hence, by arbitrariness of $\varepsilon$, PLRNN (6) has $(2k)^p$ stable equilibrium points located in $\bar{\Omega}$.

**Remark 12:** When $k = 1$, $f(r) = (r + 1) - |r - 1|/2$. In [14] and [15], it has been proven that an n-neuron neural network can have $2^p$ equilibrium points or periodic orbits located in saturation regions and these equilibrium points or periodic orbits are locally exponentially attractive by using the activation function $f(r) = (r + 1) - |r - 1|/2$. Hence, Theorems 3 and 4 are the improvement and extension of the existing stability results in the literature.

**IV. ILLUSTRATIVE EXAMPLE**

**Example 3:** Consider a PLRNN for $k = 2, n = 2$

$$
x_1(t) = -x_1(t) + 0.5f(x_1(t)) + 0.5f(x_2(t)) + 0.5f(x_1(t - 0.1)) + 0.1 \\
x_2(t) = -x_2(t) - 0.62f(x_1(t)) + 0.6f(x_2(t)) + 0.6f(x_1(t - 0.2)) + 0.7f(x_2(t - 0.2)) - 0.2.
$$

(34)

According to Theorems 3 and 4, the PLRNN (34) has $2^2$ isolated equilibrium points and $2^2$ of them are locally expo-
nentially stable. The simulation result of PLRNN (34) with 200 random initial values is depicted in Fig. 5.

V. CONCLUSION

In this brief, we have shown that n-dimensional PLRNNs can have \((4k-1)^n\) isolated equilibrium points and \((2k)^n\) of them are locally exponentially stable. The analytical results have improved and extended the existing stability results in the literature. Multistability in recurrent neural networks is an important issue when it comes to associative memories. Increasing storage capacity is a fundamental problem, hence, the importance of the work undertaken in this brief is obvious. Several extensions would be welcome as future work: such as, studying regions of attraction of the exponentially stable steady states, generalizing these results to Hopfield or Cohen–Grossberg neural networks (with nonlinear activation functions).

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