ABOUT UNIQUELY COLORABLE MIXED HYPERTREES

ANGELA NICULITSA∗

Department of Mathematical Cybernetics
Moldova State University
Mateevici 60, Chişinău, MD-2009, Moldova

AND

VITALY VOLOSHIN†

Institute of Mathematics and Informatics
Moldovan Academy of Sciences
Academiei, 5, Chişinău, MD-2028, Moldova

Abstract

A mixed hypergraph is a triple $H = (X, C, D)$ where $X$ is the vertex set and each of $C$, $D$ is a family of subsets of $X$, the $C$-edges and $D$-edges, respectively. A $k$-coloring of $H$ is a mapping $c : X \to [k]$ such that each $C$-edge has two vertices with the same color and each $D$-edge has two vertices with distinct colors. $H = (X, C, D)$ is called a mixed hypertree if there exists a tree $T = (X, E)$ such that every $D$-edge and every $C$-edge induces a subtree of $T$. A mixed hypergraph $H$ is called uniquely colorable if it has precisely one coloring apart from permutations of colors. We give the characterization of uniquely colorable mixed hypertrees.

Keywords: colorings of graphs and hypergraphs, mixed hypergraphs, unique colorability, trees, hypertrees, elimination ordering.

1991 Mathematics Subject Classification: 05C15.

∗Partially supported by DAAD, TU-Dresden.
†Partially supported by DFG, TU-Dresden.
Preliminaries

We use the standard concepts of graphs and hypergraphs from [1, 2] and updated terminology on mixed hypergraphs from [4, 5, 6, 7].

A mixed hypergraph is a triple $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ where $X$ is the vertex set, $|X| = n$, and each of $\mathcal{C}$, $\mathcal{D}$ is a family of subsets of $X$, the $\mathcal{C}$-edges and $\mathcal{D}$-edges, respectively.

A proper $k$-coloring of a mixed hypergraph is a mapping $c : X \to [k]$ from the vertex set $X$ into a set of $k$ colors so that each $\mathcal{C}$-edge has two vertices with the same color and each $\mathcal{D}$-edge has two vertices with different colors. The chromatic polynomial $P(\mathcal{H}, k)$ gives the number of different proper $k$-colorings of $\mathcal{H}$.

A strict $k$-coloring is a proper coloring using all $k$ colors. By $c(x)$ we denote the color of vertex $x \in X$ in the coloring $c$. The maximum number of colors in a strict coloring of $\mathcal{H}$ is the upper chromatic number $\bar{\chi}(\mathcal{H})$; the minimum number is the lower chromatic number $\chi(\mathcal{H})$.

If for a mixed hypergraph $\mathcal{H}$ there exists at least one coloring, then it is called colorable. Otherwise $\mathcal{H}$ is called uncolorable. Throughout the paper we consider colorable mixed hypergraphs.

If $\mathcal{H} = (X, \mathcal{C}, \mathcal{D})$ is a mixed hypergraph, then the subhypergraph induced by $X' \subseteq X$ is the mixed hypergraph $\mathcal{H}' = (X', \mathcal{C}', \mathcal{D}')$ defined by setting $\mathcal{C}' = \{ C \in \mathcal{C} : C \subseteq X' \}$, $\mathcal{D}' = \{ D \in \mathcal{D} : D \subseteq X' \}$ and denoted by $\mathcal{H}' = \mathcal{H}/X'$.

The mixed hypergraph $\mathcal{H} = (X, \emptyset, \emptyset)$ ($\mathcal{H} = (X, \emptyset, \emptyset)$) is called "$\emptyset$-hypergraph" ("$\emptyset$-hypergraph") and denoted by $\mathcal{H}_{\emptyset}$ ($\mathcal{H}_{\emptyset}$). If $\mathcal{H}_{\emptyset}$ contains only $\emptyset$-edges of size 2 then from the coloring point of view it coincides with classical graph ([2]). We call it $\emptyset$-graph.

For each $k$, let $r_k$ be the number of partitions of the vertex set into $k$ nonempty parts (color classes) such that the coloring constraint is satisfied on each $\mathcal{C}$- and $\mathcal{D}$- edge. In fact $r_k$ equals the number of different strict $k$-colorings of $\mathcal{H}$ if we disregard permutations of colors. The vector $R(\mathcal{H}) = (r_1, \ldots, r_n) = (0, \ldots, 0, r_\chi(\mathcal{H}), \ldots, r_{\bar{\chi}}(\mathcal{H}), 0, \ldots, 0)$ is the chromatic spectrum of $\mathcal{H}$.

For the simplicity we assume that two strict $k$-colorings are considered the same if they can be obtained from each other by permutation of colors. In this case the number of different strict $k$-colorings coincides with $r_k(\mathcal{H})$.

A mixed hypergraph $\mathcal{H}$ is called a uniquely colorable (uc for short) [5] if it has just one strict coloring.
A mixed hypergraph \( \mathcal{H} = (X, \mathcal{C}, \mathcal{D}) \) is called \textit{uc-orderable} [5] if there exists the ordering of the vertex set \( X \), say \( X = \{x_1, x_2, \ldots, x_n\} \), with the following property: each subhypergraph \( \mathcal{H}_i = \mathcal{H}/X_i \) induced by the vertex set \( X_i = \{x_i, x_{i+1}, \ldots, x_n\} \) is uniquely colorable. The corresponding sequence \( x_1, \ldots, x_n \) will be called a \textit{uc-ordering} of \( \mathcal{H} \).

A sequence \( x_0, x_1, \ldots, x_{t+1} \) of vertices is called a \textit{D-path} if \( (x_i, x_{i+1}) \in \mathcal{D}, 0 \leq i \leq t \). A mixed hypergraph \( \mathcal{H} = (X, \mathcal{C}, \mathcal{D}) \) is called \textit{reduced} if \( |\mathcal{C}| \geq 3 \) for each \( C \in \mathcal{C} \), and \( |\mathcal{D}| \geq 2 \) for each \( D \in \mathcal{D} \), and moreover, no one \( C \)-edge (\( D \)-edge) is included in another \( C \)-edge (\( D \)-edge).

As it follows from the splitting-contraction algorithm [6, 7] colorings properties of arbitrary mixed hypergraph may be obtained from some reduced mixed hypergraph. Therefore, throughout the paper we consider reduced mixed hypergraphs.

Let \( \mathcal{C}(x)(\mathcal{D}(x)) \) denote the set of \( C \)-edges (\( D \)-edges) containing vertex \( x \in X \). Call the set \( N(x) = \{y : y \in X, y \neq x, \mathcal{C}(x) \cap \mathcal{C}(y) \neq \emptyset, \text{ or } \mathcal{D}(x) \cap \mathcal{D}(y) \neq \emptyset\} \) the \textit{neighbourhood} of the vertex \( x \) in a mixed hypergraph \( \mathcal{H} \). In other words, the neighbourhood of a vertex \( x \) consists of those vertices which are contained in common \( C \)-edges or \( D \)-edges with \( x \) except \( x \) itself.

A vertex \( x \) is called \textit{simplicial} [8] in a mixed hypergraph if its neighbourhood induces a uniquely colorable mixed subhypergraph. A mixed hypergraph \( \mathcal{H} = (X, \mathcal{C}, \mathcal{D}) \) is called \textit{pseudo-chordal} [8] if there exists an ordering \( \sigma \) of the vertex set \( X \), \( \sigma = (x_1, x_2, \ldots, x_n) \), such that the vertex \( x_j \) is simplicial in the subhypergraph induced by the set \( \{x_j, x_{j+1}, \ldots, x_n\} \) for each \( j = 1, 2, \ldots, n-1 \).

\textbf{Definition} [8]. A mixed hypergraph \( \mathcal{H} = (X, \mathcal{C}, \mathcal{D}) \) is called a \textit{mixed hypertree} if there exists a tree \( T = (X, \mathcal{E}) \) such that every \( C \)-edge induces a subtree of \( T \) and every \( D \)-edge induces a subtree of \( T \).

Such a tree \( T \) is called further a \textit{host tree}. The edge set of a host tree \( T \) is denoted by \( \mathcal{E} = \{e_1, e_2, \ldots, e_{n-1}\} \), \( e_i = (x, y), x, y \in X, i = 1, 2, \ldots, n-1 \).

\section{Uniquely Colorable Mixed Hypertrees}

Let \( \mathcal{H} = (X, \mathcal{C}, \mathcal{D}) \) be an arbitrary mixed hypergraph.

\textbf{Definition}. A sequence of vertices of \( \mathcal{H} \), \( x = x_0, x_1, \ldots, x_k = y, k \geq 1 \), is called \((x, y)\)-invertor iff:
(1) \(x_i \neq x_{i+1}, \ i = 0, 1, \ldots, k - 1\);
(2) \((x_i, x_{i+1}) \in D, \ i = 0, 1, \ldots, k - 1\);
(3) if \(x_j \neq x_{j+2}\) then \((x_j, x_{j+1}, x_{j+2}) \in C, \ j = 0, 1, \ldots, k - 2\).

In \(\mathcal{H}\) for two vertices \(x, y \in X\) there may exist many \((x, y)\)-invertors. The shortest \((x, y)\)-inverter contains minimal number of vertices. Two \((x, y)\)-invertors are different if they have at least one distinct vertex. A \((x, y)\)-inverter with \(x = y\) is called cyclic inverter.

Definition. In a mixed hypertree, a cyclic inverter is called simple if all \(C\)-edges are different and every \(D\)-edge appears consecutively precisely two times.

Let \(\mu = (z_0, z_1, \ldots, z_k = z_0), \ k \geq 6,\) be some simple cyclic inverter in a mixed hypertree. Without loss of generality assume that \(z_0 \neq z_1 \neq z_2 \neq z_0\). From the definition of simple cyclic inverter it follows that \(z_0 \neq z_1 \neq \ldots \neq z_{k-2}\) and \(z_1 = z_3 = \ldots = z_{k-1} = y,\) where \(y\) is the center of some star in the host tree \(T\).

Theorem 1. If \(\mathcal{H} = (X, C, D)\) is a mixed hypertree then

1. \(\chi(\mathcal{H}) \leq 2;\)
2. if, in addition, \(|D| \leq n - 2\) then \(r_2(\mathcal{H}) \geq 2.\)

Proof. (1) It follows from the possibility to start at any vertex and to color \(\mathcal{H}\) alternatively by the colors 1 and 2 along the host tree \(T\).

(2) Let \(T = (X, \mathcal{E})\) be a host tree of the mixed hypertree \(\mathcal{H}\). Since \(|D| \leq n - 2\) in \(T\) there exists an edge \(e = (x, y) \notin D\). Starting with the vertices \(x, y\) we can construct two different colorings with two colors in the following way. First, put \(c(x) = c(y) = 1\) and color all the other vertices alternatively along the tree \(T\) with the colors 2, 1, 2, \ldots. Second, apply the same procedure starting with \(c(x) = 1\) and \(c(y) = 2.\)

Theorem 2. A mixed hypertree \(\mathcal{H} = (X, C, D)\) is uniquely colorable if and only if for every two vertices \(x, y \in X\) there exists an \((x, y)\)-inverter.

Proof. \(\Rightarrow\) Let \(c\) be the unique strict coloring of the mixed hypertree \(\mathcal{H}\). We show that for any two vertices \(x, y \in X\) there exists an \((x, y)\)-inverter.

Suppose \(\mathcal{H}\) has two vertices \(u, v \in X\) such that there is no \((u, v)\)-inverter in \(\mathcal{H}\). Consider the unique \((u, v)\)-path in the host tree \(T\) of \(\mathcal{H}\). The assumption implies that either in \(\mathcal{H}\) there is no \(D\)-path connecting \(u\) and \(v\) or in
Theorem 3. 

4 is redundant because there is no invertor containing such \( \geq \).

In a uniquely colorable mixed hypertree \( H \).

Corollary 1. 

If there is no \( D \)-path connecting \( u \) and \( v \) then by Theorem 1(2) \( H \) has two different colorings with two colors. This contradicts to the unique colorability of mixed hypertree \( H \).

Assume that in the sequence \( u = x_1, x_2, \ldots, x_p = v \) there exists a triple of pairwise different vertices \( x_j, x_{j+1}, x_{j+2} \) not belonging to \( C \).

If there is no \( D \)-path connecting \( u \) and \( v \) then by Theorem 1(2) \( H \) has two different colorings with two colors. This contradicts to the unique colorability of mixed hypertree \( H \).

Consider \( (x_j, x_{j+1}, x_{j+2}) \). From Theorem 1(1) it follows that the number of colors in the unique coloring \( c \) of \( H \) is 2. Recolor the vertex \( x_{j+2} \) and all vertices on even distance from \( x_{j+2} \) in the component \( T_2 \) with the new color. The obtained coloring is a proper coloring of \( H \) different from \( c \), a contradiction.

(2) \( c(x_j) \neq c(x_{j+2}) \). Since \( (x_j, x_{j+1}, x_{j+2}) \in \mathcal{D} \) we have that \( c(x_j) \neq c(x_{j+1}) \neq c(x_{j+2}) \). Consequently, \( H \) is colored with at least three colors. But according to Theorem 1 every mixed hypertree can be colored with two colors, a contradiction.

\( \Leftarrow \) Assume that any two vertices \( x, y \in \mathcal{X} \) are joined by an \((x, y)\)-invertor. Suppose \( H \) has at least two strict colorings \( c_1 \) and \( c_2 \). Then there exist two vertices, say \( x', y' \), such that \( c_1(x') = c_1(y') \) but \( c_2(x') \neq c_2(y') \). Consider \((x', y')\)-invertor \( x' = x_0, x_1, \ldots, x_k = y' \). From the definition of invertor follows that if \( k \) is even then in all possible colorings the vertices \( x' \) and \( y' \) have the same color. If \( k \) is odd then in all possible colorings the vertices \( x' \) and \( y' \) have distinct colors. Consider the unique \((x', y')\)-path connecting the vertices \( x', y' \) on the host tree \( T \). One can see that the parity of \( k \) coincides with the parity of length of the path. Moreover, it is true for any other \((x', y')\)-invertor. Therefore, in all colorings either \( c(x') = c(y') \) or \( c(x') \neq c(y') \), a contradiction.

\[ \text{Corollary 1. If } H \text{ is a uniquely colorable mixed hypertree then } \mathcal{D} = \mathcal{E}. \]

Definition. Let \( H = (\mathcal{X}, \mathcal{C}, \mathcal{D}) \) be a mixed hypergraph. The \( C \)-edge \( C \subseteq \mathcal{C} \) is called redundant if \( R(H) = R(H_1) \), where \( H_1 = (\mathcal{X}, \mathcal{C} \setminus \{C\}, \mathcal{D}) \).

In a uniquely colorable mixed hypertree \( H = (\mathcal{X}, \mathcal{C}, \mathcal{D}) \) any \( C \)-edge of size \( \geq 4 \) is redundant because there is no invertor containing such \( C \)-edge.

Theorem 3. In a uniquely colorable mixed hypertree \( H = (\mathcal{X}, \mathcal{C}, \mathcal{D}) \) a \( C \)-edge \( C \) of size 3 is redundant if and only if there exists a simple cyclic invertor containing \( C \).
**Proof.** Let $C = (x_1, x_2, x_3)$ be the redundant $C$-edge. By definition $\mathcal{H}' = (X, C', D)$ where $C' = C \setminus \{C\}$ is a uniquely colorable mixed hypertree. Then for the vertices $x_1$ and $x_3$ in $\mathcal{H}'$ there exists an $(x_1, x_3)$-invertor: $x_1 = z_0, z_1, \ldots, z_k = x_3$. Construct the $(x_1, x_1)$-invertor in the following way: $x_1 = z_0, z_1, \ldots, z_k = x_3, x_2, x_1$. This invertor is a simple cyclic invertor of $\mathcal{H}$ containing $C$.

Conversely, suppose that $C$-edge, $C = (x_1, x_2, x_3)$ is contained in some simple cyclic invertor $x_1 = z_0, z_1, \ldots, z_k = x_3, x_2, x_1$. Then the vertices $x_1$ and $x_3$ are joined by two different $(x_1, x_3)$-invertors: $(x_1, x_2, x_3) = C$ and $(x_1 = z_0, z_1, \ldots, z_k = x_3) = (x_1, x_3)'$-invertor. In each $(x, y)$-invertor containing $C$ replace this $C$-edge by $(x_1, x_3)'$-invertor. Thus, $\mathcal{H}' = (X, C \setminus \{C\}, D)$ is uniquely colorable, i.e., the $C$-edge is redundant.

Let us have a mixed hypergraph $\mathcal{H} = (X, C, D)$. Consider $X = X_1 \cup X_2 \cup \ldots \cup X_i$ any $i$-coloring of $\mathcal{H}$, $\chi(\mathcal{H}) = i \leq \bar{\chi}(\mathcal{H})$. Choose any $X_j$ and construct touching graph $L_j = (X_j, V_j)$ in the following way: if some $C \in C$ satisfies $C \cap X_j = \{x, y\}$ and $|C \cap X_k| \leq 1$, $k \neq j$, for some $x, y \in X_j$, then $(x, y) \in V_j$ (cf. pair graphs [3]).

**Theorem 4.** If a mixed hypertree $\mathcal{H} = (X, C, D)$ is uniquely colorable then in its 2-coloring the touching graphs $L_1$ and $L_2$ are connected.

**Proof.** By Theorem 1(2), Corollary 1 we obtain $|D| = n - 1$, $\bar{\chi} = 2$ for each uniquely colorable mixed hypertree. If at least one touching graph is disconnected, then we can construct a new coloring of $\mathcal{H}$ with 3 colors by assigning new color to the vertices of one component. This assures the proper coloring also of any $C$-edge of size $\geq 4$.

**Corollary 2.** The minimal number of $C$-edges in any uniquely colorable mixed hypertree $\mathcal{H} = (X, C, D)$ is $n - 2$.

**Proof.** Let $\mathcal{H}$ be a uniquely colorable mixed hypertree. Consider its unique 2-coloring, say $X = X_1 \cup X_2$, and construct the touching graphs $L_1 = (X_1, V_1)$, $L_2 = (X_2, V_2)$. The minimal number of edges in $L_i$ to be connected is $|X_i| - 1$, and in this case each of $L_i$ is a tree, $i = 1, 2$. Since every edge in $L_i$ corresponds to some $C$-edge of $\mathcal{H}$, we obtain that the minimal number of $C$-edges is:

$$|X_1| - 1 + |X_2| - 1 = |X| - 2.$$
Corollary 3. In a uniquely colorable mixed hypertree $\mathcal{H} = (X, C, D)$ the number of redundant $C$-edges is $|C| - n + 2$.

**Proof.** Indeed, consider touching graphs $L_i$, and construct a spanning trees $T_i$, $i = 1, 2$. Each elementary cycle in $L_i$ generates some simple cyclic inverter in $\mathcal{H}$. Therefore, each $C$-edge of $\mathcal{H}$ which has a size $\geq 4$ or corresponds to some edge of $L_i$ which is a chord with respect to $T_i$, is redundant. Hence, the assertion follows. 

**Remark.** Redundant $C$-edge may become not redundant after deleting from $C$ some another redundant $C$-edges.

**Definition.** A mixed hypertree $\mathcal{H} = (X, C, D)$ is called complete if every edge of the host tree $T$ forms a $D$-edge of $\mathcal{H}$, and every path on three vertices of $T$ forms a $C$-edge in $\mathcal{H}$.

Therefore, having the host tree $T$ for the complete mixed hypertree $\mathcal{H} = (X, C, D)$ we obtain that $D = \mathcal{E}$.

Denote by $M$ the number of $C$-edges of a complete mixed hypertree $\mathcal{H} = (X, C, D)$. Then

$$M = \sum_{x \in T} \left( \frac{d(x)}{2} \right),$$

where $d(x)$ is the degree of vertex $x$ in the host tree $T$.

Examples show that for any $k > 1$ one can construct a mixed hypertree $\mathcal{H} = (X, C, D)$ with $|D| = n - 1$, $n - 2 \leq |C| \leq M$ and $\chi(\mathcal{H}) = k$. Therefore these bounds on $|D|$ and $|C|$ are not sufficient for the mixed hypertrees to be uniquely colorable.

**Proposition 1.** A uniquely colorable mixed hypertree with $|C| = n - 2$ is a pseudo-chordal mixed hypergraph.

**Proof.** Since $\mathcal{H}$ is uniquely colorable mixed hypertree and $|C| = n - 2$ then it contains no redundant $C$-edges and, moreover, all $C$-edges have the size 3. It follows that there exists a pendant vertex, say $x$, of the host tree $T = (X, \mathcal{E})$ which belongs to precisely one $C$-edge, say $(x, y, z)$. The neighbourhood of $x$ induces a complete $D$-graph on 2 vertices, which itself is uniquely colorable. Consequently, the vertex $x$ is simplicial in $\mathcal{H}$. Deleting the vertex $x$ and $C$-edge and $D$-edge containing it, obtain $\mathcal{H}'$ which
is uniquely colorable mixed hypertree with minimal number of $C$-edges. Indeed, if $\mathcal{H}'$ would be not uniquely colorable, then two distinct colorings of $\mathcal{H}'$ formed different colorings of $\mathcal{H}$ because $c(x) = c(z)$, a contradiction. ■

**Remark.** Redundant $C$-edges enlarge the neighbourhood of some vertices without affecting any coloring. Therefore, to recognise the pseudo-chordality we need to delete the redundant $C$-edges.

From the Theorem 4, Corollaries 2–4 and Proposition 1 we conclude that a uc-orderable mixed hypertree $\mathcal{H}$ can be recognised by consecutive elmination of pendant vertices of $D$-graph $\mathcal{H}_D$ in special ordering by applying the following

**Algorithm** (uc-ordering).

**Input:** A mixed hypertree $\mathcal{H} = (X, C, D)$, $\sigma$ – $n$-dimensional empty vector.

**Idea:** Simultanious decomposition of $\mathcal{H}_D$, spanning trees $T_1$ and $T_2$ of touching graphs $L_1, L_2$, respectively, by pendant vertices.

**Iterations:**
1. If there is a vertex $x \in X$ belonging to none $C$-edge of size 3 or $D$-edge of size 2 then return NON UC. Otherwise remove from $C$ all elements of size $\geq 4$.
2. Color $D$-graph $\mathcal{H}_D$ with two colors.
3. Construct touching graphs $L_1$ and $L_2$.
4. If $L_i$, $i = 1, 2$, is not connected then return NON UC.
5. For $L_i$ construct spanning tree $T_i$, $i = 1, 2$.
6. $i := 1$.
7. While in $T_i$ there exists a vertex $x$ pendant in both $T_i$ and $\mathcal{H}_D$ then delete it from $T_i$ and $\mathcal{H}_D$ and include $x$ in $\sigma$.
8. If at least one of $T_1$ and $T_2$ is not empty then go to 9. Otherwise return UC, $\sigma$-uc-ordering.
9. If $i = 1$ then assign $i := i + 1$, otherwise $i := i - 1$. Go to 7.

**Remark.** All chords of graph $L_i$ with respect to spanning tree $T_i$, $i = 1, 2$, correspond to redundant $C$-edges in $\mathcal{H}$. The trees $T_1$ and $T_2$ provide existence of unique $(x, y)$-invertor for any $x, y \in X$. The last assures at any step of the algorithm the existence of a vertex, say $x$, pendant in both $\mathcal{H}_D$ and one
of $T_1$ or $T_2$. Notice that not every elimination of pendant vertices generates a uc-ordering in $\mathcal{H}_D$.

**Example.** Given the mixed hypertree $\mathcal{H}$ with $X = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\}$, $\mathcal{C} = \{(0, 1, 2); (0, 1, 3); (0, 2, 3); (0, 3, 4); (0, 4, 5); (0, 5, 6); (0, 4, 6); (0, 2, 7); (0, 7, 8); (7, 8, 9); (9, 8, 10); (9, 10, 11); (9, 11, 12); (9, 10, 12); (9, 12, 13); (9, 13, 14); (9, 14, 15); (9, 13, 15)\}$, and $\mathcal{D} = \{(0, 1); (0, 2); (0, 3); (0, 4); (0, 5); (0, 6); (0, 7); (7, 8); (8, 9); (9, 10); (9, 11); (9, 12); (9, 13); (9, 14); (9, 15)\}$, see the figures 1 and 2 (the $\mathcal{C}$-edges are depicted by triangles).

![Figure 1](image_url)

Apply the algorithm. Each vertex of $\mathcal{H}$ belongs to at least one $\mathcal{D}$-edge of size 2 and at least one $\mathcal{C}$-edge of size 3. Color $\mathcal{H}_D$ with 2 colors. Denote by $X_1 = \{0, 8, 10, 11, 12, 13, 14, 15\}$ and $X_2 = \{1, 2, 3, 4, 5, 6, 7, 9\}$ two color classes of $\mathcal{H}_D$. Construct the following touching graphs $L_1 = (X_1, V_1)$ and $L_2 = (X_2, V_2)$, where $V_1 = \{(0, 8); (8, 10); (10, 11); (10, 12); (11, 12); (12, 13); (13, 14); (13, 15); (14, 15)\}$ and $V_2 = \{(1, 2); (1, 3); (2, 3); (2, 7); (3, 4); (4, 5); (4, 6); (5, 6); (7, 9)\}$. Choose the respective trees $T_1$ and $T_2$ (Figure 3).

Consecutively applying the algorithm we obtain one of uc-orderings of $\mathcal{H}$: $\sigma = \{15, 14, 13, 11, 12, 10, 1, 5, 9, 6, 4, 3, 2, 8, 0, 7\}$. At the 7-th step of the algorithm, after including of vertex 10 in $\sigma$, we alternate the trees because $T_1$ has no pendant vertex which is also pendant in $\mathcal{H}_D$. The next alternations of trees are made after adding to $\sigma$ of vertices 2 and 0. From the above algorithm we have
Figure 2. $\mathcal{H}_D = (X, \emptyset, D)$

$L_1 = (X_1, V_1)$

$T_1 = (X_1, E_1)$

$L_2 = (X_2, V_2)$

$T_2 = (X_2, E_2)$
Theorem 5. A mixed hypertree is uniquely colorable if and only if it is uc-orderable.

Therefore, combining the Theorems 2, 5 and relation between chromatic polynomial and chromatic spectrum [6, 7], we obtain the following

Theorem 6. Let $\mathcal{H} = (X, C, D)$ be a mixed hypertree. Then the following five statements are equivalent:

1. $R(\mathcal{H}) = (0, 1, 0, \ldots, 0)$;
2. $P(\mathcal{H}, k) = k(k - 1)$;
3. $\mathcal{H}$ is uniquely colorable;
4. Every two vertices $x, y \in X$ are joined by an $(x, y)$-inverter;
5. $\mathcal{H}$ is uc-orderable.

References


Received 16 April 1999
Revised 24 March 2000