Sparse Linear Prediction and Its Applications to Speech Processing

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Abstract—The aim of this paper is to provide an overview of Sparse Linear Prediction, a set of speech processing tools created by introducing sparsity constraints into the linear prediction framework. These tools have shown to be effective in several issues related to modeling and coding of speech signals. For speech analysis, we provide predictors that are accurate in modeling the speech production process and overcome problems related to traditional linear prediction. In particular, the predictors obtained offer a more effective decoupling of the vocal tract transfer function and its underlying excitation, making it a very efficient method for the analysis of voiced speech. For speech coding, we provide predictors that shape the residual according to the characteristics of the sparse encoding techniques resulting in sparse linear predictive coding. The proposed estimators are all solutions to convex optimization problems, which can be solved efficiently and reliably using, e.g., interior-point methods. Extensive experimental results are provided to support the effectiveness of the proposed methods, showing the improvements over traditional linear prediction in both speech analysis and coding.

Index Terms—1-norm minimization, compressed sensing, linear prediction, sparse representation, speech analysis, speech coding.

I. INTRODUCTION

LINEAR prediction (LP) has been successfully applied in many modern speech processing systems in such diverse applications as coding, analysis, synthesis and recognition (see, e.g., [1]). The speech model used in many of these applications is the source-filter model where the speech signal is generated by passing an excitation through an all-pole filter, the predictor in the feedback loop. Typically, the prediction coefficients are identified such that the 2-norm of the residual, the difference between the observed signal and the predicted signal, is minimized. This works well when the excitation signal is Gaussian and independent and identically distributed (i.i.d.) [2], consistent with the equivalent maximum-likelihood approach to determine the coefficients [3]. However, when the excitation signal does not satisfy these assumptions, problems arise [2]. This is the case for voiced speech where the excitation can be considered to be a spiky excitation of a quasi-periodic nature [1]. In this case, the spectral cost function associated with the minimization of the 2-norm of the residual can be shown to suffer from certain well-known problems such as overemphasis on peaks and cancellation of errors [2]. In general, the shortcomings of LP in spectral envelope modeling can be traced back to the 2-norm minimization approach: by minimizing the 2-norm, the LP filter cancels the input voiced speech harmonics causing the envelope to have a sharper contour than desired with poles close to the unit circle. A wealth of methods have been proposed to mitigate these effects. Some of the proposed techniques involve a general rethinking of the spectral modeling problem (see, e.g., [4]–[6], and [7]) while others are based on changing the statistical assumptions made on the prediction error in the minimization process (notably [8], [9], and [10]).

The above-mentioned deficiencies of the 2-norm minimization in LP modeling have also repercussions in the speech coding scenario. In fact, while the 2-norm criterion is consistent with achieving minimal variance of the residual for efficient coding,1 sparse techniques are employed to encode the residual. Examples of this can be seen since early GSM standards with the introduction of multi-pulse excitation (MPE [12]) and regular-pulse excitation (RPE [13]) methods and, more recently, in sparse algebraic codes in code-excited linear prediction (ACELP [14]). In these cases, the sparsity of the RPE and ACELP excitation was motivated, respectively, by psychoacoustic and by the dimensionality reduction of the excitation vector space. Therefore, a better suited predictor for these two coding schemes, arguably, is not the one that minimizes the 2-norm, but the one that leaves the fewest nonzero pulses in the residual, i.e., the sparsest residual. Early contributions (notably [9], [15], and [16]) have followed this line of thought questioning the fundamental validity of the 2-norm criterion.

1The fundamental theorem of predictive quantization [11] states that the mean squared reproduction error in predictive encoding is equal to the mean squared quantization error when the residual signal is presented to the quantizer. Therefore, by minimizing the 2-norm of the residual, these variables have a minimal variance whereby the most efficient coding is achieved.
with regards to speech coding. Despite this research effort, to
the authors’ best knowledge, 2-norm minimization is the only
criterion used in commercial speech applications.

Traditional usage of LP is confined to modeling only the spec-
tral envelope capturing the short-term redundancies of speech.
Hence, in the case of voiced speech, the predictor does not fully
decorrelate the speech signal because of the long-term redu-
dancies of the underlying pitch excitation. This means that the
residual will still have pitch pulses present. The usual approach
is then to employ a cascaded structure where LP is initially
applied to determine the short-term prediction coefficients to
model the spectral envelope and, subsequently, a long-term pre-
dictor is determined to model the harmonic behavior of the spec-
trum [1]. Such a structure is arguably suboptimal since it ig-
nores the interaction between the two different stages. Also in
this case, while early contributions have outlined gains in per-
fomance in jointly estimating the two filters (the work in [17]
is perhaps the most successful attempt), the common approach
is to distinctly separate the two steps.

The recent developments in the field of sparse signal pro-
cessing, backed up by significant improvements in convex op-
timization algorithms (e.g., interior point methods [18], [19]),
have recently encouraged the authors to explore the concept of
sparsity in the LP minimization framework [20]. In particular,
while reintroducing well-known methods to seek a short-term
predictor that produces a residual that is sparse rather than min-
umum variance, we have also introduced the idea of employing
high-order sparse predictors to model the cascade of short-term
and long-term predictors, engendering a joint estimation of the
two [21]. This preliminary work has led the way for the exploita-
tion of the sparse characteristics of the high-order predictor and
the residual to define more efficient coding techniques. Specifi-
cally, in [22], we have demonstrated that the new model achieves
a more parsimonious description of a speech segment with inter-
esting direct applications to low bit-rate speech coding. While in
these early works, the 1-norm has been reasonably chosen as a
convex approximation of the so-called 0-norm,2 in [23] we have
applied the reweighted 1-norm algorithm in order to produce a
more focused solution to the original problem that we are trying
to solve. In this work, we move forward, introducing the nov-
ety of a compressed sensing formulation [24] in sparse LP, that
will not only offer important information on how to retrieve the
sparse structure of the residual, but will also help reduce the size
of the minimization problem, with a clear impact on the com-
putational complexity.

The contribution of this paper is then twofold. First, we put
our earlier contributions in a common framework giving an in-
trductory overview of sparse linear prediction and we also in-
troduce its compressed sensing formulation. Second, we pro-
vide a detailed experimental analysis of its usefulness in mod-
ing and coding applications transcending the well-known lim-
itations related to traditional LP.

The paper is organized as follows. In Section II, we pro-
vide a prologue that defines the mathematical formulations of
the proposed sparse linear predictors. In Section III, we de-
fine the sparse linear predictors and, in Section IV, we provide
their compressed sensing formulations. The results of the exper-
imental evaluation of the analysis properties of the short-term
predictors are outlined in Section V, while the experimental re-

cults of the coding properties and applications are outlined in
Section VI. We provide a discussion on some of the drawbacks
of sparse linear prediction in Section VII. Finally, Section VIII
concludes our work.

II. FUNDAMENTALS OF LINEAR PREDICTION

We consider the following speech production model, where a
sample of speech \( x(n) \) is written as a linear combination of \( K \)
past samples:

\[
x(n) = \sum_{k=1}^{K} a_k x(n-k) + r(n)
\]

(1)

where \( \{a_k\} \) are the prediction coefficients and \( r(n) \) is the
prediction error. In particular, we consider the optimization
problem associated with finding the prediction coefficient vector
\( a \in \mathbb{R}^K \) from a set of observed real samples \( x(n) \) for
\( n = 1, \ldots, N \) so that the prediction error is minimized [18].

Considering the speech production model for a segment of \( N \)
speech samples \( x(n) \), for \( n = 1, \ldots, N \), in matrix form:

\[
x = Xa + r
\]

(2)

the problem becomes

\[
a = \arg \min_a |x - Xa|_p^p + \gamma |a|_0^p
\]

(3)

where

\[
x = \begin{bmatrix} x(N_1) \\ \vdots \\ x(N_2) \end{bmatrix}, \quad X = \begin{bmatrix} x(N_1 - 1) & \cdots & x(N_1 - K) \\ \vdots \\ x(N_2 - 1) & \cdots & x(N_2 - K) \end{bmatrix}
\]

(4)

The \( p \)-norm operator \( \| \cdot \|_p \) is defined as \( \|x\|_p = (\sum_{n=1}^{N} |x(n)|^p)^{1/p} \). The starting and ending points \( N_1 \) and
\( N_2 \) can be chosen in various ways by assuming \( x(n) = 0 \) for
\( n < 1 \) and \( n > N \). In this paper we will use the most common
choice of \( N_1 = 1 \) and \( N_2 = N + K \), which is equivalent, when
\( p = 2 \) and \( \gamma = 0 \), to the autocorrelation method [25].

The introduction of the regularization term \( \gamma \) in (3) can be seen as
being related to the prior knowledge of the coefficients vector
\( a \); problem (3) then corresponds to the maximum a posteriori
(MAP) approach for finding \( a \) under the assumptions that \( a \) has
a Generalized Gaussian Distribution [26]. In finding a sparse
signal representation, there is the somewhat subtle problem of
how to measure sparsity. Sparsity is often measured as the
 cardinality, corresponding to the so-called 0-norm \( \| \cdot \|_0 \). Our
optimization problem (3) would then become

\[
a = \arg \min_a |x - Xa|_0 + \gamma |a|_0
\]

(5)

with the particular case in which we are only considering the
sparsity in the residual (\( \gamma = 0 \))

\[
a = \arg \min_a |x - Xa|_0
\]

(6)
Unfortunately, these are combinatorial problems which generally cannot be solved in polynomial time. Instead of the cardinality measure, we will then use the more tractable 1-norm $\|r\|_1$, which is known throughout the sparse recovery literature (see, e.g., [27]) to perform well as a relaxation of the 0-norm. We will also consider more recent variations of the 1-norm minimization criterion such as the reweighted 1-norm [28] to enhance the sparsity measure and moving the solution closer to the original 0-norm problem (5).

III. SPARSE LINEAR PREDICTORS

In this section, we will define the different sparse linear predictors and show their application in the context of speech processing. In particular, we will introduce the problem of determining a short-term predictor that engenders a sparse residual and the problem of finding a high-order sparse predictor that also engenders a sparse residual. Since in Section II, we have introduced the 1-norm minimization as the sparsity measure, here we will also give a brief overview of the reweighted 1-norm algorithm to enhance this sparsity measure, moving closer to the original problem (0-norm minimization).

A. Finding a Sparse Residual

We consider the problem of finding a prediction coefficient vector $\mathbf{a}$ such that the resulting residual is sparse. Having identified the 1-norm as a suitable convex relaxation of the cardinality, the cost function for this problem is a particular case of (3). By setting $p = 1$ and $\gamma = 0$ we obtain the following optimization problem:

$$\min_\mathbf{a} \| \mathbf{x} - \mathbf{Xa} \|_1. \tag{7}$$

This formulation of the LP problem has been considered since the early works on speech analysis [9], [15], [16] and becomes particularly relevant for the analysis of voiced speech. In particular, compared to the traditional 2-norm minimization, the cost function associated with the 1-norm minimization deemphasizes the impact of the spiky underlying excitation associated with voiced speech on the solution $\mathbf{a}$. Thus, there is an interesting connection between recovering a sparse residual vector and applying robust statistics methods to find the predictor [8]. An example of the more accurate recovery of the voiced excitation is shown in Fig. 1. The effect of putting less emphasis on the outliers of the spiky excitation associated with voiced speech will reflect on the spectral envelope that will avoid the overemphasis on peaks generated in the effort to cancel the pitch harmonics. An example of this property is shown in Fig. 2.

While the 1-norm has been shown to outperform the 2-norm in finding a more proper LP model in speech analysis, in the case of unvoiced speech both approaches seem to provide appropriate models. However, by using the 1-norm minimization, we provide a residual that is sparser. In particular in [29] it is shown that, the residual vector provided by 1-norm minimization will have at least $K$ components equal to zero.

B. Finding a High-Order Sparse Predictor

We now consider the problem of finding a high-order sparse predictor that also engenders a sparse residual. This problem is particularly relevant when considering the usual modeling approach adopted in low bit-rate predictive coding for voiced speech segments. This corresponds to a cascade of a short-term linear predictor $F(z)$ and a long-term linear predictor $P(z)$ to remove respectively near-sample redundancies, due to the presence of formants, and distant-sample redundancies, due to the presence of a pitch excitation. The cascade of the predictors corresponds to the multiplication in the $z$-domain of the their transfer functions:

$$A(z) = F(z)P(z) = 1 - \sum_{k=1}^{K} a_k z^{-k}$$

$$= \left(1 - \sum_{k=1}^{N_f} f_k z^{-k}\right) \left(1 - \sum_{k=1}^{N_i} g_k z^{-(T_{v}+k-1)}\right). \tag{8}$$

Fig. 1. Example of prediction residuals obtained by 2-norm and 1-norm error minimization. The speech segment analyzed is shown in the top box. The prediction order is $K = 10$ and the frame length is $N = 160$. It can be seen that the spiky pitch excitation is retrieved more accurately when 1-norm minimization is employed.

Fig. 2. Example of LP spectral model obtained by 1-norm and 2-norm error minimization for a segment of voiced speech. The prediction order is $K = 10$ and the frame length is $N = 160$. The lower emphasis on peaks in the envelope, when 1-norm minimization is employed, is a direct consequence of the ability to retrieve the spiky pitch excitation.
The resulting prediction coefficient vector $\mathbf{a} = \{a_k\}$ of the high-order polynomial $\hat{A}(z)$ will therefore be highly sparse.\(^3\) Taking this into account in our minimization process, and again considering the 1-norm as convex relaxation of the 0-norm, our original problem (5) becomes

$$\min_{\mathbf{a}} \|\mathbf{x} - \mathbf{Xa}\|_1 + \gamma\|\mathbf{a}\|_1$$  \hspace{1cm} (9)$$

where the dimension of the prediction coefficient vector $\mathbf{a}$ (the order of the predictor) has to be sufficiently large to model the filter cascade ($K > N_f + T_p + N_p$) in (8). This approach, although maintaining resemblances to (7) looking for a sparse residual, is fundamentally different. While the predictor in (7) aims at modeling the spectral envelope, the purpose of the high-order sparse predictor is to model the whole spectrum, i.e., the spectral envelope and the spectral harmonics. This can be easily achieved due to the strong ability of high-order LP to resolve closely spaced sinusoids [30], [31]. Furthermore, considering the construction of the observation matrix $\mathbf{X}$, finding a high-order sparse predictor is equivalent to identify which columns of $\mathbf{X}$, and in turn, which samples in $\mathbf{x}$ are important in the linear combination to predict a sample of speech (1). Thus, when a segment of voiced speech is analyzed with the predictive framework in (9), the nonzero coefficients roughly coincide with the structure of the cascade in Fig. 3. An example of the spectral modeling properties is shown in Fig. 4. An example of the spectral modeling properties is shown in Fig. 4.

There are mainly two problems associated with exploiting the modeling properties of the sparse high-order predictor: determining an appropriate value of $\gamma$ to solve (9) and using an approximate factorization to obtain again the initial formulation composed by the two predictors (8). Below we address these two issues.

\(^3\)Traditionally, for speech sampled at 8 kHz, $N_f = 10$, $N_p = 1$, and $T_p$ usually belongs in the range [16, 120].
\( \hat{Q}(z) \) we can then extract the \( N_p \) taps predictor. In this paper, we will consider the most common pitch predictor where \( N_p = 1 (P(z) = 1 - g_p z^{-T_p}) \), then we merely identify the minimum value and its position in the coefficients vector of \( \hat{Q}(z) \):

\[
\begin{align*}
g_p &= \min \{ q_k \}, \\
T_p &= \arg \min \{ q_k \}.
\end{align*}
\]

(12)

It is clear that, while heuristic, this factorization procedure is highly flexible. A different numbers of taps for both the short-term and long-term can be selected and also a voiced/unvoiced classification can be included, based on the presence or absence of long-term information, as described in [21], [22].

It should be noticed that the structure of the cascade can also be incorporated into the minimization scheme and can be potentially beneficial in reducing the size of the problem. This approach is then similar to the One-Shot Combined Optimization presented in [17] which is implicitly a sparse method looking for a similar high-order factorizable predictor. The joint estimation in this case requires prior knowledge on the position of the pitch contributions (a pitch estimate) and the model order of both the short-term and long-term predictors. Differently from this method, in our approach, we obtain information on the model order of both short-term and long-term contribution and a pitch estimate, just by a simple postprocessing the solution of (9).

C. Enhancing Sparsity by Reweighted 1-Norm Minimization

As shown throughout this section, the 1-norm is used as a convex relaxation of the 0-norm, because 0-norm minimization yields a combinatorial problem (NP-hard). We are therefore interested in adjusting the error weighting difference between the 1-norm and the 0-norm. A variety of recently introduced methods have dealt with reducing the error weighting difference between the 1-norm and the 0-norm by relying on the iterative reweighted 1-norm minimization (see, e.g., [34] and references therein). In particular, the iteratively reweighted 1-norm minimization may be used for estimating \( \mathbf{a} \) and enhancing the sparsity of \( \mathbf{r} \) (and \( \mathbf{a} \)), while keeping the problem solvable with convex tools [28], [23]. The predictor can then be seen as a solution of the following minimization problem:

\[
\mathbf{a} = \arg \min_{\mathbf{a}} \lim_{p \to 0, k \to 0} \left\{ \| \mathbf{x} - \mathbf{Xa} \|_p^p + \gamma \| \mathbf{a} \|_k^k \right\}
\]

(13)

where each iteration of the reweighting process brings us closer to the 0-norm.

The mismatch between the 0-norm and the 1-norm minimization can be seen more clearly in Fig. 5, where larger coefficients are penalized more heavily by the 1-norm than small ones. From an optimization point of view, when \( p = 1 \), the cost functions will have lower emphasis on large values and sharper slopes near zero compared to the \( p = 1 \) case. In turn, from a statistical point of view, the density functions will have heavier tails and a sharper slope near zero. This means that the minimization will encourage small values to become smaller while enhancing the amplitude of larger values. The limit case for \( p = 0 \) will have an infinitely sharp slope in zero and equally weighted tails. This will introduce as many zeros as possible as these are infinitely weighted. In this sense, the 0-norm can be seen as more “impartial” by penalizing every nonzero coefficient equally. It is clear that if a very small value would be weighted as much as a large value, the minimization process will eliminate the smaller ones and enhance the larger ones.

The algorithm to obtain a short-term predictor engendering a sparser residual, a reweighted formulation of (7), is shown in Algorithm 1. This approach, as we shall see, becomes beneficial in finding a predictor that produces a sparser residual, providing a tighter coupling between the prediction estimation and the search for the approximated sparse excitation. An example of the reweighted residual estimate is shown in Fig. 6.

When we impose sparsity both on the residual and on the high-order predictor, as in (9), the algorithm is modified as shown in Algorithm 2. This formulation is relevant as it enhances the components that contain the information regarding the near-end and far-end redundancies in the high-order predictor making the approximate factorization presented in Section III-B2 more accurate. In particular, the reweighting allows to reduce the spurious near-zero components in the high-order predictor obtained (see Fig. 3) while enhancing the larger components that contain information of near-end and far-end redundancies.

It has been shown in [28] that \( \| \mathbf{F}^{t+1} \|_1 \leq \| \mathbf{F}^t \|_1 \), meaning that this is a descent algorithm. The halting criterion can therefore be chosen as either a maximum number of iterations or as a
convergence criterion. In the experimental analysis we will give
details on how many iterations are required in our setting. In
both algorithms, the parameter $\epsilon > 0$ is used to provide stability
when a component of $\hat{r}$ goes to zero.
As a general remark, in [28] and [34], it is also shown that
the reweighted 1-norm algorithm, at convergence, is equivalent
to the minimization of the log-sum penalty function. This is
relevant to what we are trying to achieve in (13): the log-sum
cost function has a sharper slope near zero compared to the
1-norm, providing more effective sparsity inducing properties.
Furthermore, since the log-sum is not convex, the iterative algo-
rithm corresponds to minimizing a sequence of linearizations of
the log-sum around the previous solution estimate, providing at
each step a sparser solution (until convergence).

Algorithm 1 Iteratively Reweighted 1-Norm Minimization of
the Residual

Inputs: speech segment $\mathbf{x}$
Outputs: predictor $\mathbf{\hat{a}}$, residual $\mathbf{\hat{r}}$
$i = 0$, initial weights $\mathbf{W}^{i=0} = \mathbf{I}$
while halting criterion false do
    $\mathbf{\hat{a}}, \mathbf{\hat{r}} \leftarrow \arg\min_{\mathbf{a}, \mathbf{r}} \|\mathbf{W}\mathbf{r}\|_1$ s.t. $\mathbf{r} = \mathbf{x} - \mathbf{Xa}$
    $\mathbf{W}^{i+1} \leftarrow \text{diag} \left( \frac{|\mathbf{r}|}{|\mathbf{r}| + \epsilon} \right)^{-1}$
    $i \leftarrow i + 1$
end while

Algorithm 2 Iteratively Reweighted 1-Norm Minimization of
Residual and Predictor

Inputs: speech segment $\mathbf{x}$
Outputs: predictor $\mathbf{\hat{a}}$, residual $\mathbf{\hat{r}}$
$i = 0$, initial weights $\mathbf{W}^{i=0} = \mathbf{I}$ and $\mathbf{D}^{i=0} = \mathbf{I}$
while halting criterion false do
    $\mathbf{\hat{a}}, \mathbf{\hat{r}} \leftarrow \arg\min_{\mathbf{a}, \mathbf{r}} \|\mathbf{W}\mathbf{r}\|_1 + \gamma\|\mathbf{D}\mathbf{a}\|_1$ s.t. $\mathbf{r} = \mathbf{x} - \mathbf{Xa}$
    $\mathbf{W}^{i+1} \leftarrow \text{diag} \left( \frac{|\mathbf{r}|}{|\mathbf{r}| + \epsilon} \right)^{-1}$
    $\mathbf{D}^{i+1} \leftarrow \text{diag} \left( \frac{|\mathbf{a}|}{|\mathbf{a}| + \epsilon} \right)^{-1}$
    $i \leftarrow i + 1$
end while

IV. COMPRESSED SENSING IN SPARSE LINEAR PREDICTION

The CS formulation is particularly relevant in our sparse
recovery problems: by exploiting prior knowledge about the
sparsity of the signal $\mathbf{x}$ we will show that a limited number
of random measures are sufficient to recover our predictors and
spare residual with high accuracy. In particular, it has been
shown [24], [35] that a random projection of a high-di-
dimensional but sparse or compressible signal vector onto a
lower-dimensional space contains enough information to be
able to reconstruct, with high probability, the signal with small
or zero error. The random measures in CS literature are usually
obtained by projecting the considered measurement vectors
onto a lower dimensional space, using random matrices.
In recent work [36], [37], CS formulations in the context of
speech analysis and coding have been formulated in order to
find a sparse approximation of the residual, given the predictor.
It is then interesting to extend this work to the case where we
want to find directly the predictor that engenders intrinsically
a sparse residual. In particular, given the sparsity level of the
spare representation that we wish to retrieve in a given domain,
we can determine an efficient shrinkage of the minimization
problem in a lower dimensional space, with a clear impact on
the computational complexity.
If we wish to perform CS, two main ingredients are needed:
a domain where the analyzed signal is sparse and the sparsity
level of this signal $T$. In our case, the residual is the domain
where the signal is sparse, while the linear transform that maps
the original speech signal to the sparse residual is the sparse
predictor. The sparsity in the residual domain is then imposed
by our needs [35]. Let us now review the formulation presented
in [37]:

$$\mathbf{r} = \arg\min_{\mathbf{r}} \|\mathbf{r}\|_1 \text{ s.t. } \mathbf{\Phi x} = \mathbf{\Phi H r}$$  (14)

where $\mathbf{x}$ is the $N \times 1$ analyzed segment of speech, $\mathbf{H}$ the $N \times (N + K)$ synthesis matrix, constructed from the truncated im-
pulse response of the known predictor [38], $\mathbf{r}$ is the residual
vector to be estimated (supposedly sparse) and $\mathbf{\Phi}$ is the sensing
matrix of dimension $M \times N$. The dimensionality of the random
linear projection $\mathbf{M}$ stems from the sparsity level $T$ that one
wishes to impose on the residual. In particular, based on empirical
results, the number of projections is set equal to four times the
sparsity, i.e., $M = 4T$. Furthermore, when the incoherence
between the synthesis matrix and the random basis matrix $\mathbf{\Phi}$
holds ($\mu(\mathbf{\Phi}, \mathbf{H}) \approx 1$), even if $\mathbf{H}$ is not orthogonal the recovery
of the sparse residual $\mathbf{r}$ is still possible and the linear program in
(14) gives an accurate reconstruction of $\mathbf{x}$ with very high proba-
bility [24], [37]. As a general remark, the entries of the random
matrix can be drawn from many different processes [39], in our
case we will use a i.i.d. Gaussian process, as done in [36], [37].

To adapt CS principles to the estimation of the predictor as
well, let us now consider the relation between the synthesis ma-
tix $\mathbf{H}$ and the analysis matrix $\mathbf{A}$ where one is the pseudo-in-
verse of the other [40]:

$$\mathbf{A} = \mathbf{H}^+. \quad \text{(15)}$$

We can now replace the constraint $\mathbf{\Phi x} = \mathbf{\Phi H r}$ in (14) as

$$\mathbf{\Phi r} = \mathbf{\Phi A x} \quad \text{(16)}$$

where $\mathbf{A}$ is the $(N + K) \times N$ analysis matrix that performs the
whitening of the signal, constructed from the coefficients of the
predictor $\mathbf{a}$ of order $K$ [40], the dimension of the sensing matrix
$\mathbf{\Phi}$ is now adjusted accordingly to $M \times (N + K)$. Notice that,
due to the structure of $\mathbf{A}$ this can be rewritten equivalently to

$$\mathbf{\Phi r} = \mathbf{\Phi A x} = \mathbf{\Phi} [\mathbf{x} | \mathbf{X}] [1] \mathbf{a}^T \quad \text{(17)}$$

where $[\mathbf{x} | \mathbf{X}]$ is the matrix obtained by stacking the vector $\mathbf{x}$
to the left of $\mathbf{X}$ in (4). The minimization problem can then be
rewritten as

$$\min_{\mathbf{a}, \mathbf{r}} \|\mathbf{r}\|_1 \text{ s.t. } \mathbf{\Phi r} = \mathbf{\Phi} (\mathbf{x} - \mathbf{Xa}). \quad \text{(18)}$$

We can now see that (18) is equivalent to (7), the only difference
being the projection onto the random basis in the constraint.
Therefore, (7) can be seen as a particular case of the formulation
in (18) where $\mathbf{\Phi} = \mathbf{I}$ and $\mathbf{I}$ is a identity matrix of size $(N + K) \times$
The formulation (9) and (20), similarly to (7) and (18), become equivalent when \( \Phi = I \) and the minimization constraint is then (19). Both formulations (18) and (20), can also be modified to involve iterative reweighting (Algorithm 3 shows the general case for \( \gamma > 0 \)).

**Algorithm 3 CS Formulation of the Iteratively Reweighted 1-norm Minimization of Residual and Predictor**

Inputs: speech segment \( x \), desired residual sparsity level \( T \)
Outputs: predictor \( \hat{a}^i \), residual \( \hat{r}^i \)
\( i = 0 \), initial weights \( W^{i=0} = I \) and \( D^{i=0} = I \), random matrix \( \Phi \) of size \( M \times (N + K) \), \( M = 4T \)
\( \text{while halting criterion false do} \)
\( \hat{a}^i, \hat{r}^i \leftarrow \arg \min_{a, r} ||W^i r||_1 + \gamma ||D^i a||_1 \) \( \text{s.t.} \)
\( \Phi r = \Phi (x - X a) \)
\( W^{i+1} \leftarrow \text{diag} (||r^i|| + \epsilon)^{-1} \)
\( D^{i+1} \leftarrow \text{diag} (||a^i|| + \epsilon)^{-1} \)
\( i \leftarrow i + 1 \)
\( \text{end while} \)

V. PROPERTIES OF SPARSE LINEAR PREDICTION

As mentioned in the introduction, many problems appearing in traditional 2-norm LP modeling of voiced speech can be traced back to the inability of the predictor to decouple the vocal tract transfer function from the pitch excitation. This results in a lower spectral modeling accuracy and a strong dependence on the placement of the analysis window. In this section, we provide some experiments to illustrate how the sparse linear predictors presented in the previous sections manage to overcome these problems. As a general remark, it is well-known that the \( p \)-norm LP estimate with \( p \neq 2 \) is not guaranteed to be stable [43]. Nevertheless, the results presented in this section concentrate on the spectral modeling properties of sparse LP; thus, the stability of the predictor is simply imposed by pole reflection which stabilizes the filter without modifying the magnitude of the frequency response. We will provide a thorough discussion of the stability issues in the Sections VII and in VI where the speech coding properties are analyzed and stability is critical.

The experimental analysis was done on 20 000 frames of length \( N = 160 \) (20 ms) of clean voiced speech coming from several different speakers with different characteristics (gender, age, pitch, regional accent) taken from the TIMIT database, downsampled at 8 kHz. The prediction methods we compare in this section are shown in Table I. The optimality of the methods BE and RLP, presented in [6], comes from the selection of the parameters which provided the lowest distortion compared with the reference envelope. For brevity and clarity of the presented results, we omitted the predictors obtained as solutions of the iterative reweighted algorithms presented in Section III-C and the CS formulation presented in Section IV. These methods, while presenting very similar modeling properties to SpLP10 and SpLP11, produce predictors estimates with slightly higher variance, thus requiring few more bits to be encoded. Therefore, while it is hard to provide a fair comparison in terms of modeling, their properties become more interesting in the coding scenario that will thoroughly analyzed in Section VI; in particular, the
differences in their bit allocation necessary for efficient coding and the information required in the residual will be analyzed.

A. Spectral Modeling

In this section, we provide results to the modeling properties of the short-term predictors. As a reference, we used the envelope obtained through a cubic spline interpolation between the harmonics peaks of the logarithmic periodogram. This method was presented in [6] and provided an approximation of the vocal tract transfer function, without the fine structure corresponding to the pitch excitation. We then calculated the log spectral distortion between our reference envelope $S_{\text{ref}}(\omega)$ and the estimated predictive model $S(\omega, \hat{a})$ as

$$\text{SD}_m = \sqrt{\frac{1}{2\pi} \int_{-\pi}^{\pi} [10\log_{10} S_{\text{ref}}(\omega) - 10\log_{10} S(\omega, \hat{a})]^2 d\omega}, \quad (21)$$

where the numerator gain is calculated as the variance of the residual.

The coefficients of the short-term predictors presented have also shown to be smoother and therefore they have a lower sensitivity to quantization. We also compared the log spectral distortion between our reference envelope $S_{\text{ref}}(\omega)$ and the quantized predictive model $S(\omega, \hat{a})$ for every predictor obtained with the presented methods. The quantizer used is the one presented in [44], with the number of bits fixed at 20 for the different prediction orders, providing in all the method presented a transparent coding.\(^4\) The results are shown in Table II for different prediction orders. A critical analysis of the results showed the improved modeling properties of SpLP11. This was given by its ability to take into consideration the whole speech production model, thus decoupling more effectively the short-term contribution that provides the spectral envelope from the contribution given by the pitch excitation. SpLP10 and RLP achieved similar performance, providing evidence supporting the generally good spectral modeling properties of the minimization problem in (7).

B. Shift Invariance

In speech analysis, a desirable property for an estimator is to be invariant to the small shifts of the analysis window, since speech, and voiced speech in particular, is assumed to be short-term stationary. However, standard LP is well-known not to be shift invariant [8]. This is a direct consequence of the coupling between the vocal tract transfer function and the underlying pitch excitation that standard LP introduces in the estimate. To analyze the invariance of the LP methods to window shifts, we took the same 20,000 frames of clean voiced speech and we expanded them to the left and to the right with 20 samples, giving a total length $N = 200$. In each frame of length $N = 200$ we defined a $M = 160$ samples boxcar window and we shifted the window by $s = 1, 2, 5, 10, 20$ samples. The average log spectral difference of the tenth-order AR estimate between $S_{\hat{a}}(\omega)$ and $S_{\hat{a}}(\omega)$ was analyzed. The average differences obtained for the methods in Table I are shown in Table III. In Fig. 9, we show an example of the shift invariance property. The results obtained indicate clearly the sparse predictors robustness to small shifts in the analyzed window. While the decay in performance for increasing shift in the analysis window is comparable for all methods, the sparse predictors still retains better performance. Also in this case, the change in the frequency response in traditional LP is clearly given by the pitch bias in the estimate of the predictor, particularly dependent on the location of the spikes of the pitch excitation.

\(^4\) According to [45], transparent coding of LP parameters is achieved when the two versions of coded speech, obtained using unquantized LP parameters and quantized LP parameters, are indistinguishable through listening. This is usually achieved with an average log distortion between quantized and unquantized spectra lower than 1 dB, with no outliers with log distortion greater than 4 dB and a number of outliers with 2–4 dB distortion lower than 2%. Furthermore, according to [46] the quality threshold for the model naturally follows from a distortion measure for the signal, the result being independent of rate, and giving the same well-known 1 dB without invoking notions of perception.

<table>
<thead>
<tr>
<th>Method</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>LP</td>
<td>Traditional 2-norm LP with 10Hz bandwidth expansion ($\gamma = 0.996$) and Hamming windowing.</td>
</tr>
<tr>
<td>SpLP10</td>
<td>1-norm LP presented in (II-A), solution of (7). Stability is imposed by pole reflection if unstable. No prediction is performed.</td>
</tr>
<tr>
<td>SpLP11</td>
<td>1-norm LP presented in (II-B). The order of (9) is $K = 110$ (covering accurately pitch delays in the interval $[N_f + 1, K - N_f - 1]$). $\gamma$ is chosen as the point of maximum curvature in the $L$-curve. The short-term predictor coefficients are the first $N_f$ coefficients of the high order polynomial. Stability is imposed by pole reflection if unstable. No windowing is performed.</td>
</tr>
<tr>
<td>BE</td>
<td>Optimally bandwidth expanded 2-norm LP as shown in [6]. Hamming window is used.</td>
</tr>
<tr>
<td>RLP</td>
<td>Optimally regularized 2-norm LP as shown in [6]. Hamming window is used.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Method</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>LP</td>
<td>Optimally bandwidth expanded 2-norm LP as shown in [6]. Hamming window is used.</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>METHOD</th>
<th>$K$</th>
<th>$SD_{\alpha}$</th>
<th>$SD_{\beta}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>LP</td>
<td>8</td>
<td>2.11±0.06</td>
<td>3.24±0.11</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>1.97±0.03</td>
<td>2.95±0.09</td>
</tr>
<tr>
<td></td>
<td>12</td>
<td>1.98±0.05</td>
<td>2.72±0.12</td>
</tr>
<tr>
<td>SpLP10</td>
<td>8</td>
<td>1.78±0.01</td>
<td>2.53±0.02</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>1.61±0.01</td>
<td>2.31±0.04</td>
</tr>
<tr>
<td></td>
<td>12</td>
<td>1.59±0.01</td>
<td>2.13±0.03</td>
</tr>
<tr>
<td>SpLP11</td>
<td>8</td>
<td>1.64±0.00</td>
<td>2.65±0.01</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>1.69±0.00</td>
<td>2.37±0.01</td>
</tr>
<tr>
<td></td>
<td>12</td>
<td>1.39±0.01</td>
<td>2.13±0.01</td>
</tr>
<tr>
<td>BE</td>
<td>8</td>
<td>2.04±0.03</td>
<td>3.11±0.08</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>1.88±0.02</td>
<td>2.92±0.07</td>
</tr>
<tr>
<td></td>
<td>12</td>
<td>1.83±0.10</td>
<td>2.71±0.04</td>
</tr>
<tr>
<td>RLP</td>
<td>8</td>
<td>1.89±0.02</td>
<td>2.93±0.04</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>1.72±0.01</td>
<td>2.51±0.03</td>
</tr>
<tr>
<td></td>
<td>12</td>
<td>1.53±0.02</td>
<td>2.22±0.04</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>METHOD</th>
<th>$SD_1$</th>
<th>$SD_2$</th>
<th>$SD_3$</th>
<th>$SD_{10}$</th>
<th>$SD_{20}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>LP</td>
<td>0.113</td>
<td>0.128</td>
<td>0.223</td>
<td>0.452</td>
<td>1.262</td>
</tr>
<tr>
<td>SpLP10</td>
<td>0.003</td>
<td>0.003</td>
<td>0.011</td>
<td>0.017</td>
<td>0.032</td>
</tr>
<tr>
<td>SpLP11</td>
<td>0.001</td>
<td>0.002</td>
<td>0.005</td>
<td>0.006</td>
<td>0.009</td>
</tr>
<tr>
<td>BE</td>
<td>0.097</td>
<td>0.117</td>
<td>0.197</td>
<td>0.238</td>
<td>0.328</td>
</tr>
<tr>
<td>RLP</td>
<td>0.015</td>
<td>0.089</td>
<td>0.180</td>
<td>0.201</td>
<td>0.323</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>METHOD</th>
<th>$SD_1$</th>
<th>$SD_2$</th>
<th>$SD_3$</th>
<th>$SD_{10}$</th>
<th>$SD_{20}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>LP</td>
<td>0.113</td>
<td>0.128</td>
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<td>0.003</td>
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<tr>
<td>SpLP11</td>
<td>0.001</td>
<td>0.002</td>
<td>0.005</td>
<td>0.006</td>
<td>0.009</td>
</tr>
<tr>
<td>BE</td>
<td>0.097</td>
<td>0.117</td>
<td>0.197</td>
<td>0.238</td>
<td>0.328</td>
</tr>
<tr>
<td>RLP</td>
<td>0.015</td>
<td>0.089</td>
<td>0.180</td>
<td>0.201</td>
<td>0.323</td>
</tr>
</tbody>
</table>
TABLE IV
AVERAGE SPECTRAL DISTORTION FOR THE CONSIDERED METHODS WITH DIFFERENT UNDERLYING PITCH EXCITATION. A 95% CONFIDENCE INTERVAL IS GIVEN FOR EACH VALUE

<table>
<thead>
<tr>
<th>METHOD</th>
<th>low</th>
<th>mid</th>
<th>high</th>
</tr>
</thead>
<tbody>
<tr>
<td>LP</td>
<td>0.81±0.12</td>
<td>1.04±0.23</td>
<td>1.32±0.56</td>
</tr>
<tr>
<td>SpLP10</td>
<td>0.02±0.00</td>
<td>0.09±0.00</td>
<td>0.11±0.01</td>
</tr>
<tr>
<td>SpLP11</td>
<td>0.00±0.00</td>
<td>0.00±0.00</td>
<td>0.01±0.00</td>
</tr>
<tr>
<td>BE</td>
<td>0.45±0.07</td>
<td>0.65±0.19</td>
<td>0.89±0.34</td>
</tr>
<tr>
<td>RLP</td>
<td>0.05±0.02</td>
<td>0.16±0.10</td>
<td>0.19±0.09</td>
</tr>
</tbody>
</table>

VI. CODING APPLICATIONS OF SPARSE LINEAR PREDICTION

By introducing sparsity in the residual, we can reasonably assume that only a small portion of the residual samples are sufficient to reconstruct the speech signal with high accuracy. We will corroborate our intuition by providing some experiments on the coding applications of sparse linear prediction. Specifically, in Section VI-A, we will first give experimental proof of the sparsity inducing effectiveness of the short-term predictors in the Analysis-by-Synthesis (AbS) scheme [38]. In this case, we used a very simple excitation model coding without long-term prediction where we exploit directly the information on the location of the nonzero samples. In Section VI-B, we will present a simple coding procedure that exploits the properties of the combined high-order sparse LP and sparse residual. As we shall see in Section VI-C, this approach presents interesting properties such as noise robustness for which we give both objective and subjective evaluation.

As a general remark, since the stability of the short-term predictors is not assured, we consistently performed a stability check and, if the short-term predictor was found to be unstable, we performed a pole reflection. Note that this approach necessarily modifies the time domain behavior of the residual as well as the predictor coefficients. Nevertheless, since the rate of unstable filters is low and the instability is very mild (i.e., the magnitude of the poles is only very slightly higher than one), this can be considered as an adequate solution to this problem. We will return to the stability issue in Section VII.

A. Coding Properties of the Short-Term Sparse Linear Predictor

The first experiment regards the use of the short-term predictor in speech coding. In particular, we compared the use of the multipulse encoding procedure in the case of bandwidth expanded linear prediction (LP) with a fixed bandwidth expansion of 60 Hz (done by lag-windowing the autocorrelation function) and Hamming windowing.

TABLE V
PREDICTION METHODS COMPARED IN THE CODING PROPERTIES EVALUATION

<table>
<thead>
<tr>
<th>Method</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>LP</td>
<td>Traditional 2-norm LP with a fixed bandwidth expansion of 60 Hz (done by lag-windowing the autocorrelation function) and Hamming windowing.</td>
</tr>
<tr>
<td>SpLP10</td>
<td>1-norm LP solution of (7).</td>
</tr>
<tr>
<td>RWLP10</td>
<td>Reweighted 1-norm LP presented in Section III-C using Algorithm 1. Four reweighting iterations are performed (sufficient for convergence).</td>
</tr>
<tr>
<td>CSLP10</td>
<td>Compressed sensing formulation presented in Section IV, solution of (18). The size of the sensing matrix is given by the number of samples we want to retrieve in the residual.</td>
</tr>
<tr>
<td>RWCSP10</td>
<td>Reweighted compressed sensing formulation of CSLP10 using Algorithm 1. Four reweighting iterations are performed (sufficient for convergence).</td>
</tr>
</tbody>
</table>
estimate of the predictor requires a slightly larger number of bits. Here we will show this providing a comparison also in terms of bits needed for transparent quantization of the predictor. The methods BE and RLP, presented in the previous section (Table I) while offering better modeling properties than traditional LP, do not provide any significant improvement in the coding scenario; thus, they will be omitted from the current experimental analysis.

We have performed the analysis on the same speech signals database considered in Section V. The frame size is $N = 40$, the 10th order predictors were quantized transparently using the LSFs coding method in [44] while the $T$ pulses are left unquantized. In the CS formulations the sensing matrix has $M = 4T$ rows; this means that just a slight reduction in the size of the problem was obtained when $T = 10$. Nevertheless, we were able to obtain important information on the location of the pulses. In the reweighted schemes, the number of iterations is four, which was sufficient to reach convergence in all the analyzed frames.

In Table VI, we present the results in terms of segmental SNR, mean opinion score (obtained through PESQ evaluation) and empirical computational time $t$ in elapsed CPU seconds for $T = 5$ and $T = 10$, and number of bits necessary to transparently encode the predictor ($\tilde{a}$) using LSFs [44]. The results demonstrate the effectiveness of the sparse linear predictors. These results also show that the predictors in the reweighted cases (RWLP10 and RWCSLP10), need a larger number of bits for transparent quantization due to the larger variance of their estimates. This result is particularly interesting when considering the model in (2). In particular, the description of a segment of speech is distributed between its predictive model and the corresponding excitation. Thus, we can observe that the complexity of the predictor necessarily increases when the complexity of the residual decreases (less significant pulses). This also leaves open questions on the optimal bit distribution between the two descriptions. As a proof of concept, the results show how only five bits of difference between LP and RWCSLP10 in the representation of the filter result in a significant improvement in performance: only five pulses in the residual are necessary in RWCSLP10 to obtain similar performance to LP using ten pulses.

A critical analysis of the results leads to another interesting conclusion. In fact, while 1-norm-based minimization, with or without the shrinkage of the problem provided by the CS formulation in (18), is computationally more costly, than 2-norm minimization, it greatly simplifies the next stage where the excitation is selected in a closed-loop AbS scheme. In particular, the empirical computational time in Table VI refers to both the LP analysis stage and the search for the MPE excitation. Since the MPE search for the location is not performed in our sparse LP methods and we exploit directly the information regarding the $T$ pulses of largest magnitude, the AbS procedure is merely a small least square problem where we find the $T$ pulse amplitudes. We will come back to the discussion regarding complexity in Section VII-B. Furthermore, it should be noted that the CS formulation improves the selection of the $T$ largest pulses. This is remarkable since while the predictor obtained with or without the random projection is similar, the reduction of the constraints helps us find a more specific solution for the level of sparsity $T$ that we would like to retrieve in the residual. As mentioned above, the price to pay is a slightly higher bit allocation for the predictors obtained through CS formulation.

### B. Speech Coding Based on Sparse Linear Prediction

As a proof of concept, we will now present a very simple coding scheme that incorporates all the previously introduced methods. We will use the method presented in Section III-B, exploiting the sparse characteristics of the high-order predictor and the sparse residual. In order to reduce the number of constraints, we cast the problem in a CS formulation (20) that provides a shrinkage of the constraints according to the number of samples we wish to retrieve in the residual. Furthermore, in order to refine the initial sparse solution, we apply the reweighting algorithm. The core scheme is summarized in Algorithm 3. Differently from multistage coders, this method, with its joint estimation of a short-term and a long-term predictor and the presence of a sparse residual, provides a one-step approach to speech coding. In synthesis, given a segment of speech, a way to encode the speech signal can be as follows:

1. Define the desired level of sparsity of the residual $T$ and define the sensing matrix dimensionality accordingly $M = 4T$.
2. Perform $n$ steps of the CS reweighted minimization process (Algorithm 3).
3. Factorize the prediction coefficients into a short-term and long-term predictor using the procedure in Section III-B2.
5. Select the $T$ positions where the values of largest magnitude are located.
6. Solve the analysis-by-synthesis equation keeping only the $T$ nonzero positions.
7. Quantize the residual.

We have again analyzed about one hour of clean speech taken from the TIMIT database. In order to obtain comparable results, the frame length is now $N = 160$ (20 ms). The order of the high-order predictor in (20) is $K = 110$ (meaning that we can cover accurately pitch delays in the interval $[N_f + 1, K - N_f - 1]$, including the usual range for the pitch frequency [70 Hz, 500 Hz]). The fixed regularization parameter is $\gamma = 0.12$ and the defined level of sparsity is $T = 20$. Four iterations of the reweighting minimization process are performed, sufficient to reach convergence in all the analyzed frames. The orders of the

<table>
<thead>
<tr>
<th>METHOD</th>
<th>$T$</th>
<th>$\tilde{a}$</th>
<th>SSNR</th>
<th>MOS</th>
<th>$t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>LP</td>
<td>5</td>
<td>19</td>
<td>14.1±2.2</td>
<td>2.85±0.23</td>
<td>0.1±0.1</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>19</td>
<td>19.1±2.9</td>
<td>3.01±0.16</td>
<td>0.9±0.3</td>
</tr>
<tr>
<td>SplLP10</td>
<td>5</td>
<td>18</td>
<td>15.3±2.1</td>
<td>2.87±0.12</td>
<td>1.3±0.2</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>18</td>
<td>20.1±1.7</td>
<td>3.11±0.11</td>
<td>1.3±0.2</td>
</tr>
<tr>
<td>RWLP10</td>
<td>5</td>
<td>22</td>
<td>17.2±1.6</td>
<td>3.01±0.06</td>
<td>4.1±0.3</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>22</td>
<td>21.4±1.5</td>
<td>3.19±0.03</td>
<td>4.1±0.3</td>
</tr>
<tr>
<td>CSLP10</td>
<td>5</td>
<td>19</td>
<td>16.9±1.9</td>
<td>2.97±0.04</td>
<td>0.4±0.0</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>19</td>
<td>20.9±1.5</td>
<td>3.25±0.03</td>
<td>0.6±0.2</td>
</tr>
<tr>
<td>RWCSLP10</td>
<td>5</td>
<td>24</td>
<td>20.2±0.9</td>
<td>3.15±0.03</td>
<td>1.3±0.3</td>
</tr>
<tr>
<td></td>
<td>10</td>
<td>24</td>
<td>24.4±0.4</td>
<td>3.43±0.01</td>
<td>1.9±0.2</td>
</tr>
</tbody>
</table>
short-term and long-term predictors obtained from the factorization of the high-order predictor are \(N_f = 10\) and \(N_p = 1\), respectively. Twenty-five bits are used to transparently encode the LSF vector, seven bits are used to quantize the pitch period \(T_p\) and six bits to quantize the pitch gain \(g_p\). The stability of the overall cascade is imposed by pole reflection on the short-term predictor, and by limiting the pitch gain to be less that unity. As for the residual, the quantizer normalization factor is logarithmically encoded with six bits while an eight-level uniform quantizer is used to quantize the normalized amplitudes; the signs are coded with 1 bit per each pulse. The upper bound given by the information content of the pulse location (\(\log_2(16)\) bits) is used as an estimate of the number of bits used for distortionless encoding of the location. No perceptual weighting is performed in our case. The total number of bits per frame used are 202, producing a 10.1-kbps rate. We will compare this method (SpLP) with the AMR coder in the 10.2-kbps mode (AMR102) [47]. The results in terms of MOS (obtained through PESQ evaluation) and empirical computation time are shown in Table VII and demonstrate similar performance but with a more straightforward approach to coding than AMR. The CS formulation also helps to generally keep the problem solvable in reasonable time.

C. Noise Robustness

This study is motivated by the ability of a sparse coder to identify more effectively the features of the residual signal that are important for its reconstruction, discarding those which probably are a result of the noise. The traditional encoding formulation, based on minimum variance analysis and residual encoding through pseudo-random sequences (i.e., algebraic codes), makes the identification of these important features basically impossible and requires, for low SNRs, noise reduction in the preprocessing. Interestingly enough, sparse LP-based coding appears to be quite robust in the presence of noise. An example of the different performance in terms of MOS for different SNR under additive white Gaussian noise is given in Table VIII.

D. Subjective Assessment of Speech Quality

To further investigate the properties of our methods, we have conducted two MUSHRA listening tests [48] with 16 non-expert listeners. Ten speech clips were used in the listening test. In the first MUSHRA test we investigate what we have shown in Sections VI-B, about the similarity in quality between the AMR coder and our method. In the second MUSHRA test, the noise robustness of our method, discussed in Section VI-C, is proved. The test results are presented in Fig. 10 where the score 100 corresponds to “Imperceptible” and the score 0 corresponds to “Very annoying” according to the six-grade impairment scale. From the results, we can see that our method does not affect greatly the quality of the signal, given that our method is conceptually simpler and substantially less optimized compared to AMR. For example, we are not taking into account some of the main psychoacoustic criteria usually implemented in the AMR, such as the adaptive postfilter to enhance the perceptual quality of the reconstructed speech and the perceptual weighting filter employed in the analysis by synthesis search of the codebooks. Nevertheless, in clean condition the average score was 89 for AMR102, and 82 for SpLP. The most significant results though, are the one related to the coding of noisy signals. In particular, we can see from Fig. 10 that our method scores considerably better than the AMR showing how a sparse encoding technique can be more effective in noise robust speech coding. In fact, in noisy conditions, the average score was 62 for AMR102, and 75 for SpLP.

VII. DISCUSSION

A. Stability

In the presented applications of sparse linear predictors, the percentage of unstable filters was found to be low (around 2%) and the instability “mild.”5 This suggested the use of a simple stability check and pole reflection in our experimental analysis. Theorems exist to determine the maximum absolute value of the roots of a monic polynomial given the norm operator used in the minimization [43] but the bounds are generally too high to gain any real insight on how to create a intrinsic stable minimization problem, as done in [50].

The stability problem in (7) was already tackled in [9] by introducing the Burg method for prediction parameters estimation

\[ r_{\text{max}} = 1.0259. \]
based on the least absolute forward–backward error. In this approach, however, the sparsity is not preserved. This is mostly due to the decoupling of the main $K$-dimensional minimization problem in $K$ one-dimensional minimization subproblems. Therefore, this method is suboptimal and produces results, as we have observed, somewhere in between those of the 2-norm and 1-norm approach. Also, the approach is only valid in (7) and not in all the other minimization schemes presented.

B. Computational Cost

As for the computational cost, finding the solution of the overdetermined system of equations in (7) using a modern interior point algorithm [19] can be shown to be equivalent to solving around 20–30 least square problems. Nevertheless, implementing this procedure in an AbS coder, as done in Section VI-A, is shown to greatly simplify the search for the sparse approximation of the residual in a closed-loop configuration, without compromising the overall quality. Furthermore, in the case of (9), the advantage is that a one step approach is taken to calculate both the short-term and the long-term predictors while the encoding of the residual is facilitated by its sparse characteristics.

The introduction of a compressed sensing formulation for the prediction problem has helped reduce dramatically the computational costs. An example of this can be seen in the coding scheme presented in Section VI-B. Retrieving $T = 20$ samples reduces the number of constraints of the minimization problem from 270 $(N+K)$ to 80 $(M = 4T)$. Since for each constraint we have a dual variable, by reducing the number of the constraints we also reduce the number of the dual variables [18]. In turn, the whole coding scheme, as shown empirically, is only about one order of magnitude more expensive than a 2-norm LP-based coder, although with added improvements such as noise robustness and a fairly high conceptual simplicity.

C. Uniqueness

The minimization problems considered do not necessarily have a unique solution. In these rare cases with multiple solutions, due to the convexity of the cost function, we can immediately state that all the possible multiple solutions will still be optimal [18]. Viewing the non-uniqueness of the solution as a weakness is also arguable: in the set of possible optimal solutions we can probably find one solution that offers better properties for our modeling or coding purposes. A theorem to verify uniqueness is discussed in [52].

D. Frequency Domain Interpretation

The standard linear prediction method exhibits spectral matching properties in the frequency domain due to Parseval’s theorem [2]

$$\sum_{n=-\infty}^{\infty} |e(n)|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |E(e^{j\omega})|^2 d\omega. \tag{22}$$

It is also interesting to note that minimizing the squared error in the time domain and in the frequency domain leads to the same set of equations, namely the Yule–Walker equations [25]. To the best of our knowledge, the only relation existing between the time and frequency domain error using the 1-norm is the trivial Hausdorff–Young inequality [53]:

$$\sum_{n=-\infty}^{\infty} |e(n)| < \frac{1}{2\pi} \int_{-\pi}^{\pi} |E(e^{j\omega})|d\omega$$

which implies that time domain minimization does not correspond to frequency domain minimization. It is therefore difficult to say if the 1-norm based approach is always advantageous compared to the 2-norm based approach for spectral modeling, since the statistical character of the frequency errors is not clear. However, the numerical results in Tables II–IV clearly show better spectral modeling properties of the sparse formulation.

VIII. Conclusion

In this paper, we have given an overview of several linear predictors for speech analysis and coding obtained by introducing sparsity into the linear prediction framework. In speech analysis, the sparse linear predictors have been shown to provide a more efficient decoupling between the pitch harmonics and the spectral envelope. This translates into predictors that are not corrupted by the fine structure of the pitch excitation and offer interesting properties such as shift invariance and pitch invariance. In the context of speech coding, the sparsity of residual and of the high-order predictor provides a more synergistic new approach to encode a speech segment. The sparse residual obtained allows a more compact representation, while the sparse high-order predictor engenders joint estimation of short-term and long-term predictors. A compressed sensing formulation is used to reduce the size of the minimization problem, and hence to keep the computational costs reasonable. The sparse linear prediction-based robust encoding technique provided a competitive approach to speech coding with a synergistic multistage approach and a slower decaying quality for decreasing SNR.

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REFERENCES


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