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Research Article

The Evolutionary $p(x)$-Laplacian Equation with a Partial Boundary Value Condition

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Consider a diffusion convection equation coming from the electrorheological fluids. If the diffusion coefficient of the equation is degenerate on the boundary, generally, we can only impose a partial boundary value condition to ensure the well-posedness of the solutions. Since the equation is nonlinear, the partial boundary value condition cannot be depicted by Fichera function. In this paper, when $\alpha < p^- - 1$, an explicit formula of the partial boundary on which we should impose the boundary value is firstly depicted. The stability of the solutions, dependent on this partial boundary value condition, is obtained. While $\alpha > p^+ - 1$, the stability of the solutions is obtained without the boundary value condition. At the same time, only if $\alpha > 0$ and $p^- > 1$ can the uniqueness of the solutions be proved without any boundary value condition.

1. Introduction and the Main Results

The evolutionary $p(x)$-Laplacian equation

$$u_t = \operatorname{div}\left(|\nabla u|^{p(x)-2} \nabla u\right), \quad (x,t) \in Q_T = \Omega \times (0,T) \tag{1}$$

comes from a new interesting kind of fluids, the so-called electrorheological fluids (see [1, 2]), where $\Omega \subset \mathbb{R}^N$ is a bounded domain with C^2 smooth boundary $\partial\Omega$ and $p(x)$ is a measurable function. Equation (1) with the initial value

$$u(x,0) = u_0(x), \quad x \in \Omega \tag{2}$$

and the homogeneous boundary value

$$u(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T) \tag{3}$$

has been researched widely; one can refer to [3–6] et al.

If $p(x) = p$,

$$\frac{\partial u}{\partial t} - \operatorname{div}\left(a(x)|\nabla u|^{p-2} \nabla u\right) - b^i(x) D_i u + c(x,t) u = f(x,t), \quad (x,t) \in Q_T \tag{4}$$

was considered by Yin and Wang [7], where $a(x)$ may be degenerate. Instead of the usual boundary condition (3), they classified the boundary into three parts: the nondegenerate boundary, the weakly degenerate boundary, and the strongly degenerate boundary, by means of a reasonable integral description. The boundary value condition should be supplemented definitely on the nondegenerate boundary and the weakly degenerate boundary. Even earlier, they had studied a simpler equation

$$u_t = \operatorname{div}\left(\rho^\alpha |\nabla u|^{p-2} \nabla u\right), \quad (x,t) \in Q_T, \tag{5}$$

in [8]. Here $\rho(x) = \operatorname{dist}(x, \partial\Omega)$, $\alpha > 0$, and $p > 1$. They showed that only if $\alpha < p - 1$, the usual boundary value condition (3) can be imposed; while $\alpha \geq p-1$, the uniqueness of the solution can be proved without any boundary value condition.

In this paper, we will consider the evolutionary equation

$$u_t = \operatorname{div}\left(\rho^\alpha |\nabla u|^{p(x)-2} \nabla u\right) + g^i(x) D_i(b(u)), \quad (x,t) \in Q_T, \tag{6}$$

with the initial value (2) and with a partial boundary condition

$$u(x,t) = 0, \quad (x,t) \in \Sigma_p \times (0,T), \tag{7}$$

where $g^i(x) \in C^1(\overline{\Omega})$, $D_i = \partial/\partial x_i$, $b(s) \in C^1(\mathbb{R})$, and

$$\Sigma_p = \left\{x \in \partial\Omega : g^i(x) \neq 0\right\}. \tag{8}$$

In the context, $p(x)$ is a $C^1(\overline{\Omega})$ function and the definitions of the function spaces with variable exponents $L^{p(x)}(\Omega), W^{1,p(x)}(\Omega), W_0^{1,p(x)}(\Omega)$ can be found in [9–16] et al. We denote that

$$\begin{aligned} p^+ &= \max_{x\in\overline{\Omega}} p(x), \\ p^- &= \min_{x\in\overline{\Omega}} p(x). \end{aligned} \tag{9}$$

The weak solutions are defined as follows.

Definition 1. If a function $u(x,t)$ satisfies

$$\begin{aligned} & u \in L^\infty(Q_T), \\ & u_t \in L^2(Q_T), \\ & \rho^\alpha |\nabla u|^{p(x)} \in L^1(Q_T), \\ & \iint_{Q_T} \left[\frac{\partial u}{\partial t}\varphi + \rho^\alpha |\nabla u|^{p(x)-2} \nabla u \cdot \nabla\varphi + b(u) g^i(x) \varphi_{x_i} \right. \\ & \quad \left. + b(u) g^i_{x_i} \varphi\right] dx\,dt = 0, \end{aligned} \tag{10}$$

for any function $\varphi \in L^\infty(0,T; W_0^{1,p(x)}(\Omega))$, then we say that $u(x,t)$ is the weak solution of (6). The initial value (2) is true in the sense of

$$\lim_{t\to 0} \int_\Omega \left|u(x,t) - u_0(x)\right| dx = 0. \tag{11}$$

The partial boundary condition (7) is true in the sense of trace.

If

$$\begin{aligned} & u_0 \in L^\infty(\Omega), \\ & \rho^\alpha \left|\nabla u_0\right|^{p^+} \in L^1(\Omega), \end{aligned} \tag{12}$$

we add some restrictions on $b(s)$, by a similar method to that in [17, 18] and the existence of the weak solution of (6) with the initial value (2) can be proved. If $0 < \alpha < p^- - 1$, we can prove that the weak solution $u \in L^\infty(0,T; W^{1,\gamma}(\Omega))$ for some $\gamma > 1$. Then the existence of the weak solutions of (6) with the initial-boundary value conditions can be obtained. The main aim of this paper is to study the stability of the weak solutions.

Firstly, we mainly pay close attention to the stability of the weak solutions based on the partial boundary value conditions.

Theorem 2. *Let $b(s)$ be a Lipschitz function and $u(x,t)$ and $v(x,t)$ be two weak solutions of (6) with the different initial values $u_0(x)$ and $v_0(x)$, respectively, and with the same partial homogeneous boundary value*

$$u|_{\Sigma_p\times(0,T)} = 0 = v|_{\Sigma_p\times(0,T)}. \tag{13}$$

If $b(s)$ is a Lipschitz function and

$$\begin{aligned} & p^+ > 2, \\ & p^- - 1 > \alpha \geq \frac{p^- - 1}{p^+ - 2}, \end{aligned} \tag{14}$$

then

$$\begin{aligned} \int_\Omega |u(x,t) - v(x,t)|\, dx \leq \int_\Omega \left|u_0(x) - v_0(x)\right| dx, \\ \forall t \in [0,T), \end{aligned} \tag{15}$$

where Σ_p is the part of the boundary expressed as (8).

Secondly, if $\alpha \geq p^- - 1$, the weak solutions of (6) cannot be defined as the trace on the boundary and the boundary value condition cannot be used. In order to overcome this difficulty, we will introduce a new kind of the weak solutions in Section 3 and the stability of the weak solutions can be proved when $\alpha > p^+ - 1$.

Theorem 3. *Let u and v be two weak solutions of (6) with the different initial values $u(x,0)$ and $v(x,0)$, respectively. If $\alpha > p^+ - 1$, the constant $\beta \geq \max\{\alpha/p^-, 1\}$, and*

$$\left|g^i(x)\right| \leq c\rho^\beta(x), \tag{16}$$

then for any $t \in [0,T)$, there holds

$$\begin{aligned} & \int_\Omega [u(x,t) - v(x,t)]^2\, dx \\ & \quad \leq c \int_\Omega [u(x,0) - v(x,0)]^2\, dx. \end{aligned} \tag{17}$$

Last but not least, no matter whether $\alpha < p^- - 1$ or not, the uniqueness of the weak solutions is always true. Actually, similar as [19], we can prove the following theorem.

Theorem 4. *If $b_i(s)$ is a Lipschitz function, $\alpha > 0$, $p^- > 1$, then the solution of (6) with the initial value (2) is unique.*

However, for the simplicity of the paper, we will not give the details of the proof of Theorem 4 in what follows.

The rest of the paper is arranged as follows. In Section 2, Theorem 2 is proved. In Section 3, Theorem 3 is proved. In the last section, we give an explanation of the partial boundary value condition (8), and some conclusions similar to Theorem 2 are obtained without condition (14).

2. The Stability of Solutions When $\alpha<p^{-}-1$

Lemma 5 (see [9]). *(i) Let $p(x)$ and $q(x)$ be real functions with $1/p(x)+1/q(x)=1$ and $p(x)>1$. Then, for any $u\in L^{p(x)}(\Omega)$ and $v\in L^{q(x)}(\Omega)$, we have*

$$\left|\int_{\Omega} uv\,dx\right| \leq 2\left\|u\right\|_{L^{p(x)}(\Omega)}\left\|v\right\|_{L^{q(x)}(\Omega)}. \tag{18}$$

(ii)

$$\begin{aligned} &\text{If} \quad \|u\|_{L^{p(x)}(\Omega)}=1,\\ &\text{then} \quad \int_{\Omega}|u|^{p(x)}dx=1,\\ &\text{If} \quad \|u\|_{L^{p(x)}(\Omega)}>1,\\ &\text{then} \quad |u|_{L^{p(x)}}^{p^{-}}\leq\int_{\Omega}|u|^{p(x)}dx\leq|u|_{L^{p(x)}}^{p^{+}},\\ &\text{If} \quad \|u\|_{L^{p(x)}(\Omega)}<1,\\ &\text{then} \quad |u|_{L^{p(x)}}^{p^{+}}\leq\int_{\Omega}|u|^{p(x)}dx\leq|u|_{L^{p(x)}}^{p^{-}}. \end{aligned} \tag{19}$$

Proof of Theorem 2. For a small positive constant $\lambda>0$, let

$$\phi(x)=\begin{cases}1, & \text{if } x\in\Omega_{\lambda},\\ \dfrac{\rho}{\lambda}, & \text{if } x\in\Omega\setminus\Omega_{\lambda},\end{cases} \tag{20}$$

where

$$\Omega_{\lambda}=\left\{x\in\Omega:\rho(x)=\operatorname{dist}(x,\partial\Omega)>\lambda\right\}. \tag{21}$$

Then

$$\nabla\phi=\frac{1}{\lambda}\nabla\rho,\quad x\in\Omega\setminus\Omega_{\lambda}. \tag{22}$$

For small $\eta>0$, let

$$\begin{aligned} S_{\eta}(s)&=\int_{0}^{s}h_{\eta}(\tau)\,d\tau,\\ h_{\eta}(s)&=\frac{2}{\eta}\left(1-\frac{|s|}{\eta}\right)_{+}. \end{aligned} \tag{23}$$

Obviously, $h_{\eta}(s)\in C(\mathbb{R})$ and

$$\begin{aligned} &h_{\eta}(s)\geq 0,\\ &\left|sh_{\eta}(s)\right|\leq 1,\\ &\left|S_{\eta}(s)\right|\leq 1,\\ &\lim_{\eta\to 0}S_{\eta}(s)=\operatorname{sgn}s,\\ &\lim_{\eta\to 0}sS_{\eta}'(s)=0. \end{aligned} \tag{24}$$

If u and v are two weak solutions of (6) with the same partial homogeneous boundary value (13) and $S_{\eta}(\phi(u-v))$ is chosen to be the test function, then

$$\begin{aligned} &\int_{\Omega}S_{\eta}(\phi(u-v))\frac{\partial(u-v)}{\partial t}dx\\ &+\int_{\Omega}\rho^{\alpha}\left(|\nabla u|^{p(x)-2}\nabla u-|\nabla v|^{p(x)-2}\nabla v\right)\\ &\cdot\phi\nabla(u-v)S_{\eta}'(\phi(u-v))\,dx\\ &+\int_{\Omega}\rho^{\alpha}\left(|\nabla u|^{p(x)-2}\nabla u-|\nabla v|^{p(x)-2}\nabla v\right)\\ &\cdot\nabla\phi(u-v)S_{\eta}'(\phi(u-v))\,dx+\int_{\Omega}g_{x_i}^{i}(x)\\ &\cdot(b(u)-b(v))S_{\eta}(\phi(u-v))\,dx+\int_{\Omega}g^{i}(x)\\ &\cdot(b(u)-b(v))(u-v)_{x_i}S_{\eta}'(\phi(u-v))\,\phi dx\\ &+\int_{\Omega}g^{i}(x)(b(u)-b(v))\phi_{x_i}(u-v)\\ &\cdot S_{\eta}'(\phi(u-v))\,dx=0. \end{aligned} \tag{25}$$

Thus

$$\begin{aligned} &\lim_{\eta\to 0}\lim_{\lambda\to 0}\int_{\Omega}S_{\eta}(\phi(u-v))\frac{\partial(u-v)}{\partial t}dx=\frac{d}{dt}\\ &\quad\cdot\int_{\Omega}|u-v|\,dx,\\ &\int_{\Omega}\rho^{\alpha}\left(|\nabla u|^{p(x)-2}\nabla u-|\nabla v|^{p(x)-2}\nabla v\right)\\ &\quad\cdot\phi\nabla(u-v)S_{\eta}'(\phi(u-v))\,dx\geq 0. \end{aligned} \tag{26}$$

Obviously, we have

$$\begin{aligned} &\left|(u-v)S_{\eta}'(\phi(u-v))\right|=\left|\phi(u-v)S_{\eta}'(\phi(u-v))\right|\frac{1}{\phi}\\ &\leq\frac{c}{\phi},\\ &\frac{|\nabla\phi|}{\phi}\leq\frac{c}{\lambda}. \end{aligned} \tag{27}$$

Using the Young inequality, we have

$$\begin{aligned} &\left|\int_{\Omega}\rho^{\alpha}\left(|\nabla u|^{p(x)-2}\nabla u-|\nabla v|^{p(x)-2}\nabla v\right)\right.\\ &\left.\quad\cdot\nabla\phi(u-v)S_{\eta}'(\phi(u-v))\,dx\right| \end{aligned}$$

$$\leq \int_{\Omega\setminus\Omega_\lambda} \rho^\alpha \left(|\nabla u|^{p(x)-1} + |\nabla v|^{p(x)-1}\right) \cdot \frac{|\nabla\phi|}{\phi} |\phi(u-v)| S'_\eta(\phi(u-v))\,dx$$
$$\leq c\int_{\Omega\setminus\Omega_\lambda} \frac{1}{\lambda}\rho^\alpha \left(|\nabla u|^{p(x)-1} + |\nabla v|^{p(x)-1}\right)|\phi(u-v)| \cdot S'_\eta(\phi(u-v))\,dx \leq \frac{c}{\lambda} \cdot \int_{\Omega\setminus\Omega_\lambda} \rho^{\alpha-\alpha/(p(x)-1)}\rho^{\alpha/(p(x)-1)}\left(|\nabla u|^{p(x)-1} + |\nabla v|^{p(x)-1}\right)dx$$
$$\leq c\int_{\Omega\setminus\Omega_\lambda}\left[\rho^\alpha\left(|\nabla u|^{p(x)} + |\nabla v|^{p(x)}\right) + \frac{1}{\lambda^{p(x)}}\rho^{p(x)(\alpha-\alpha/(p(x)-1))}\right]dx, \tag{28}$$

which goes to 0 as $\lambda \to 0$, due to the assumption that $p^- - 1 > \alpha \geq (p^- - 1)/(p^+ - 2)$ implies that

$$\frac{1}{\lambda^{p(x)}}\rho^{p(x)(\alpha-\alpha/(p(x)-1))} \leq \lambda^{[\alpha-1-\alpha/(p(x)-1)]p(x)} \longrightarrow 0, \tag{29}$$

while

$$\left|\int_\Omega g^i(x)(b(u)-b(v))S'_\eta(\phi(u-v))(u-v) \cdot \phi_{x_i}(x)\,dx\right| \leq c\int_{\Omega\setminus\Omega_\lambda}|b(u)-b(v)|\frac{|g^i(x)|}{\lambda}dx. \tag{30}$$

Since $b(s)$ is a Lipschitz function, $|b(u) - b(v)| \leq c|u - v|$. According to the definition of the trace, by the partial boundary value condition (7),

$$u(x,t) = v(x,t) = 0, \quad x \in \Sigma_1 = \left\{x \in \partial\Omega : g^i(x) \neq 0\right\}, \tag{31}$$
$$g^i(x) = 0, \quad x \in \Sigma_2 = \left\{x \in \partial\Omega : g^i(x) = 0\right\},$$

we have

$$\lim_{\lambda\to 0}\left|\int_\Omega g^i(x)(b(u)-b(v))S'_\eta(\phi(u-v))(u-v) \cdot \phi_{x_i}(x)\,dx\right| \leq c\int_{\partial\Omega}\left|g^i(x)\right||u-v|\,d\Sigma = c\int_{\Sigma_1\cup\Sigma_2}\left|g^i(x)\right||u-v|\,d\Sigma = 0. \tag{32}$$

Moreover, as in [17], we can prove that

$$\lim_{\eta\to 0}\lim_{\lambda\to 0}\int_\Omega g^i(x)(b(u)-b(v))S'_\eta(\phi(u-v))(u-v)_{x_i} \cdot \phi(x)\,dx = 0. \tag{33}$$

The details of the proof of (33) are omitted here.

Once again,

$$\lim_{\eta\to 0}\lim_{\lambda\to 0}\left|\int_\Omega g^i_{x_i}(x)(b(u)-b(v))S_\eta(\phi(u-v))\,dx\right| \leq c\int_\Omega |u-v|\,dx. \tag{34}$$

Now, after letting $\lambda \to 0$, let $\eta \to 0$ in (25). Then, by (26), (28), (32), (33), and (34), we have

$$\frac{d}{dt}\int_\Omega |u-v|\,dx \leqslant c\int_\Omega |u-v|\,dx, \tag{35}$$

and by the Gronwall inequality, we have

$$\int_\Omega |u(x,t)-v(x,t)|\,dx \leqslant c\int_\Omega \left|u_0(x)-v_0(x)\right|dx, \quad \forall t \in [0,T). \tag{36}$$

Theorem 2 is proved. □

3. The Stability of Solutions without the Boundary Value Condition

As we have said in the introduction, when $\alpha \geq p^- - 1$, since the weak solutions of (6) generally lack the regularity, we cannot define the trace on the boundary. Thus, we cannot use the boundary value condition to research the stability or the uniqueness of the weak solution. In order to overcome this difficulty, we introduce another kind of the weak solutions as follows.

Definition 6. A function $u(x,t)$ is said to be a weak solution of (6) with the initial value (2), if u satisfies

$$u \in L^\infty(Q_T), \quad \frac{\partial u}{\partial t} \in L^2(Q_T), \quad \rho^\alpha|\nabla u|^{p(x)} \in L^1(Q_T), \tag{37}$$

and for any function $\varphi_1 \in L^1(0,T;C_0^1(\Omega))$, $\varphi_2 \in L^\infty(Q_T)$ such that for any given $t \in [0,T)$, $\varphi_2(x,\cdot) \in W^{1,p(x)}_{\text{loc}}(\Omega)$,

$$\iint_{Q_T}\left[\frac{\partial u}{\partial t}(\varphi_1\varphi_2) + \rho^\alpha|\nabla u|^{p(x)-2}\nabla u \cdot \nabla(\varphi_1\varphi_2) + g^i(x)b(u)(\varphi_1\varphi_2)_{x_i} + b(u)g^i_{x_i}(\varphi_1\varphi_2)\right]dx\,dt = 0, \tag{38}$$

and the initial value (2) is satisfied in the sense of

$$\lim_{t\to 0}\int_\Omega \left|u(x,t)-u_0(x)\right|dx = 0. \tag{39}$$

We first introduced this kind of the weak solutions in our previous paper [19], in which the following equation was studied:

$$\frac{\partial u}{\partial t} - \operatorname{div}\left(a(x)|\nabla u|^{p-2}\nabla u\right) - b^i(x) D_i u + c(x,t)u = f(x,t), \quad (x,t) \in \Omega \times (0,T), \tag{40}$$

where $0 \le a(x) \in C^1(\overline{\Omega})$ with $a(x)|_{x\in\partial\Omega} = 0$. It is not difficult to prove the existence of the weak solution in the sense of Definition 6.

Proof of Theorem 3. For any fixed $s, \tau \in [0,T)$, after an approximate procedure, we may choose $\chi_{[\tau,s]}(u-v)\phi$ as a test function in equality (38), where $\chi_{[\tau,s]}$ is the characteristic function on $[\tau, s]$ and $\phi(x)$ is defined as (20). Thus we have

$$\begin{aligned}
&\iint_{Q_{\tau s}} (u-v)\phi \frac{\partial(u-v)}{\partial t} dx\,dt \\
&= -\iint_{Q_s} \rho^\alpha \left(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v\right) \cdot \nabla\left[(u-v)\phi\right] dx\,dt \\
&\quad - \iint_{Q_{\tau s}} g^i(x)\left[b(u)-b(v)\right]\left[(u-v)\phi\right]_{x_i} dx\,dt \\
&\quad - \iint_{Q_{\tau s}} \left[b(u)-b(v)\right] g^i_{x_i}\phi(u-v)\,dx\,dt,
\end{aligned} \tag{41}$$

where $Q_{\tau s} = \Omega \times [\tau, s]$.

We can rewrite (41) as follows:

$$\begin{aligned}
&\int_\Omega \left[u(x,s)-v(x,s)\right]^2 \phi\,dx = \int_\Omega \left[u(x,0)-v(x,0)\right]^2 \cdot \phi\,dx \\
&\quad - 2\iint_{Q_{\tau s}} \phi\rho^\alpha \left(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v\right) \cdot \nabla(u-v)\,dx\,dt - 2\iint_{Q_{\tau s}} (u-v) \\
&\quad \cdot \rho^\alpha \left(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v\right)\nabla\phi\,dx\,dt \\
&\quad - 2\iint_{Q_{\tau s}} g^i(x)\left[b(u)-b(v)\right] \cdot \left[(u-v)\phi\right]_{x_i} dx\,dt - 2\iint_{Q_{\tau s}} \left[b(u)-b(v)\right] \cdot g^i_{x_i}\phi(u-v)\,dx\,dt \\
&\le \int_\Omega \left[u(x,0)-v(x,0)\right]^2 \phi dx - 2\iint_{Q_{\tau s}} (u-v) \cdot \rho^\alpha \left(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v\right)\nabla\phi\,dx\,dt \\
&\quad - 2\iint_{Q_{\tau s}} g^i(x)\left[b(u)-b(v)\right] \cdot \left[(u-v)\phi\right]_{x_i} dx\,dt - 2\iint_{Q_{\tau s}} \left[b(u)-b(v)\right] \cdot g^i_{x_i}\phi(u-v)\,dx\,dt.
\end{aligned} \tag{42}$$

In the first place, since

$$\begin{aligned}
&\left|-2\iint_{Q_{\tau s}} (u-v)\rho^\alpha \left(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v\right) \cdot \nabla\phi\,dx\,dt\right| \le 2\iint_{Q_{\tau s}} |u-v|\rho^\alpha \left(|\nabla u|^{p(x)-1} + |\nabla v|^{p(x)-1}\right)|\nabla\phi|\,dx\,dt \\
&\le c\int_\tau^s \int_{\Omega\setminus\Omega_\lambda} \left[\frac{p(x)-1}{p(x)}\rho^\alpha\left(|\nabla u|^{p(x)} + |\nabla v|^{p(x)}\right) + \frac{1}{p(x)}\rho^\alpha |\nabla\phi|^{p(x)}\right] dx\,dt \\
&\le c\int_\tau^s \int_{\Omega\setminus\Omega_\lambda} \left[\frac{p(x)-1}{p(x)}\rho^\alpha\left(|\nabla u|^{p(x)} + |\nabla v|^{p(x)}\right) + \frac{1}{p(x)}\lambda^{\alpha-p(x)}\right] dx\,dt,
\end{aligned} \tag{43}$$

by $\alpha > p^+ - 1$ leading to $\alpha - p(x) > -1$, (43) yields

$$\lim_{\lambda\to 0}\left|-2\iint_{Q_{\tau s}} (u-v) \cdot \rho^\alpha \left(|\nabla u|^{p(x)-2}\nabla u - |\nabla v|^{p(x)-2}\nabla v\right)\nabla\phi\,dx\,dt\right| = 0. \tag{44}$$

In the second place, since $|\rho_{x_i}| \le |\nabla\rho| = 1$, by (16), $|g^i(x)| \le c\rho^\beta(x)$, then

$$\begin{aligned}
&\lim_{\lambda\to 0}\left|\iint_{Q_{\tau s}} g^i(x)\left[b(u)-b(v)\right](u-v)\phi_{x_i}\,dx\,dt\right| \\
&= \lim_{\lambda\to 0}\left|\int_\tau^s \int_{\Omega\setminus\Omega_\lambda} g^i(x)\left[b(u)-b(v)\right](u-v)\phi_{x_i}\,dx\,dt\right| \\
&\le c\lim_{\lambda\to 0}\int_\tau^s \int_{\Omega\setminus\Omega_\lambda} |u-v|^2\,dx\,dt = 0.
\end{aligned} \tag{45}$$

By $u, v \in L^\infty(Q_T)$ and the condition (16), we have

$$\begin{aligned}
&\left|\iint_{Q_{\tau s}} g^i(x)\,[b(u)-b(v)]\,(u-v)_{x_i}\,\phi\,dx\,dt\right| \\
&\quad \leq \sum_{i=1}^{N}\left(\int_\tau^s\int_\Omega \left|g^i(x)\right|^{p(x)/(p(x)-1)}\rho^{-\alpha/(p(x)-1)}\,|b(u)-b(v)|^{p(x)/(p(x)-1)}\,dx\,dt\right)^{1/q_1} \\
&\quad \cdot\left(\int_\tau^s\int_\Omega \rho^{\alpha}\left(|\nabla u|^{p(x)}+|\nabla v|^{p(x)}\right)dx\,dt\right)^{1/p_1} \leq c\sum_{i=1}^{N}\left(\int_\tau^s\int_\Omega \rho^{(\beta p(x)/(p(x)-1)-\alpha/(p(x)-1))}\,|u-v|^{p(x)/(p(x)-1)}\,dx\,dt\right)^{1/q_1} \\
&\quad \leq c\sum_{i=1}^{N}\left(\int_\tau^s\int_\Omega |u-v|^{p(x)/(p(x)-1)}\,dx\,dt\right)^{1/q_1},
\end{aligned} \tag{46}$$

by which $\beta \geq \alpha/p^-$. Here $p_1 = p^+$ or p^- and $q_1 = \max\{p(x)/(p(x)-1)\}$ or $\min\{p(x)/(p(x)-1)\}$ according to (iii) of Lemma 5.

We denote that

$$\begin{aligned}
\Omega_1 &= \left\{x\in\Omega : \frac{p(x)}{p(x)-1}\geq 2\right\}, \\
\Omega_2 &= \left\{x\in\Omega : \frac{p(x)}{p(x)-1}< 2\right\},
\end{aligned} \tag{47}$$

and then

$$\begin{aligned}
&\int_{\Omega_1} |u-v|^{p(x)/(p(x)-1)}\,dx \leq c\int_{\Omega_1} |u-v|^2\,dx, \\
&\int_{\Omega_2} |u-v|^{p(x)/(p(x)-1)}\,dx \leq c\left(\int_{\Omega_1} |u-v|^2\,dx\right)^{1/q_2},
\end{aligned} \tag{48}$$

where $q_2 = \max\{2(p(x)-1)/p(x)\}$ or $\min\{2(p(x)-1)/p(x)\}$ according to (iii) of Lemma 5.

Combining (48) with (46), we have

$$\begin{aligned}
&\left|\iint_{Q_{\tau s}} g^i(x)\,[b(u)-b(v)]\,(u-v)_{x_i}\,\phi\,dx\,dt\right| \\
&\quad \leq c\left(\int_\tau^s\int_\Omega |u-v|^2\,dx\,dt\right)^{l},
\end{aligned} \tag{49}$$

where $l < 1$.

Once more,

$$\begin{aligned}
&\left|\iint_{Q_{\tau s}} [b(u)-b(v)]\,\phi\,(u-v)\,g^i_{x_i}\,dx\,dt\right| \\
&\quad \leq c\int_\tau^s\int_\Omega |u-v|^2\,dx\,dt.
\end{aligned} \tag{50}$$

Let $\lambda \to 0$ in (42). By (44), (45), (49), and (50), we have

$$\begin{aligned}
&\int_\Omega [u(x,s)-v(x,s)]^2\,dx \\
&\quad \leq \int_\Omega [u(x,0)-v(x,0)]^2\,dx \\
&\qquad + c\int_\tau^s\int_\Omega |u-v|^2\,dx\,dt \\
&\qquad + c\left(\int_\tau^s\int_\Omega |u-v|^2\,dx\,dt\right)^{l}.
\end{aligned} \tag{51}$$

Let $\kappa(s) = \int_\Omega |u(x,s)-v(x,s)|^2dx$. Without loss of the generality, we may assume that there exist $\tau \in [0,T)$ and $\kappa(\tau) > 0$. Then for any $s > \tau$, $\int_\tau^s k(t)dt > 0$. If we denote that

$$\begin{aligned}
&\tau_0 = \sup\{t\in[\tau,s],\ \kappa(t)>0\}, \\
&\int_\tau^{\tau_0} k(t)\,dt = c_1,
\end{aligned} \tag{52}$$

then $\tau < \tau_0 \leq s$ and

$$\int_\tau^s k(t)\,dt \geq \int_\tau^{\tau_0} k(t)\,dt = c_1. \tag{53}$$

By $u, v \in L^\infty(Q_T)$, there exists a constant $C > 0$ such that

$$\begin{aligned}
\frac{c\left(\int_\tau^s k(t)\,dt\right)^l}{\int_\tau^s k(t)\,dt} &\leq \frac{c\left(\int_\tau^s k(t)\,dt\right)^l}{c_1} \leq C \\
&= C(c,c_1,T,q).
\end{aligned} \tag{54}$$

By (51) and (54), we have

$$\kappa(s)-\kappa(\tau) \leq (C+c)\int_\tau^s k(t)\,dt, \tag{55}$$

and using the Gronwall inequality, we have

$$\begin{aligned}
&\int_\Omega |u(x,s)-v(x,s)|^2\,dx \\
&\quad \leq c\int_\Omega |u(x,\tau)-v(x,\tau)|^2\,dx,
\end{aligned} \tag{56}$$

where c depends on C. Thus, we have

$$\int_{\Omega} |u(x,s) - v(x,s)|^2 \, dx \leq c \int_{\Omega} \left|u_0(x) - v_0(x)\right| dx. \quad (57)$$

The proof is complete. □

4. The Partial Boundary Value Condition

Consider the linear degenerate equation

$$\frac{\partial u}{\partial t} - \operatorname{div}(a(x)\nabla u) - \sum_{i=1}^{N} f^i(x) D_i u = 0, \quad (58)$$

which is a particular case of (6) (where $p = 2$, $g^i = f^i, b(s) = s$). Rewrite it as

$$\frac{\partial u}{\partial t} - a(x)\Delta u - \sum_{i=1}^{N} \left(a_{x_i}(x) + f^i(x)\right) D_i u = 0. \quad (59)$$

According to the Fichera-Oleinik theory [20, 21], besides the initial value condition (2), since $a(x)|_{\partial\Omega} = 0$, the partial boundary, where we should impose the boundary value condition, is

$$\Sigma_1 = \left\{x \in \partial\Omega : \sum_{i=1}^{N} f^i(x) n_i(x) < 0\right\}, \quad (60)$$

where $\vec{n} = \{n_i\}$ is the inner normal vector of $\partial\Omega$.

By reviewing the formula of (8), One can see that

$$\Sigma_1 \subseteq \Sigma_p \subseteq \partial\Omega. \quad (61)$$

By this token, condition (13) in Theorem 2 is reasonable.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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