$H_\infty$ Control of Singular Systems via Delta Operator Approach

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Abstract—This paper investigates the problem of state feedback $H_\infty$ control for singular systems through delta operator approach. Firstly, a bounded real lemma corresponding to a singular continuous system under the framework of the delta operator model is obtained. Then, the existence condition and explicit expression of a desirable $H_\infty$ controller for the singular delta operator system are presented. As special cases, the results of admissible control for the singular delta operator system and $H_\infty$ control as well as admissible control for the singular continuous system are also derived. All required conditions in this paper are characterized in the form of strict linear matrix inequalities whose feasible solutions can be obtained easily and directly. Finally, some numerical examples are provided to illustrate the effectiveness of the obtained theoretical results in the paper.

Keywords—singular continuous systems, admissibility, delta operator model, $H_\infty$ control, state feedback, linear matrix inequality (LMI)

I. INTRODUCTION

Compared with state-space systems, singular systems can describe dynamics and algebraic constraints simultaneously and therefore are more natural and suitable to describe many practical systems in applications. This type of systems has been appeared in many different research areas, such as economic systems, electrical networks and highly interconnected large-scale systems, constrained mechanical systems, robot systems [1], [2], [3], etc. Many fundamental results of singular systems, such as controllability and observability [1], stability and stabilization [2], [4], [5], $H_\infty$ control [2], [6], [7], [8], [9], have been obtained. Moreover, singular system methods are also adopted to study state-space systems in order to obtain less conservative results [10]. It is well-known that both analysis and synthesis of singular systems are much more complicated than those of state-space systems, since admissibility (i.e., regularity, stability and impulse elimination for the continuous case or regularity, stability and causality for the discrete case) is a basic requirement in singular systems.

Discrete systems are often obtained from continuous systems through sampling. Most research results used the standard shift operator to study discrete systems. But there exists a problem that the dynamic response of a discrete system does not converge smoothly to its continuous counterpart when the sampling period tends to zero [11]. In order to avoid the above problem, a delta operator method was proposed in [12]. It was shown that the delta operator method is significantly less sensitive than the shift operator method at high sampling rates [13]. Furthermore, the delta operator model can provide a unified expression of state-space continuous systems and their corresponding discrete models and the sampling period can be characterized in a more direct way. There have been many research results for state-space systems through the delta operator method, including Lyapunov and Riccati equations [14], stabilization [15], $H_\infty$ control [16], [17] and sliding mode control [18], etc. Recently, the delta operator method has been introduced to singular systems and some results about singular delta operator systems (SDOSs) have been appeared in [19], [20], [21], [22], [23]. The delta operator model has been set up for a singular continuous system in [19] and [20] through two different methods, respectively, which was obtained directly from a suitable singular discrete model and proved to tend to the corresponding continuous system as the sampling period tends to zero. Thus, the delta operator model provided a unified description of a singular continuous system and its discrete model. The analysis results of controllability and observability for SDOSs were provided in [20] and [22], respectively. Various admissibility conditions were presented for SDOSs in [21]. Some results about state feedback admissible control for SDOSs, given in terms of matrix inequalities, can be found in [19] and [23].

The problem of $H_\infty$ control for singular systems is of both practical and theoretical importance and has received considerable attention in the past decades. A number of significant results have been reported in the literatures. For example, [6] presented a bounded real lemma and then solved the problem of $H_\infty$ control for singular continuous systems, where the results all involved non-strict LMIs. As to singular discrete systems, [7] obtained a bounded real lemma which also contained a non-strict LMI. [8] provided a bounded real lemma and then a sufficient condition to solve the state feedback $H_\infty$ control problem in terms of strict LMIs. [9] derived another strict LMI-based bounded real lemma. Many difficulties still exist in designing a feedback controller for singular systems such as extreme computation, system transformation, et al.

In this paper, we will consider the problem of state feedback $H_\infty$ control for singular systems via delta operator method. The contributions of this paper are as follows. (1) A bounded real lemma is presented for the SDOS which is obtained from a singular continuous system directly through sampling. (2) The existence condition and explicit expression of a desirable $H_\infty$ controller are given for the SDOS. (3) Based on the above
result, the conditions are also given for the admissible control of SDOS, $H_\infty$ control and admissible control for the singular continuous system. It is worthy of mentioning here that all the conditions in this paper are in the form of strict LMIs whose feasible solutions can be derived easily by existing softwares.

The remainder of this paper is organized as follows. Section 2 presents some preliminaries and problem formulation. Section 3 gives the main results. Section 4 provides some numerical examples and Section 5 concludes this paper.

Throughout this paper, the following notations are adopted. $R^n$ and $R^{m \times n}$ denote the spaces of $n$-dimensional real vectors and $m \times n$ real matrices, respectively. Matrix $P > 0$ (or $P < 0$, $P \geq 0$, respectively) means that $P$ is symmetric and positive definite (or negative definite, semi-positive definite, respectively). The superscript $\ast$ denotes a block diagonal matrix with diagonal blocks being the means the half left open part of the complex plane. The delta operator $\delta$ is denoted by $\delta \omega = \{z \mid \det(zE - A) = 0\}$. The shorthand diag$(M_1, M_2, \ldots, M_r)$ denotes a block diagonal matrix with diagonal blocks being the matrices $M_1, M_2, \ldots, M_r$. The identity matrix with dimension $r \times r$ is denoted by $I_r$. The delta operator $\delta$ is defined by

$$\delta x(t) = \begin{cases} \frac{d}{dt} x(t), & h = 0 \\ \frac{x((t + h) - x(t)}{h}, & h \neq 0 \end{cases}$$

where $h$ is the sampling period.

II. PRELIMINARIES AND PROBLEM FORMULATION

Consider the following singular continuous system

$$\begin{align*}
E \dot{x}(t) &= A_s x(t) + F_s \omega(t) + B_s u(t) \\
z(t) &= C x(t)
\end{align*}$$

(1)

where $x(t) \in R^n$ is the state, $\omega(t) \in R^p$ is the disturbance input, $u(t) \in R^m$ is the control input, $z(t) \in R^q$ is the regulated output, $E \in R^{m \times n}$ and $rank(E) = r \leq n$, $A_s, F_s, B_s, C$ and $D$ are known constant matrices with appropriate dimensions.

Consider the following unfurmed singular system

$$E \dot{x}(t) = A_s x(t)$$

(2)

**Definition 1:** [6] The system (2) is said to be regular if $\text{det}(sE - A_s)$ is not identically zero. The system (2) is said to be impulse-free if $\text{deg} (\text{det}(sE - A_s)) = \text{rank}(E)$. The system (2) is said to be stable if $\lambda(E, A_s) < 0$. The system (2) is said to be admissible if it is regular, impulse-free and stable.

**Lemma 1:** [2] The system (2) is admissible if and only if there exist matrices $W > 0$ and $Q$ satisfying

$$A_s^T P + P A_s < 0$$

(3)

where $P = WE + SQ$ and $S$ is any matrix of full column rank and satisfies $E^T S = 0$.

The unfurmed system of the system (1) is

$$\begin{align*}
E \dot{x}(t) &= A_s x(t) + F_s \omega(t) \\
z(t) &= C x(t)
\end{align*}$$

(4)

**Lemma 2:** [2] The system (4) is admissible and with a prescribed $H_\infty$ performance $\gamma$ if and only if there exist matrices $W > 0$ and $Q$ satisfying

$$\begin{bmatrix}
A_s^T P + P A_s + C^T C & P^T F_s \\
F_s^T P & -\gamma^2 I_r
\end{bmatrix} < 0$$

(5)

where $P = WE + SQ$ and $S$ is any matrix of full column rank and satisfies $E^T S = 0$.

For a prescribed sampling period $h > 0$, according to the method introduced in [19] or [22], we can set up the delta operator model for the system (1) as follows

$$E \delta x(t_k) = Ax(t_k) + F \omega(t_k) + Bu(t_k)$$

$$z(t_k) = C x(t_k)$$

(6)

where $x(t_k) \in R^n$ is the state, $u(t_k) \in R^m$ is the control input, $\omega(t_k) \in R^p$ is the disturbance input, $z(t_k) \in R^q$ is the regulated output. $t_k$ denotes the time $t = kh$. $E$ and $C$ are the same as that in the system (1), $A, B$ and $F$ are constant matrices obtained from $h$ and the system (1).

From [19] and [22] we know that the system (6) has the property that it will tend to the corresponding continuous system (1) when the sampling period $h$ tends to zero.

Consider the following system

$$E \delta x(t_k) = Ax(t_k)$$

(7)

**Definition 2:** [19], [21] The system (7) is said to be regular if $\text{det}(\delta E - A)$ is not identically zero. The system (7) is said to be causal if $\text{deg} (\text{det}(\delta E - A)) = \text{rank}(E)$. The system (7) is said to be stable if $\lambda(E, A) < 0$. The system (7) is said to be admissible if it is regular, causal and stable.

**Lemma 3:** [21], [23] The system (7) is admissible if and only if there exist matrices $W > 0$ and $Q$ such that

$$h A^T W A + A^T P + P A < 0$$

(8)

where $P = WE + SQ$ and $S$ is any matrix of full column rank and satisfies $E^T S = 0$.

**Remark 1:** From Lemmas 1, 3 and $\lim_{h \to 0} A = A_s$ we can obtain that when the sampling period $h$ tends to zero, the admissibility condition of the system (7) will tend to that of the corresponding continuous system (2). Thus Lemma 3 provides a unified admissibility condition for both singular discrete systems and singular continuous systems.

Let the controller to be designed in this paper is a state feedback one as

$$u(t_k) = K x(t_k)$$

(9)

Then the closed-loop system of the system (6) under the controller (9) is

$$\begin{align*}
E \delta x(t_k) &= A_c x(t_k) + F \omega(t_k) \\
z(t_k) &= C x(t_k)
\end{align*}$$

(10)

where $A_c = A + BK$.

The purpose of this paper is to give the design method of the gain matrix $K$ in the controller (9), such that the closed-loop system (10) is admissible with a prescribed $H_\infty$ performance $\gamma$, i.e., the system (10) satisfies the following requirements:

1) When $\omega(t_k) = 0$, the system (10) is admissible.
In this case, the controller (9) is said to be an $H_\infty$ controller of the system (6). If the system (10) satisfies only 1), then the controller (9) is said to be an admissible controller of the system (6).

For the derivation of our main results, we present the following lemmas.

**Lemma 4:** [24] (Schur complement lemma) For matrices $Q = Q^T$, $R = R^T$ and $S$, the inequality

$$
\begin{bmatrix}
Q & S \\
S^T & R
\end{bmatrix} < 0
$$

is equivalent to $R < 0$ and $Q - SR^{-1}S^T < 0$.

**Lemma 5:** [18] For any time function $x(t)$ and $y(t)$, there exists

$$
\delta(x(t)y(t)) = \delta(x(t)y(t)) + x(t)\delta y(t) + h\delta x(t)\delta y(t)
$$

**III. MAIN RESULTS**

Consider the unforced system of the system (6) as

$$
E\delta x(t_k) = Ax(t_k) + F\omega(t_k) \\
z(t_k) = Cx(t_k)
$$

(11)

**Theorem 1:** The system (11) is admissible with a prescribed $H_\infty$ performance $\gamma$ if and only if there exist matrices $W > 0$ and $Q$ such that

$$
\begin{bmatrix}
A^TP + P^TA + C^TC & P^TF & A^TW \\
FTP & -\gamma^2I_p & FTW \\
WA & WF & -h^{-1}W
\end{bmatrix} < 0
$$

(12)

where $P = WE + SQ$ and $S$ is any matrix of full column rank and satisfies $E^TS = 0$.

Proof (Sufficiency) Assume that the inequality (12) holds. Then from $h > 0$, $W > 0$ and Lemma 4 we have that (12) is equivalent to the following inequality

$$
\Xi = \begin{bmatrix}
A^TP + P^TA + C^TC & P^TF \\
FTP & -\gamma^2I_p
\end{bmatrix} - h\begin{bmatrix}
A^TF \\
FT
\end{bmatrix}W\begin{bmatrix}
A & F
\end{bmatrix} < 0
$$

(13)

From (13) we can obtain

$$
hA^TW\Xi + A^TP + PT A + C^TC < 0
$$

which means

$$
hA^TW\Xi + A^TP + PT A < 0
$$

from $C^TC > 0$. Then from Lemma 3 we know that the system (11) is admissible.

Let

$$
V(x(t_k)) = x(t_k)^TE^TWEx(t_k)
$$

Then from $W > 0$ we have that $V(x(t_k)) \geq 0$ holds for any $k \geq 0$.

From Lemma 5 we have

$$
\delta V(x(t_k)) = \delta x(t_k)^TE^TWEx(t_k) + x(t_k)^TE^TW\delta x(t_k)
$$

$$
= (Ax(t_k) + F\omega(t_k))^TWEx(t_k) + x(t_k)^TE^TW(F\omega(t_k) + F\omega(t_k))
$$

$$
= (Ax(t_k) + F\omega(t_k))^TWEx(t_k) + h(Ax(t_k) + F\omega(t_k))^TW(Ax(t_k) + F\omega(t_k))
$$

$$
= \zeta(t_k)^T\zeta(t_k)
$$

where

$$
\Sigma = \Xi + \Pi + \text{diag}(C^TC, -\gamma^2I_p) = \Xi < 0
$$

From $\Sigma < 0$ we can obtain $J < 0$. Therefore the system (11) satisfies a prescribed $H_\infty$ performance $\gamma$. Thus we have proved that the system (11) is admissible with a prescribed $H_\infty$ performance $\gamma$.

(Necessity) Assume that the system (11) is admissible with a prescribed $H_\infty$ performance $\gamma$. Then from the definition of $\delta x(t_k)$, the system (11) can also be written as

$$
Ex(t_{k+1}) = Ax(t_k) + F\omega(t_k) \\
z(t_k) = Cx(t_k)
$$

(14)

where $A_x = E + hA$ and $F_x = hF$.

Then from [8] we know that if the system (14) (i.e. the system (11)) is admissible with a prescribed $H_\infty$ performance $\gamma$ (here we use $hJ < 0$ instead of $J < 0$), then there exist matrices $W > 0$ and $Q$ satisfying

$$
\Omega = Q^TSTF - A^TW
$$

$$
W = A_xW_z + F_xW_f + hI_p - W
$$

(15)

where $\Omega = Q^TST A_z + A_x^T S Q - E^TW E + hC^T C$ and $S$ is any matrix of full column rank and satisfies $E^TS = 0$.

From $h > 0$, $W > 0$ and Lemma 4, the inequality (15) is equivalent to

$$
\Omega = Q^TST F_z - A^T W
$$

$$
W = A_xW_z + F_xW_f - W
$$

(16)
Let $P = WE + SQ$. Then from $A_z = E + hA$, $F_z = hF$ and $E^T S = 0$, the inequality (16) is also the same as

$$h \begin{bmatrix} A^T P + PT A + CT C & PT F \\ F^T P & -\gamma^2 I_p \end{bmatrix} + h^2 \begin{bmatrix} A^T F \\ F^T F \end{bmatrix} W \begin{bmatrix} A & F \end{bmatrix} < 0 \quad (17)$$

From $W > 0$ and Lemma 4, it is easy to know that (17) is equivalent to the inequality (12). This completes the proof.

**Remark 2:** From Lemma 2, Theorem 1, $\lim_{h \to 0} A = A_s$ and $\lim_{h \to 0} F = F_s$ we know that when the sampling period $h$ tends to zero, the bounded real lemma for the system (11) will tend to the bounded real lemma of the continuous system (4). Thus Theorem 1 provides a unified bounded real lemma for both singular discrete systems and singular continuous systems.

**Remark 3:** Since Lemma 3 and Theorem 1 have given unified expressions of admissibility condition and bounded real lemma for singular discrete systems and singular continuous systems, respectively, we can then obtain that the methods to design an $H_{\infty}$ controller or an admissible controller for SDOSs should also be suitable to design the same controller for singular continuous systems.

Based on Theorem 1, we know that the closed-loop system (10) is admissible with a prescribed $H_{\infty}$ performance $\gamma$ if and only if there exist matrices $W > 0$ and $Q$ satisfying (12) where the matrix $A$ is replaced by the matrix $A_c = A + BK$. In this case, the unknown matrix $K$ is contained in $A_c$ and $A_c$ is companied by different matrices as $W$ and $P$, thus it is difficult to obtain the design method of the matrix $K$. Therefore we do not use Theorem 1 directly to the closed-loop system (10). To solve the problem of $H_{\infty}$ control, we rewrite the system (10) as

$$\begin{align*}
E \delta x(t_k) &= Ax(t_k) + F \omega(t_k) + Bu(t_k) \\
0 &= K x(t_k) - u(t_k) \\
z(t_k) &= C x(t_k)
\end{align*} \quad (18)$$

The system (18) is also the same as

$$\begin{align*}
\bar{E} \delta y(t_k) &= \bar{A} y(t_k) + \bar{F} \omega(t_k) \\
z(t_k) &= \bar{C} y(t_k)
\end{align*} \quad (19)$$

where

$$\bar{E} = \begin{bmatrix} E & 0 \\ 0 & 0 \end{bmatrix}, \quad \bar{A} = \begin{bmatrix} A & B \\ K & -I_m \end{bmatrix} \quad (20)$$

$$\bar{y}(t_k) = \begin{bmatrix} x(t_k) \\ u(t_k) \end{bmatrix}, \quad \bar{F} = \begin{bmatrix} F \\ 0 \end{bmatrix}, \quad \bar{C} = \begin{bmatrix} C & 0 \end{bmatrix} \quad (21)$$

Then we have the following result.

**Theorem 2:** There exists an $H_{\infty}$ controller (9) for the system (6) if there exist matrices $W > 0, V > 0, Q_1, Q_2$ and $Y$ such that

$$\begin{bmatrix}
\Psi_{11} & \Psi_{21}^T F \\
\Psi_{21} & \Psi_{22} \\
F^T P_1 & 0 & -\gamma^2 I_p \\
W A & WB & WF & -h^{-1} W & 0
\end{bmatrix} < 0 \quad (22)$$

where $P_1 = WE + SQ_1$, $S$ is any matrix of full column rank and satisfies $E^T S = 0$, $J_1, J_2$ are known matrices with appropriate dimensions, and

$$\begin{align*}
\Psi_{11} &= A^T P_1 + P_1^T A + J_1 Y + Y J_1^T + C^T C \\
\Psi_{21} &= B^T P_1 + J_2 Y - V J_1^T + Q_2^T S^T A \\
\Psi_{22} &= Q_2^T S^T B + B^T SQ_2 - J_2 Y - V J_2^T
\end{align*}$$

In this case, the gain matrix $K$ of the controller (9) can be designed as $K = V^{-1} Y$.

Proof From Theorem 1 we know that the system (19) (i.e. the system (10)) is admissible with a prescribed $H_{\infty}$ performance $\gamma$ if and only if there exist matrices $\bar{W} > 0$ and $\bar{Q}$ such that

$$\begin{bmatrix}
A^T \bar{P} + \bar{P}^T A + C^T C & \bar{P}^T F & \bar{A}^T \bar{W} \\
F^T \bar{P} & -\gamma^2 I_p & \bar{F}^T \bar{W} \\
\bar{W} A & \bar{W} F & -h^{-1} \bar{W}
\end{bmatrix} < 0 \quad (23)$$

where $\bar{P} = W E + SQ$ and $\bar{S}$ is any matrix of full column rank and satisfies $E^T \bar{S} = 0$.

Let

$$\bar{Q} = \begin{bmatrix} Q_1 & J_1 \\ Q_2 & J_2 \end{bmatrix} \quad (24)$$

$$\bar{W} = diag(W, V), \quad \bar{S} = diag(S, V) \quad (25)$$

where the partitions are compatible with the structure of $E$.

Then we can easily obtain that $\bar{W} > 0$ from $W > 0$ and $V > 0$, and the matrix $\bar{S}$ is of full column rank from that $S$ is of full column rank and the matrix $V$ is invertible. From (20), (25) and $E^T \bar{S} = 0$ we also have $E^T \bar{S} = 0$.

By substituting (20), (21) (24) and (25) into the inequality (23) and denoting $Y = VK$, we can obtain the inequality (22) immediately. From $V > 0$, we can get $K = V^{-1} Y$ easily.

Based on Theorem 2, we can derive the following corollaries directly.

**Corollary 1:** There exists an admissible controller (9) for the system (6) if there exist matrices $W > 0, V > 0, Q_1, Q_2$ and $Y$ such that

$$\begin{bmatrix}
\Psi_{11} - C^T C & \Psi_{21} & A^T W & Y^T \\
\Psi_{21} & \Psi_{22} & B^T W & -V \\
W A & WB & -h^{-1} W & 0 \\
Y & -V & 0 & -h^{-1} V
\end{bmatrix} < 0 \quad (26)$$

where $\Psi_{11}, \Psi_{21}, \Psi_{22}, S, J_1, J_2$ are the same as that in Theorem 2. In this case, the gain matrix $K$ of the controller (9) can be designed as $K = V^{-1} Y$.

**Corollary 2:** There exists an $H_{\infty}$ controller (9) for the system (1) if there exist matrices $W > 0, V > 0, Q_1, Q_2$ and $Y$ such that

$$\begin{bmatrix}
\Pi_{11} & \Pi_{21}^T F_s \\
\Pi_{21} & 0 \\
F_s^T P_1 & 0 & -\gamma^2 I_p
\end{bmatrix} < 0 \quad (27)$$

where $P_1 = WE + SQ_1$, $S$ is any matrix of full column rank and satisfies $E^T S = 0$, $J_1, J_2$ are known matrices with appropriate dimensions, and

$$\begin{align*}
\Pi_{11} &= A^T P_1 + P_1^T A_s + J_1 Y + YJ_1^T + C^T C \\
\Pi_{21} &= B^T P_1 + J_2 Y - V J_1^T + Q_2^T S^T A_s \\
\Pi_{22} &= Q_2^T S^T B_s + B^T SQ_2 - J_2 Y - V J_2^T
\end{align*}$$
In this case, the gain matrix $K$ of the controller (9) can be designed as $K = V^{-1}Y$.

**Corollary 3**: There exists an admissible controller (9) for the system (1) if there exist matrices $W > 0, V > 0, Q_1, Q_2$ and $Y$ such that

$$
\begin{bmatrix}
\Pi_{11} - CT^C & \Pi_{21} \\
\Pi_{21} & \Pi_{22}
\end{bmatrix} < 0
$$

where $\Pi_{11}, \Pi_{21}, \Pi_{22}, J_1, J_2$ and $S$ are the same as that in Corollary 2. In this case, the gain matrix $K$ of the controller (9) can be designed as $K = V^{-1}Y$.

IV. EXAMPLES

In this section, we give some numerical examples to demonstrate the theoretical results that we have obtained in the above sections.

**Example 1** Consider the singular continuous system (1) with the following coefficient matrices

$$
E = \begin{bmatrix}
1 & 2 \\
1 & 2
\end{bmatrix}, \quad A_s = \begin{bmatrix}
-5 & 3 \\
2 & -3
\end{bmatrix}
$$

$$
B_s = \begin{bmatrix}
2 \\
3
\end{bmatrix}, \quad F_s = \begin{bmatrix}
1 \\
2
\end{bmatrix}, \quad C = \begin{bmatrix}
1 & 1
\end{bmatrix}
$$

Let $S = \begin{bmatrix}
1 & -1
\end{bmatrix}^T$. From Lemma 1 one can show that the system (1) is admissible. By Lemma 2 we can get the minimal scalar $\gamma$ as $\gamma = 2.34$ such that the open-loop system of the system (1) is admissible with an $H_\infty$ performance $\gamma$.

Let us consider to design an $H_\infty$ controller (9) for the system (1). Choose $J_1 = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ and $J_2 = 1$. By Corollary 2 we can derive the minimal scalar $\gamma$ as $\gamma = 0.17$ such that the closed-loop system is admissible with an $H_\infty$ performance $\gamma$. Let $\gamma = 0.2$ and solve the inequality (27), then we can obtain that the gain matrix $K_1$ of the $H_\infty$ controller (9) can be designed as

$$
K_1 = \begin{bmatrix}
-0.8346 & -1.3508
\end{bmatrix}
$$

Now we consider the delta operator model of the system (1). Let the sampling period be $h = 0.1$. By using the method in [19] or [22] we can set up the delta operator model (6) of the system (1) with the following coefficient matrices

$$
A = \begin{bmatrix}
-4.9927 & 3.0219 \\
2.0073 & -2.9781
\end{bmatrix}
$$

$$
B = \begin{bmatrix}
1.9416 \\
2.9416
\end{bmatrix}, \quad F = \begin{bmatrix}
0.9635 \\
1.9635
\end{bmatrix}
$$

By Lemma 3 we can obtain that the system (6) is admissible. From Theorem 1 we can derive the minimal scalar $\gamma$ as $\gamma = 2.34$ such that the open-loop system of the system (6) is admissible with an $H_\infty$ performance $\gamma$.

Now we consider to design an $H_\infty$ controller (9) for the system (6). By Theorem 2, we can obtain the minimal scalar $\gamma$ as $\gamma = 0.28$ such that the closed-loop system (10) is admissible with an $H_\infty$ performance $\gamma$. Select $\gamma = 0.3$. Then by solving the inequality (22), we can find that the gain matrix $K_2$ of the $H_\infty$ controller (9) can be designed as

$$
K_2 = \begin{bmatrix}
-0.2157 & -1.8409
\end{bmatrix}
$$

**Example 2** Consider a singular continuous system given by

$$
E\dot{x}(t) = A_s x(t) + B_s u(t)
$$

(29)

with the following coefficient matrices

$$
E = \begin{bmatrix}
1 & 2 \\
1 & 2
\end{bmatrix}, \quad A_s = \begin{bmatrix}
-2 & -5 \\
2 & 3
\end{bmatrix}, \quad B_s = \begin{bmatrix}
1 \\
3
\end{bmatrix}
$$

Consider the unforced system (2) of the system (29). Then from Definition 1 we can obtain that the system (2) is regular but neither impulse-free nor stable. Thus it is not admissible. Solve the system (2) and we can find that its solution is just a fixed point $(0, 0)$.

Select $S = \begin{bmatrix}
1 & -1
\end{bmatrix}^T, J_1 = \begin{bmatrix}
1 & 1
\end{bmatrix}^T$ and $J_2 = 1$. Then by Corollary 3, we can obtain that an admissible controller (9) for the system (29) exists and the gain matrix $K_3$ of the controller (9) can be designed as

$$
K_3 = \begin{bmatrix}
-8.2088 & -14.0618
\end{bmatrix}
$$

Let $x_1(0) = 6.0404$. Then, under the compatible initial conditions we can get $x_2(0) = -3.7273$. The state trajectory of the closed-loop system with the gain matrix $K_3$ is shown in Figure 1.

![Figure 1](image-url)
Let $x_1(0) = -5.2027$. Then, under the compatible initial conditions we can get $x_2(0) = 1.5407$. The state trajectory of the closed-loop system with the gain matrix $K_4$ is shown in Figure 2.

![State trajectory with $K_4$](image)

**Fig. 2.** The state trajectory of the closed-loop system with $K_4$

V. CONCLUSION

In this paper, the problem of state feedback $H_{\infty}$ control have been investigated for singular systems via delta operator method. Based on the results in this paper we can now provide unified expressions of admissibility condition and bounded real lemma for singular discrete systems and singular continuous systems. Thus, the design methods of a desirable $H_{\infty}$ controller or an admissible controller for SDOs are also suitable for singular continuous systems. This paper has presented the results about $H_{\infty}$ control and admissible control for both SDOs and singular continuous systems. It should be pointed out that all the results are in the form of strict LMIs and therefore can be solved easily and directly by the existing softwares.

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