Metrics of Vector Logic Algebra for Cyber Space

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Abstract - The algebraic structure determining the vector-matrix transformation in the discrete vector Boolean space for the analyzing information based on logical operations on associative data.

Keywords - vector-matrix transformation, discrete vector Boolean space, information analysis.

I. INTRODUCTION

The purpose of this article is significant decreasing the analysis time of associative data structures through the developing metrics of vector logic algebra for parallel implementation of vector operations on dedicated multiprocessor device. The problems are: 1. Develop a signature, satisfying a system of axioms, identities and laws for the carrier, which is represented by a set of associative vectors of equal length in the logic vector space. 2. Create a signature of the relations for the carrier, represented by a pair: an associative vector – an associative matrix. 3. Develop a signature of the transformations for the carrier, represented by a pair of associative matrices of equal length.

The research subject is the algebraic structures and logic spaces, focused to creating the mathematical foundations of effective parallel computing processes, implemented in a multiprocessor dedicated product.


II. B-METRIC OF THE VECTOR DIMENSION

Vector discrete logic (Boolean) space determines the interaction of objects through the use of three axioms (identity, symmetry and triangle) forming a nonarithmetic B-metric of vector dimension:

\[ \begin{align*}
& (d(a,b) = a \oplus b = (a_i \oplus b_i), i = 1, n; \\
& d(a,b) = [0 \iff \exists d_i (d_i = 0)] \iff a = b; \\
& B = \{d(a,b) = d(b,a); \\
& d(a,b) \oplus d(b,c) = d(a,c); \\
& \ominus = [d(a,b) \times d(b,c)] \vee [d(a,b) \wedge d(b,c)].
\end{align*} \]

Vertices of the transitive triangle \((a,b,c)\) are vectors (Fig. 1), which identify the objects in the n-dimensional Boolean B-Space; the sides of triangle \(d(a,b), d(b,c), d(a,c)\) are the distances between vertices, which are also represented by vectors of the length \(n\), where each bit is defined in the same alphabet as the coordinates of the vectors-vertices.

Fig. 1. Triangle of the vector transitive closure

Vector transitive triangle is characterized by complete analogy with the numerical measurement of the distance in the metric M-space, which is specified by the system of axioms, determining the interaction between one, two and three points of any space:

\[ \begin{align*}
& (d(a,b) = 0 \iff a = b; \\
& M = \{d(a,b) = d(b,a); \\
& (d(a,b) \oplus d(b,c) \geq d(a,c).)
\end{align*} \]

The specific of metric triangle axiom lies in numerical (scalar) comparison the distances of three objects, where the interval uncertainty of the result – two sides of a triangle can be greater or equal to a third one – not really suitable for determining the exact length of the last side. Removal of this disadvantage is possible only in a logical vector space, which can form a deterministic view for each characteristic of the process or phenomenon state. Then the numerical uncertainty of the third triangle side in a vector logical space takes the form of the exact binary vector, which characterizes the distance between two objects.
and is calculated on the basis of knowledge of the distances for
the other two triangle sides:
\[ d(a, b) \oplus d(b, c) = d(a, c) . \]

The three axioms of the determining a metric are redundant,
at least for the vector space, where a single axiom can be used –
the interaction between three points:
\[ d(a, b) \oplus d(b, c) \oplus d(c, a) = 0 . \] Two identities are followed from
this law, which determine the relations between one and two
points in a space:
\[ d(a, b) \oplus d(b, c) \oplus d(c, a) = 0 \rightarrow d(a, b) = \emptyset \rightarrow c = \emptyset ; \]
\[ d(a, a) = 0 \rightarrow [b, c] = \emptyset . \]

The following fact is interesting. Having regard to the cycli-
cal nature of the triangle, for any two known adjacent (incident)
components the third one can be calculated. This concerns both
to states (codes) of vertices and to the distances between them:
\[ \begin{align*}
(\dfrac{c}{a}, \dfrac{c}{b}, \dfrac{b}{a}) &= (\dfrac{a}{c}, \dfrac{b}{c}, \dfrac{b}{a}) = (\dfrac{a}{c}, \dfrac{b}{c}, \dfrac{b}{a}) \oplus (\dfrac{a}{c}, \dfrac{b}{c}, \dfrac{b}{a}) = \emptyset .
\end{align*} \]

Isomorphism of the set theory concerning the algebra of logic
allows determining the vector set-theoretic S-space, where the
triangle axiom is defined by symmetric difference \( \Delta \), which is
analogous to the operation \( \oplus \) in Boolean algebra:
\[ \begin{align*}
\Delta &= \{a, b, c\} \oplus \{a, b, c\} = \{a, b, c\} \oplus \{a, b, c\} = \emptyset ,
\end{align*} \]

Here \( \Delta \) is the symmetric difference operation on the four-digit
set-theoretic alphabet \( \emptyset \), represented by the following table:
\[
\begin{array}{|c|c|c|}
\hline
\text{\( \Delta \)} & 0 & 1 & x \\
\hline
0 & \emptyset & x & 1 \\
1 & x & \emptyset & 0 \\
x & 1 & 0 & \emptyset \\
\emptyset & 0 & 1 & x \\
\hline
\end{array}
\]

When determining the distance between two vectors in the S-
space the symmetric difference is used, which is isomorphic to
the XOR-operation in the Boolean B-space. Examples of calculat-
ing the distances between vectors in both spaces (S, B) are given below:
\[
\begin{array}{|c|c|c|c|c|c|}
\hline
& a & 0 & 0 & 0 & 0 & 1 \\
\hline
b & x & x & 0 & 0 & 1 & 1 \\
c & x & x & x & 0 & 0 & 0 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|c|}
\hline
& d(a, b) & 0 & 1 & \emptyset & \emptyset & x \\
\hline
d(b, c) & \emptyset & 1 & x & x & \emptyset & x \\
d(a, c) & 0 & 1 & 1 & x & \emptyset & x \\
\hline
\end{array}
\]

\[
B = \begin{pmatrix}
a & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\
b & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\
c & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
\hline
d(a, b) & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\
d(b, c) & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 1 \\
d(a, c) & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\
\end{pmatrix}
\]

Vector equal to zero (empty set) for all coordinates means a
full match the response and query. As well as the vector equal to
1 (symbol x) for all digits indicating complete contradictoriness
the response and query. Number of gradations for a variable can
be a finite number that multiple of a power 2
\( a = 2^n \rightarrow (2^2 = 4, 2^4 = 16) \), which is determined by the power
of the Boolean on the universe of n primitive. Otherwise, the
symmetric difference can exist only in closed concerning the
set-theoretic operations alphabet. Thus, the interaction of two
objects in a vector logical space can have either binary or multi-
valued deterministic scale of measuring interaction. Hasse dia-
gram of any finite number of primitives \( (1, 2, 3, 4, \ldots) \) can be
packed to a variable of logical vector. Moreover, 16 gradations
(for instance) of vector interaction by the four primitives exactly
indicate not only the degree of proximity by the variable, but in
what way they differ – by some primitives, or their combination.
Vector operation XOR actually smooths out the changes in the
two codes or vectors, that is of interest for the creating digital
filters. If it is applied many times, we can get a binary pyramid,
where the last vertex is always the zero vector. Thus, the ob-
tained pyramid makes it possible using some redundancy to cor-
rect errors in the process of information transferring. The proce-
dure of convolution distances in order to verify the errors of data
transferring for the number of vectors equal degree 2 is pre-
sented below. 1) Compute all the distances between the binary
codes, including the last and first vectors, resulting in a closed
geometric figure 0)1in(iaac 1iii
\[ c_i = a_i \oplus a_{i+1} \rightarrow i = n \rightarrow i+1 = 0 . \]

2) Compute all distances between non-overlapping pairs of obtained in
the first stage codes \( c_i = a_{2i-1} \oplus a_{2i} \rightarrow i = 1, 2, 3, \ldots, n \). 3) Repeat pro-
cedure 2 to obtain a package equal to zero in all coordinates.
The procedure is illustrated by the following calculations:
\[
\begin{pmatrix}
0 & 1 & 0 & 1 \\
1 & 1 & 1 & 0 \\
1 & 1 & 1 & 1 \\
\hline
0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 \\
0 & 1 & 0 & 1 \\
0 & 0 & 0 & 0 \\
\hline
0 & 1 & 0 & 1 \\
1 & 0 & 1 & 1 \\
1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 \\
\hline
\end{pmatrix}
\]

Similar actions can be performed and for multivalued vectors,
where, for instance, every coordinate is defined in four-digit set-
thetical alphabet, and the procedure is reduced to the obtain-
ing a vector of empty values coordinates:
transitive triangle, which can be transformed by shifting the

tables. Digital products based on fault detection tables. 3) Searching

ingformation via communication channels. 2) Detecting faults in

problems: 1) Diagnosing and correcting errors when the transmitting

metric when \(1, 2, 3\) and \(i\) respectively.

Three points in a vector logical space is a special case of B-

complex structures of n-dimensional space. The classical metric

forms from three to one and to extend its action on an arbitrarily

points are located in line, Fig. 3. But the axiom of metric transi-

tive closure uses a structure consisting of three points on the

plane with different coordinates, which is strictly called a trian-

gle. Then a figure with sides 1, 2, 3, according to the definition

of the metric, is a triangle with two zero angles and the third

one, equal to 180 degrees, where all the conditions for the three

sides are met: \(a + b \geq c \rightarrow 1 + 2 = 3\).

Here it happen a convolution of a closed space to a single point, Fig. 2, defined in all coordinates by symbols of the empty set, by calculating the distance between vector-objects, and then – the distance between the vector-distances. Otherwise, the modulo sum of all vector-distances, closed in the cycle is equal to an empty vector

\[
\mathbf{m}_i = c_j \oplus c_j \rightarrow \mathbf{m} = \mathbf{m}_i \oplus \mathbf{m}_{i+1}.
\]

But this procedure is characterized by less error diagnosis depth – the detecting an incorrect bit is possible. While a binary tree of space convolution makes it possible to increase the diagnosis depth up to a vector pair.

Space convolution is of interest for many real-world problems: 1) Diagnosing and correcting errors when the transmitting information via communication channels. 2) Detecting faults in digital products based on fault detection tables. 3) Searching faults in digital products based on multivalued fault detection tables.

The essence of the space convolution lies in the metric of transitive triangle, which can be transformed by shifting the

right side of the equation to the left:

\[
d(a, b) \oplus d(b, c) = d(a, c) \rightarrow d(a, b) \oplus d(b, c) \oplus d(a, c) = 0.
\]

This definition assign primary importance not elements of the set, but the relations, thereby reducing the system of metric axioms from three to one and to extend its action on an arbitrarily complex structures of n-dimensional space. The classical metric definition for determining the interaction between one, two and three points in a vector logical space is a special case of B-metric when \(i = 1, 2, 3\) respectively:

\[
\begin{align*}
\text{d}_1 &= 0 \leftrightarrow a = b; \\
\text{d}_2 &= \text{d}_1 \oplus \text{d}_3 = 0 \leftrightarrow d(a, b) = d(b, a); \\
\text{d}_3 &= \text{d}_1 \oplus \text{d}_2 = 0 \leftrightarrow d(a, b) \oplus d(b, c) = d(a, c).
\end{align*}
\]

In particular, metric, functional and other kinds of spaces in the sum also give zero. For example, a figure with sides 1, 2, 3, according to all the textbooks, is not a triangle, because three points are located in line, Fig. 3. But the axiom of metric transitive closure uses a structure consisting of three points on the plane with different coordinates, which is strictly called a triangle.

Information vector logic space as a subset of a metric one determines the interaction between a finite numbers of objects by means of the introduced definitions, axioms of identity, symmetry and transitivity of the triangle. At that the last property degenerates into a strict equality, which makes it possible potentially to reduce by a third volume of binary information about the interaction of objects, due to the convolution of any closed logical space in the zero-vector.

Beta-metric of a vector logic space, presented by a zero-sum of cycle distances of binary codes, creates a fundamental basis for all logical and associative problems of synthesis and analysis related to the searching, recognition and decision-making.

Based on the beta metric and the three quality criteria of interaction between vector logical objects in the same space a beta-criterion is created. It makes it possible to determine effectively, accurately and adequately the quality of object interaction, when searching, pattern recognition and decision-making by calculating the xor-function.

Algebra of vector logic creates an infrastructure mathematical service of a vector logical space for the solving real-world problems of synthesis and analysis. It consists of three components: vector, vector-matrix and matrix algebraic structures. Signature of algebras is given by a standard set of logical vector operations AND, OR, NOT, XOR to determine the interaction between compatible objects of a carrier, which form a binary n-dimensional vectors and compatible by the dimension matrix.

V. REFERENCES


