CONJUGACY GROWTH SERIES AND LANGUAGES IN GROUPS

LAURA CIOBANU AND SUSAN HERMILLER

Abstract. In this paper we introduce the geodesic conjugacy language and geodesic conjugacy growth series for a finitely generated group. We study the effects of various group constructions on rationality of both the geodesic conjugacy growth series and spherical conjugacy growth series, as well as on regularity of the geodesic conjugacy language and spherical conjugacy language. In particular, we show that regularity of the geodesic conjugacy language is preserved by the graph product construction, and rationality of the geodesic conjugacy growth series is preserved by both direct and free products.

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1. Introduction

For a finitely generated group, several growth functions and series associated with elements or conjugacy classes of the group have been studied. In this paper, we study a conjugacy growth series first examined by Rivin ([19], [20]), and introduce a new growth series arising from the conjugacy classes, which we show admits much stronger closure properties. We also study the regularity properties of languages associated to the set of conjugacy classes.

Let $G = \langle X \rangle$ be a group generated by a finite inverse-closed generating set X. For each word $w \in X^*$, let $l(w) = l_X(w)$ denote the length of this word over X. Any language $L \subseteq X^*$ over the finite alphabet X gives rise to two growth functions : $\mathbb{N} \cup \{0\} \to \mathbb{N} \cup \{0\}$, namely the "usual" or *cumulative growth function* β_L defined by $\beta_L(n) := |\{w \in L \mid l(w) \leq n\}|$ and the *strict growth function* $\phi_L(n) := |\{w \in L \mid l(w) = n\}|$. The generating series associated to these functions are the "usual" or *cumulative growth series* $b_L(z) := \sum_{i=0}^{\infty} \beta_L(i) z^i$ and the *strict growth series* $f_L(z) := \sum_{i=0}^{\infty} \phi_L(i) z^i$. These growth functions and series are closely related, in that $\phi_L(n) = \beta_L(n) - \beta_L(n-1)$ for all $n \geq 1$, and $\phi_L(0) = \beta_L(0)$, and so these two series satisfy the identity $f_L(z) = (1-z) b_L(z)$. It is well known (see, for example, [1]) that if the language L is a regular language (i.e., the language of a finite state automaton), then both of the series b_L and f_L are rational functions. In this paper, we will focus on strict growth series associated to two languages derived from the pair (G, X).

Let $\sim_c$ denote the equivalence relation on G given by conjugacy, with set $G/\sim_c$ of equivalence classes, and let $[g]_c$ denote the conjugacy class of $g \in G$. Let $\pi : X^* \to G$ be the natural projection. Fix a total ordering $<$ of the set X, and let $<_{sl}$ be the induced shortlex ordering of X^*. For each conjugacy class $c \in G/\sim_c$, there is a *shortlex conjugacy normal form* z_c for c that is the shortlex least word over X representing an element of G lying in c. That is, $[\pi(z_c)]_c = c$, and for all $w \in X^*$ with $w \neq z_c$ and $[\pi(w)]_c = c$ we have $z_c <_{sl} w$. We call the set

$$\widetilde{\Sigma} = \widetilde{\Sigma}(G, X) := \{z_c \mid c \in G/\sim_c\}$$

the *spherical conjugacy language* for G over X. Note that if L is any other language of length-minimal normal forms for the conjugacy classes of G over the generating set X, then the growth functions (and hence also the corresponding series) for the spherical conjugacy language and L must coincide; that is, for all natural numbers n we have $\phi_{\widetilde{\Sigma}}(n) = \phi_L(n)$. The strict growth series

$$\widetilde{\sigma} = \widetilde{\sigma}(G, X) := f_{\widetilde{\Sigma}(G,X)}$$

is the *spherical conjugacy growth series* for G over X. In Section 2 we study this language and series.

The corresponding cumulative growth function $\beta_{\widetilde{\Sigma}(G,X)}$, known as the "conjugacy growth function", has been studied by several authors; see the surveys by Guba and Sapir [10] and Breuillard and de Cornulier [2] for further information on these functions. For the class of non-elementary word hyperbolic groups, Coornaert and Knieper [6] have shown bounds on the growth of the conjugacy growth function in terms of the exponential cumulative growth rate of the *spherical language*

$$\Sigma = \Sigma(G, X) := \{y_g \mid g \in G\}$$

of shortlex normal forms for the elements of G; here for each $g \in G$ the word $y_g \in X^*$ satisfies $\pi(y_g) = g$ and whenever $w \in X^*$ with $w \neq y_g$ and $\pi(w) = g$ then $y_g <_{sl} w$. Analogous to the case of the conjugacy language above, if L is any other language of geodesic normal forms for the elements of G over the generating set X, then the growth functions for the spherical language and L must coincide; that is, for all natural numbers n we have $\phi_\Sigma(n) = \phi_L(n) =$ the number of elements of G in the sphere of radius n with respect to the word metric on G induced by X. The (usual) growth function of the group G with respect to X, i.e. the cumulative growth function $\beta_{\Sigma(G,X)}$, is very well known and studied; see the texts by de la Harpe [7, Chapter VI] and Mann [17] for surveys of results and open problems for the cumulative and strict growth functions associated to this spherical language.

Chiswell [3, Corollary 1] has shown that rationality of the spherical growth series (i.e. the strict growth series of the spherical language Σ) is preserved by the construction of graph products of groups, and in their proof of [11, Corollary 5.1], Hermiller and Meier show that regularity of the spherical language is preserved by this construction as well. (The graph product construction includes both direct and free products; see Section 3 for the definition.) In Proposition 2.1 we note that regularity of the spherical conjugacy language and rationality of the spherical conjugacy growth series are preserved by direct products. By contrast, in [20, Theorem 14.6], Rivin has shown that although for the infinite cyclic group $\mathbb{Z} = \langle a \rangle$ the spherical conjugacy language is a regular language and $\widetilde{\sigma}(\mathbb{Z}, \{a^{\pm 1}\}) = \frac{1+z}{1-z}$ is rational, the spherical conjugacy growth series $\widetilde{\sigma}(F_2, \{a^{\pm 1}, b^{\pm 1}\})$ for the free group on two generators a, b is not a rational function. Combining this with the result above on rationality of the growth of regular languages, this shows that the spherical conjugacy language $\widetilde{\Sigma}(F_2, \{a^{\pm 1}, b^{\pm 1}\})$ is not regular. In Section 2, we strengthen this result to the broader class of context-free languages.

Proposition 2.2. *Let $F = F(a, b)$ be a free group on generators a, b. Then the spherical conjugacy language $\widetilde{\Sigma}(F, \{a^{\pm 1}, b^{\pm 1}\})$ is not context-free.*

Consequently neither regularity of the spherical conjugacy language nor rationality of the spherical conjugacy growth series are preserved by the free product construction. We also give further examples for which these properties of spherical conjugacy languages are not preserved by free products, in the following.

Theorem 2.4. *For finite nontrivial groups A and B with generating sets $X_A = A \setminus 1_A$ and $X_B = B \setminus 1_B$, the free product group $A * B$ with generating set $X := X_A \cup X_B$ has rational spherical conjugacy growth series $\widetilde{\sigma}(A * B, X)$ if and only if $A = B = \mathbb{Z}/2\mathbb{Z}$. Moreover, given any ordering of X satisfying $a < b$ for all $a \in X_A$ and $b \in X_B$, for the induced shortlex ordering the associated spherical conjugacy language $\widetilde{\Sigma}(A * B, X)$ is regular if and only if $A = B = \mathbb{Z}/2\mathbb{Z}$.*

The property of admitting a rational spherical conjugacy growth series appears to be extremely restrictive; indeed, Rivin [20, Conjecture 13.1] has conjectured that the only word hyperbolic groups that have a rational spherical conjugacy growth series are the virtually cyclic groups. Theorem 2.4 gives further evidence for this conjecture.

In [12], Holt, Rees, and Röver also connect languages and conjugation in groups, but from a different perspective. They consider the *conjugacy problem* set associated a group G with finite generating set X, i.e. the set of ordered pairs (u, v) such that u and v are conjugate in G. They show that various notions of context-freeness of this language can be used to characterize the classes of virtually cyclic and virtually free groups.

It is also natural to consider all of the geodesics instead of a normal form set when studying languages or growth series. For each element g of G, let $|g| (= |g|_X)$ denote the length of a shortest representative word for g over X. A *geodesic*, then, is a word $w \in X^*$ with $l(w) = |\pi(w)|$. The *length up to conjugacy* of g, denoted by $|g|_c$, is defined to be

$$|g|_c := \min\{|h| \mid h \in [g]_c\} ,$$

and the length of the conjugacy class is denoted by $|[g]_c| := |g|_c$. An element $w \in X^*$ satisfying $l(w) = |[\pi(w)]_c|$ will be called a *geodesic* of the pair (G, X) *with respect to conjugacy*, or a *conjugacy geodesic* word. We define a new series for the pair (G, X) built from these words. The set

$$\widetilde{\Gamma} = \widetilde{\Gamma}(G, X) := \{w \in X^* \mid l(w) = |\pi(w)|_c\}$$

of conjugacy geodesics is the *geodesic conjugacy language*, and the strict growth series

$$\widetilde{\gamma} = \widetilde{\gamma}(G, X) := f_{\widetilde{\Gamma}(G,X)}$$

is the *geodesic conjugacy growth series*, for G over X. In Section 3 we study this language and series.

This geodesic conjugacy language is the canonical analog in the case of conjugacy classes to the set

$$\Gamma = \Gamma(G, X) := \{w \in X^* \mid l(w) = |\pi(w)|\}$$

of geodesic words, which we will call the *geodesic language* of the group G over X. The corresponding strict growth series will be called the *geodesic growth series*, and denoted

$$\gamma = \gamma(G, X) := f_{\Gamma(G,X)}.$$

See the paper by Grigorchuk and Nagnibeda [9, Section 6] for a survey of results on this series. For word hyperbolic groups, Cannon has shown that the geodesic language for every finite generating set is regular [8, Chapter 3]; that is, the group has finitely many "cone types".

Loeffler, Meier, and Worthington [15] have shown that regularity of Γ is preserved by the graph product construction and rationality of γ is preserved by direct and free products. In Section 3 in Propositions 3.3 and 3.5 we give several characterizations of the geodesic words and conjugacy geodesic words for a graph product group in terms of the geodesic languages and geodesic conjugacy languages for the vertex (factor) groups. We use these to show that, unlike the spherical conjugacy case above, regularity of the pair of languages $\widetilde{\Gamma}$, Γ is preserved when taking graph products.

Theorem 3.1. *If G is a graph product of groups G_i with $1 \leq i \leq n$, and each G_i has a finite inverse-closed generating set X_i such that both $\Gamma(G_i, X_i)$ and $\widetilde{\Gamma}(G_i, X_i)$ are regular, then both the geodesic language $\Gamma(G, \cup_{i=1}^n X_i)$ and the geodesic conjugacy language $\widetilde{\Gamma}(G, \cup_{i=1}^n X_i)$ are also regular.*

A corollary of Theorem 3.1 is that for every right-angled Artin group, right-angled Coxeter group, and graph product of finite groups, with respect to the "standard" generating set (that is, a union of the generating sets of the vertex groups), the geodesic conjugacy language is regular and hence the geodesic conjugacy growth series is rational.

Rationality of geodesic conjugacy growth series is also preserved by both free and direct products.

Theorem 3.8. *Let G and H be groups with finite inverse-closed generating sets A and B, respectively. Let $\widetilde{\gamma}_G(z) := \widetilde{\gamma}(G,A)(z) = \sum_{i=0}^{\infty} r_i z^i$ and $\widetilde{\gamma}_H(z) := \widetilde{\gamma}(H,B)(z) = \sum_{i=0}^{\infty} s_i z^i$ be the geodesic conjugacy growth series, and let $\gamma_G := \gamma(G,A)$ and $\gamma_H := \gamma(H,B)$ be the geodesic growth series for these pairs.*

(i) The geodesic conjugacy growth series $\widetilde{\gamma}_\times$ of the direct product $G \times H = \langle G, H \mid [G,H]\rangle$ of groups G and H with respect to the generating set $A \cup B$ is given by $\widetilde{\gamma}_\times = \sum_{i=0}^{\infty} \delta_i z^i$ where $\delta_i := \sum_{j=0}^{i} \binom{i}{j} r_j s_{i-j}$.

(ii) The geodesic conjugacy growth series $\widetilde{\gamma}_*$ of the free product $G * H$ of the groups G and H with respect to the generating set $A \cup B$ is given by

$$\widetilde{\gamma}_* - 1 = (\widetilde{\gamma}_G - 1) + (\widetilde{\gamma}_H - 1) - z\frac{d}{dz} \ln\left[1 - (\gamma_G - 1)(\gamma_H - 1)\right].$$

In our last result of Section 3, we show in Proposition 3.11 that geodesic conjugacy languages and series also behave nicely for a free product with amalgamation of finite groups

Proposition 3.11. *If G and H are finite groups with a common subgroup K, then the free product $G *_K H$ of G and H amalgamated over K, with respect to the generating set $X := G \cup H \cup K - \{1\}$, has regular geodesic conjugacy language $\widetilde{\Gamma}(G *_K H, X)$ and rational geodesic conjugacy growth series $\widetilde{\gamma}(G *_K H, X)$.*

In Section 4 we conclude with a few open questions.

2. Spherical conjugacy series and languages

In this section we collect information about the closure properties of the class of pairs (G,X) for which the spherical conjugacy language is regular, or for which the spherical conjugacy growth series is rational.

Proposition 2.1. *Let G and H be groups with finite inverse-closed generating sets X and Y, respectively.*

- **(i):** *Let $<$ be a total ordering on $X \cup Y$ satisfying $x < y$ for all $x \in X$ and $y \in Y$; we take all shortlex orderings to be defined from $<$ or its restriction to X or Y. If $\widetilde{\Sigma}(G,X)$ and $\widetilde{\Sigma}(H,Y)$ are regular, then $\widetilde{\Sigma}(G \times H, X \cup Y)$ is regular.*
- **(ii):** *If $\widetilde{\sigma}(G,X)$ and $\widetilde{\sigma}(H,Y)$ are rational, then $\widetilde{\sigma}(G \times H, X \cup Y)$ is rational.*

Proof. The spherical conjugacy language for the direct product group $G \times H$, viewed as the group generated by G and H (and hence by $X \cup Y$) with relations $[g,h] = 1$ for all $g \in G$ and $h \in H$, is given by $\widetilde{\Sigma}(G \times H, X \cup Y) = \widetilde{\Sigma}(G,X)\widetilde{\Sigma}(H,Y)$. Thus (i) follows from the fact that regular languages are closed under concatenation. For part (ii), the spherical conjugacy growth series are related by the formula $\widetilde{\sigma}(G \times H, X \cup Y) = \widetilde{\sigma}(G,X)\widetilde{\sigma}(H,Y)$. □

Rivin [20, Theorem 14.6] has shown that the spherical conjugacy growth series of the free group on k generators, with respect to a free basis, is not a rational function. Combining this with the fact that growth series of regular languages are rational, then the spherical conjugacy language for any free group, again with respect to a free basis, cannot be regular. In Proposition 2.2 we give a proof that this language is not even context-free for the free group on two generators, which immediately extends to the case of a group with a free factor in Corollary 2.3. (See [13] for an exposition of the theory of context-free languages.)

Proposition 2.2. *Let $F = F(a,b)$ be a free group on generators a,b. Then the spherical conjugacy language $\widetilde{\Sigma}(F,\{a^{\pm 1},b^{\pm 1}\})$ is not context-free.*

Proof. Suppose that $\widetilde{\Sigma} = \widetilde{\Sigma}(F,\{a^{\pm 1},b^{\pm 1}\})$ is context-free and consider the intersection $I = \widetilde{\Sigma} \cap L$, where $L = a^+b^+a^+b^+$. Since L is a regular language, and the intersection of a context-free language with a regular language is context-free (see [13, p. 135, Theorem 6.5]), I is context-free. Suppose that $a < b$. Then all words in I have the form

$$a^p b^l a^q b^j \quad \text{with} \quad p \geq q, \tag{1}$$

and I can be written as the union of the two disjoint sets

$$I_1 = \{a^p b^l a^p b^j \mid p,l,j > 0, l \leq j\}$$
$$I_2 = \{a^p b^l a^q b^j \mid p,l,j > 0, p > q > 0\}$$

Now let k be the constant given by the pumping lemma (see [13, Lemma 6.1, p. 125] for a statement and details) for context-free languages applied to the set I, and consider the word $W = a^n b^n a^n b^n$, where $n > k$. One can see W as composed of four blocks, the first block being a^n, the second being b^n etc. Then by the pumping lemma W can be written as $W = uvwxy$, where $l(vx) \geq 1$, $l(vwx) \leq k$, and $uv^iwx^iy \in I$ for all $i \geq 0$. Since $l(vwx) \leq k < n$, vwx cannot be part of more than two consecutive blocks.

In a first case, suppose that vwx is just part of one block, i.e. vwx is a power of a, or a power of b. If vwx is in the first block, for $i = 0$ one obtains a word that does not satisfy (1). If it is in the second block then for $i > 2$ one gets a word uv^iwx^iy of the form $a^p b^l a^p b^j$, but $l > j$, so this word doesn't belong to either I_1 or I_2. If vwx is in the third block, for $i > 2$ uv^iwx^iy does not have the form (1). If vwx is in the fourth block, for $i = 0$ uwy has the form $a^p b^l a^p b^j$, $j < l$, and so uwy does not belong to I.

In a second case, suppose vwx contains both a's and b's. If one of v or x contains both a and b, then for $i > 2$ the word uv^iwx^iy contains many blocks alternating between powers of a and b, and so does not lie in $a^+b^+a^+b^+$. So v has to be a power of one letter only, and x a power of the other letter. If v is in first block and x in the second block, take $i = 0$, and one gets a contradiction to (1). If v is in the second block and x in the third, then for $i > 2$, uv^iwx^iy has the form $a^p b^l a^q b^j$ with $p < q$, which gives a contradiction to (1) . Finally, if v is in the third and x in fourth block, for $i > 2$ the word uv^iwx^iy does not satisfy (1).

Hence none of these cases hold, giving the required contradiction. □

In fact, whenever a group G with a finite inverse-closed generating set contains a free subgroup F such that there is a free basis $\{a_1, ..., a_n\}$ of F lying in X and such that all elements of $\widetilde{\Sigma}(F, \{a_i^{\pm 1}\})$ are also shortlex conjugacy normal forms for the pair (G, X), the proof above can be applied. As a result, we obtain the following.

Corollary 2.3. *Let F be a free group with free basis Z and let H be any group with finite inverse-closed generating set Y.*

- **(i):** *For the direct product group $F \times H$ the spherical conjugacy language $\widetilde{\Sigma}(F \times H, Z \cup Z^{-1} \cup Y)$ is not a context-free language.*
- **(ii):** *For the free product group $F * H$ the spherical conjugacy language $\widetilde{\Sigma}(F * H, Z \cup Z^{-1} \cup Y)$ is not a context-free language.*

Since the spherical conjugacy growth series for the free product of two infinite cyclic groups is not rational, it is natural to consider next the free product of two finite groups.

Theorem 2.4. *For finite nontrivial groups A and B with generating sets $X_A = A \setminus 1_A$ and $X_B = B \setminus 1_B$, the free product group $A * B$ with generating set $X := X_A \cup X_B$ has rational spherical conjugacy growth series $\widetilde{\sigma}(A * B, X)$ if and only if $A = B = \mathbb{Z}/2\mathbb{Z}$. Moreover, given any ordering of X satisfying $a < b$ for all $a \in X_A$ and $b \in X_B$, for the induced shortlex ordering the associated spherical conjugacy language $\widetilde{\Sigma}(A * B, X)$ is regular if and only if $A = B = \mathbb{Z}/2\mathbb{Z}$.*

Proof. Let $G := A * B$ and let us assume that all letters in X_A come before the letters in X_B in a fixed ordering of X. Notice that the set $\Gamma(G, X)$ of geodesics in G are simply the alternating words in X_A and X_B. To simplify notation, write $\widetilde{\Sigma}_G := \widetilde{\Sigma}(G, X)$, $\widetilde{\Sigma}_A := \widetilde{\Sigma}(A, X_A)$, and $\widetilde{\Sigma}_B := \widetilde{\Sigma}(B, X_B)$. Then $\widetilde{\Sigma}_A \subseteq X_A \cup \{\lambda\}$ and $\widetilde{\Sigma}_B \subseteq X_B \cup \{\lambda\}$, where λ denotes the empty word. Since any word alternating over X_A and X_B can be cyclically conjugated to a word of the same length starting with a letter in X_A, we have

$$\widetilde{\Sigma}_G = \widetilde{\Sigma}_A \cup \widetilde{\Sigma}_B \cup \widetilde{\Sigma}_{AB},$$

where $\widetilde{\Sigma}_{AB}$ is defined to be the set of words in $\widetilde{\Sigma}_G$ that alternate between X_A and X_B, starting with a letter from X_A and ending with a letter from X_B.

Suppose first that $A = B = \mathbb{Z}/2\mathbb{Z}$. Then $X_A = \{a\}$ and $X_B = \{b\}$ are singleton sets, and $\widetilde{\Sigma}_G = \{\lambda, a, b\} \cup \{(ab)^r \mid r \in \mathbb{N}\}$. Hence $\widetilde{\Sigma}_G$ is regular, and $\widetilde{\sigma}_G$ is rational.

For the remainder of this proof we assume that at least one of the groups A or B has order at least 3. In order to analyze the spherical conjugacy growth series of G, we follow the ideas in Theorems 14.2, 14.4 and 14.6 in [20]. We refer the reader to [5] for details of complex analytic techniques used here.

Notice that all words in $\widetilde{\Sigma}_{AB}$ have even length; we denote those words in $\widetilde{\Sigma}_{AB}$ of length $2r$ by $\widetilde{\Sigma}_{AB,2r}$. Moreover, if we denote the subset of $\Gamma(G,X)$ of alternating words of length $2r$ beginning with a letter in X_A and ending with a letter in X_B by $\Gamma_{AB,2r}$, and similarly for $\Gamma_{BA,2r}$, then $|\Gamma_{AB,2r} \cup \Gamma_{BA,2r}| = 2|X_A|^r|X_B|^r$ and the set $\widetilde{\Sigma}_{AB,2r}$ is in bijective correspondence with the cyclic conjugacy classes of $\Gamma_{AB,2r} \cup \Gamma_{BA,2r}$. That is, if we let $f(2r) = |\widetilde{\Sigma}_{AB,2r}|$, then $f(2r)$ is the number of orbits of the group $\mathbb{Z}/2r\mathbb{Z}$ acting by cyclic conjugation on the set $\Gamma_{AB,2r} \cup \Gamma_{BA,2r}$. We can compute $f(2r)$ by using Burnside's Lemma, which gives

$$\begin{aligned} f(2r) &= \frac{1}{2r} \sum_{g \in \mathbb{Z}/2r\mathbb{Z}} |Fix(g)| \\ &= \frac{1}{2r} \sum_{2|d|2r} \sum_{g \in \mathbb{Z}/2r\mathbb{Z}, \gcd(g,2r)=d} |Fix(g)| \end{aligned}$$

where the second equality uses the fact that an odd element of $\mathbb{Z}/2r\mathbb{Z}$ cannot permute an even length alternating word to itself. Now for any $g \in \mathbb{Z}/2r\mathbb{Z}$ with $2|d = gcd(g,2r)$ and any alternating word $w \in Fix(g)$, we have $w = v^{2r/d}$ for an alternating word $v \in \Gamma_{AB,d} \cup \Gamma_{BA,d}$. There are $2|X_A|^{d/2}|X_B|^{d/2}$ such words. Also using the fact that $\phi(\frac{2r}{d}) = |\{g \leq 2r \mid \gcd(g,2r) = d\}|$, where ϕ is the Euler totient function, yields

$$\begin{aligned} f(2r) &= \frac{1}{r} \sum_{2|d|2r} \phi(\frac{2r}{d})|X_A|^{d/2}|X_B|^{d/2} \\ &= \frac{1}{r} \sum_{e|r} \phi(\frac{r}{e})|X_A|^e|X_B|^e \\ &= \frac{1}{r} \sum_{e|r} \phi(e)|X_A|^{r/e}|X_B|^{r/e} \ . \end{aligned}$$

Let $\widetilde{\sigma}_G := \widetilde{\sigma}(G,X)$. Now we have

$$\begin{aligned} \widetilde{\sigma}_G(z) &= 1 + |\widetilde{\Sigma}_A \cup \widetilde{\Sigma}_B \setminus \{\lambda\}|z + \sum_{r=1}^{\infty} |\widetilde{\Sigma}_{AB,2r}|z^{2r} \\ &= 1 + |\widetilde{\Sigma}_A \cup \widetilde{\Sigma}_B \setminus \{\lambda\}|z + \sum_{r=1}^{\infty} \frac{1}{r} \sum_{e|r} \phi(e)(|X_A||X_B|)^{r/e} z^{2r} \ . \end{aligned}$$

To simplify notation, let $\alpha := |X_A||X_B|$ and $\beta := |\widetilde{\Sigma}_A \cup \widetilde{\Sigma}_B \setminus \{\lambda\}|$. Formally taking the derivative of this power series and multiplying by z gives

$$z\widetilde{\sigma}'_G(z) = \beta z + 2 \sum_{r=1}^{\infty} \sum_{e|r} \phi(e)\alpha^{r/e}(z^2)^r \ .$$

Rearrange the terms of this formal power series (e.g. as in [20, Theorem 14.4] with $c_n = \phi(n)$ and $b_n = \alpha^n$) to obtain

$$z\widetilde{\sigma}'_G(z) = \beta z + 2\sum_{d=1}^{\infty}\phi(d)\sum_{n=1}^{\infty}\alpha^n(z^{2d})^n .$$

Now $\alpha = |X_A||X_B|$ and at least one of the sets X_A, X_B contains more than one element. Hence $\alpha > 1$. For any real number $r > 0$ and any point $p \in \mathbb{C}$, let $D_r(p)$ denote the open disk of the complex plane defined by $D_r(p) := \{z \mid |p - z| < r\}$. Since $\sum_{n=1}^{\infty}\alpha^n(z^{2d})^n = \frac{\alpha z^{2d}}{1-\alpha z^{2d}}$ for all z in the disk $D_{1/\alpha}(0)$, we have $z\widetilde{\sigma}'_G(z) = h(z)$ in this disk, where

$$h(z) := \beta z + 2\sum_{d=1}^{\infty}\phi(d)\frac{\alpha z^{2d}}{1-\alpha z^{2d}} .$$

For any $0 < \epsilon < 1$ there are only finitely many even roots of $\frac{1}{\alpha}$ in the closed disk $\overline{D_{1-\epsilon}(0)}$; denote these roots by $z_1, ..., z_k$. (In particular, all $2d$-th roots of $\frac{1}{\alpha}$ in this closed disk must satisfy $d \leq -\ln(\alpha)/2\ln(1-\epsilon)$.) Let

$$\epsilon' := \frac{\epsilon}{3}\min\left[\{1 - \epsilon - |z_i| \mid 1 \leq i \leq k\} \cup \{d(z_i, z_j) \mid i \neq j\}\right],$$

and let R be the region of the complex plane defined by $R := \overline{D_{1-\epsilon}(0)} \setminus \cup_{i=1}^{k} D_{\epsilon'}(z_i)$. Then for all $z \in R$ the value of $|1 - \alpha z^{2d}|$ must be strictly greater than 0. Since R is compact, then there is a $\delta > 0$ such that for all z in the region R, we have $|1-\alpha z^{2d}| \geq \delta$. Now $|\phi(d)\frac{\alpha z^{2d}}{1-\alpha z^{2d}}| \leq d\frac{\alpha}{\delta}(1-\epsilon)^{2d}$. Since the series $\sum_{d=1}^{\infty} d\frac{\alpha}{\delta}(1-\epsilon)^{2d}$ converges to a finite number, the Weierstrass M-test [5, II.6.2] says that the series $h(z)$ is uniformly convergent on this region R. Since each partial sum in the expression for h is continuous on R, then h is also continuous on this region [5, II.6.1], and since each partial sum is analytic, the function h is also analytic on R [5, VII.2.1]. Allowing ϵ to shrink to 0, this shows that h is analytic on the disk $D_1(0)$ outside of the infinite set of points that are $2d$-th roots of $\frac{1}{\alpha}$. Moreover, whenever y is a $2d_0$-th root of $\frac{1}{\alpha}$, a similar argument shows that the sum $\sum_{d=1, d\neq d_0}^{\infty}\phi(d)\frac{\alpha y^{2d}}{1-\alpha y^{2d}}$ is analytic, and so the function h has a pole at the point y. That is, $h(z)$ is analytic on the unit disk except for infinitely many poles.

Now assume that the power series $z\widetilde{\sigma}'_G(z)$ is a rational function $p(z)$. Then p must be analytic in the unit disk outside of finitely many poles. Let R' be the unit disk with all of the poles of both h and p removed. Then we have p and h are analytic on R', and $p = h$ in an open disk $D_{1/\alpha}(0)$; thus $p = h$ on R' [5, IV.3.8]. This shows that infinitely many of the poles of h must be removable singularities, which is a contradiction.

Thus $z\widetilde{\sigma}'_G(z)$ cannot be a rational function. Therefore the function $\widetilde{\sigma}'_G$ also is not rational. Since the derivative of any rational function is also rational, this shows that $\widetilde{\sigma}_G$ also is not a rational function. Since the growth series of the set $\widetilde{\Sigma}_G$ is not rational in this case, we must also have that this set is not a regular language. □

3. Geodesic conjugacy series and languages

Given a finite simplicial graph with a group attached to each vertex, the associated *graph product* is the group generated by the vertex groups with the added relations that elements of distinct adjacent vertex groups commute. This construction often preserves geometric, algebraic or algorithmic properties of groups. In particular Loeffler, Meier, and Worthington [15] have shown that regularity of the full language Γ of geodesics (with respect to the union of the generating sets for the vertex groups) is preserved by the graph product construction. Theorem 3.1 gives a further illustration of this behavior, showing that regularity of the pair of languages Γ and $\widetilde{\Gamma}$ is preserved by graph products.

Theorem 3.1. *If G is a graph product of groups G_i with $1 \leq i \leq n$, and each G_i has a finite inverse-closed generating set X_i such that both $\Gamma(G_i, X_i)$ and $\widetilde{\Gamma}(G_i, X_i)$ are regular, then both the geodesic language $\Gamma(G, \cup_{i=1}^n X_i)$ and the geodesic conjugacy language $\widetilde{\Gamma}(G, \cup_{i=1}^n X_i)$ are also regular.*

Before proceeding with the proof of Theorem 3.1, we need some preliminary notation and results. Let Λ be a finite simplicial graph with n vertices $v_1, ..., v_n$ and suppose that for each $1 \leq i \leq n$ the vertex v_i is labeled by a group G_i that has a finite inverse-closed generating set X_i with geodesic language $\Gamma_i := \Gamma(G_i, X_i)$ and geodesic conjugacy language $\widetilde{\Gamma}_i := \widetilde{\Gamma}(G_i, X_i)$. (Note that two vertices of Λ are considered to be adjacent here if the vertices are distinct and joined by an edge.) For two words $u, w \in X_i^*$, we write $u =_{G_i} w$ if u and w represent the same element of G_i, and $u \sim_i w$ if u and w represent conjugate elements of G_i. Let G be the associated graph product with generating set $X := \cup_{i=1}^n X_i$. For words $y, z \in X^*$, write $y =_G z$ if y and z represent the same element of G, and $y \sim_G z$ if y and z represent conjugate elements of G. Also let Γ and $\widetilde{\Gamma}$ denote the geodesic language and geodesic conjugacy language, respectively, for G over X.

For each i, we define the *centralizing set* C_i to be the union of the sets X_j such that the vertices v_i and v_j are adjacent in the graph Λ. Given a word $w = a_1 \cdots a_m$ with each $a_i \in X$, the *centralizing set* $C(w)$ associated to w is defined by $C(w) := \cap_{i=1}^m C_{j_i}$, where for each $1 \leq i \leq m$, the letter a_i lies in the set X_{j_i}. That is, $C(w)$ is the subset of X that commutes with every letter of w from the graph product construction.

We define several types of rewriting operations on words over X as follows.

- **(x0):** *Local reduction*: $yuz \to ywz$ with $y, z \in X^*$, $u, w \in X_i^*$, $u =_{G_i} w$, and $l(u) > l(w)$.
- **(x1):** *Local exchange*: $yuz \to ywz$ with $y, z \in X^*$, $u, w \in X_i^*$, $u =_{G_i} w$, and $l(u) = l(w)$.
- **(x2):** *Shuffle*: $yuwz \to ywuz$ with $y, z \in X^*$, $u \in X_i^*$, $w \in X_j^*$, and v_i, v_j adjacent in Λ.
- **(x3):** *Cyclic conjugation*: $yz \to zy$ with $y, z \in X^*$.
- **(x4):** *Conjugacy exchange*: $uy \to wy$ with $y \in X^*$, $u, w \in X_i^*$, $X_i \subseteq C(y)$, $u \sim_i w$, and $l(u) = l(w)$.

(x5): *Conjugacy reduction*: $uy \to wy$ with $y \in X^*$, $u, w \in X_i^*$, $X_i \subseteq C(y)$, $u \sim_i w$, and $l(u) > l(w)$.

For $0 \leq j \leq 5$, we write $y \overset{xj}{\to} z$ if y is rewritten to z with a single application of operation (xj). Given a subset α of $0-5$, we write $y \overset{x\alpha *}{\to} z$ if z can obtained from y by a finite (possibly zero) number of rewritings of the types (xj) with j in the set α. Note that rewriting operations (x1)-(x4) preserve word length, and (x0), (x5) decrease word length.

A word $y \in X^*$ is *trimmed* if whenever $y \overset{x0-2*}{\to} z$, no operation of type (x0) can ever occur. The word y is *conjugationally trimmed* if whenever $y \overset{x0-5*}{\to} z$, no operation of type (x0) or (x5) can occur.

For each i, we define a monoid homomorphism $\pi_i : X^* \to (X_i \cup \{\$\})^*$, where \$ denotes a letter not in X, by defining

$$\pi_i(a) := \begin{cases} a & \text{if } a \in X_i \\ \$ & \text{if } a \in X \setminus (X_i \cup C_i) \\ 1 & \text{if } a \in C_i \end{cases}$$

Given any subset t of X, we define the *support* $supp(t)$ of t to be the set of all vertices v_i of Λ such that t contains an element of X_i. Let T be the set of all nonempty subsets t of X satisfying the properties that $supp(t)$ is a clique of Λ, and for each $v_i \in supp(t)$, the intersection $t \cap X_i$ is a single element of X_i. Note that for each $t = \{a_1, ..., a_k\} \in T$, whenever $a_{i_1}, ..., a_{i_k}$ is another arrangement of the letters in t, then $a_1 \cdots a_k =_G a_{i_1} \cdots a_{i_k}$. Hence t denotes a well-defined element of G. Also note that for each $a \in X$, we have $\{a\} \in T$, and a is the element of G associated to $\{a\}$. By slight abuse of notation, we will consider $X \subseteq T \subseteq G$. Now T is another inverse-closed generating set for G.

Example 3.2. For the graph product of three infinite cyclic groups $G_i = \langle a_i \rangle$ $(1 \leq i \leq 3)$, if the only adjacent pair of vertices is v_2, v_3, then the graph product group $G = G\Lambda = G_1 * (G_2 \times G_3)$ has generating sets
$X = \{a_1, a_1^{-1}, a_2, a_2^{-1}, a_3, a_3^{-1}\}$ and
$T = \{ \ \{a_1\}, \{a_1^{-1}\}, \{a_2\}, \{a_2^{-1}\}, \{a_3\}, \{a_3^{-1}\}, \{a_2a_3\}, \{a_2^{-1}a_3\}, \{a_2a_3^{-1}\}, \{a_2^{-1}a_3^{-1}\} \ \}$.

Analogous to the operations above on words over X, we define three sets of rewriting rules on words over T as follows. For each index i fix a total ordering on X_i, and let $<_i$ denote the corresponding shortlex ordering on X_i^*. To ease notation, we let the empty set $\{\}$ denote the empty word over T. Whenever $t \in T \cup \{\emptyset\}$, $a \in X$, $t \cup \{a\} \in T$, and $t \cap \{a\} = \emptyset$, we let $\{a, t\}$ denote the set $t \cup \{a\}$.

(R0): $\{a_1, t_1\} \cdots \{a_m, t_m\} \to \{b_1, t_1\} \cdots \{b_k, t_k\} t_{k+1} \cdots t_m$ whenever there is an index $1 \leq i \leq n$ such that for each $1 \leq j \leq m$, $a_j, b_j \in X_i$, $t_j \in T \cup \{\emptyset\}$, and $\{a_j, t_j\} \in T$; $k < m$; and $a_1 \cdots a_m =_{G_i} b_1 \cdots b_k$.

(R1): $\{a_1,t_1\}\cdots\{a_m,t_m\} \to \{b_1,t_1\}\cdots\{b_m,t_m\}$ whenever there is an index $1 \le i \le n$ such that for each $1 \le j \le m$, $a_j,b_j \in X_i$, $t_j \in T \cup \{\emptyset\}$, $\{a_j,t_j\} \in T$; $a_1\cdots a_m =_{G_i} b_1\cdots b_m$; and $a_1\cdots a_m >_i b_1\cdots b_m$.

(R2): $t\{a,t'\} \to \{a,t\}t'$ whenever $a \in X$, $t \in T$, $t' \in T\cup\{\emptyset\}$, and $\{a,t\},\{a,t'\} \in T$.

Also in analogy with the operations above on X^*, whenever $0 \le j \le 2$, $\alpha \subseteq \{0,1,2\}$ and $w,x \in T^*$, we write $w \overset{Rj}{\to} x$ if w rewrites to x via exactly one application of rule (Rj), and $w \overset{R\alpha*}{\to} x$ if x can be obtained from w by a finite (possibly zero) number of rewritings using rules of the type (Rj) for $j \in \alpha$.

Let R denote the set of all rewriting rules of the form (R0), (R1), and (R2). Note that the generating set T of G together with the relations given by the rules of R form a monoid presentation of G; hence, (T,R) is a *rewriting system* for the graph product group G. We refer the reader to Sims' text [21] for definitions and details on rewriting systems for groups which we use in this section.

Define a partial ordering on T by $\{a,t\} > \{b,t\}$ whenever $a,b \in X_i$ for some i, $t \in T\cup\emptyset$, $\{a,t\} \in T$, and $a >_i b$; and by $\{a,t\} < t$ whenever $a \in X$ and $t,\{a,t\} \in T$. For each t in T, define the *weight* $wt(t)$ of t to be the number of elements of t as a subset of X (equivalently, $wt(t)$ is the number of vertices in $supp(t)$). Then all of the rules in the rewriting system R decrease the associated weightlex ordering on T^*. Since the weightlex ordering is compatible with concatenation and well-founded, no word $w \in T^*$ can be rewritten infinitely many times; that is, the system R is *terminating*. One can check via the Knuth-Bendix algorithm [21, Chapter 2] that this system is also *confluent*, i.e., that whenever a word w rewrites two words $w \to w'$ and $w \to w''$ using these rules, then there is a word w''' such that $w' \overset{R0-2*}{\to} w'''$ and $w' \overset{R0-2*}{\to} w'''$. (This check is provided by the present authors in [4, Appendix]; an alternative proof of this can be found in [11, Theorem C]. See Example 3.6 below for details of this rewriting system for an example of a right-angled Coxeter group.)

Hence for each word $w \in X^*$ there is a unique word $irr(w)$ in T^* that is irreducible (i.e. cannot be rewritten) such that $w \overset{R0-2*}{\to} irr(w)$, and each element $g \in G$ is represented by a unique word $irr(g)$ in T^* that is irreducible with respect to the rewriting rules in R [21, Prop. 2.4, p. 54]. That is, the set

$$irr(R) := \{irr(g) \mid g \in G\}$$

is a set of weightlex normal forms for G over T.

For each $t \in T$, let $a_1,...,a_k$ be a choice of an ordered listing of the elements of the set t, and let $h(t) := a_1\cdots a_k$. Then h determines a monoid homomorphism $h : T^* \to X^*$. For each $w \in X^*$, let $\Theta(w) := h(irr(w))$. Then the set

$$h(irr(R)) = \{\Theta(w) \mid w \in X^*\}$$

is a set of normal forms for G over the original alphabet X.

In Proposition 3.3 below, we show that the geodesic words for the graph product group G over X are exactly the trimmed words, and can be characterized as an intersection of homomorphic inverse images via these π_i maps. Although the equivalence of (i), (ii), and (iii) of this Proposition follows from results in [11] and [15], we include some details here for a condensed exposition and because the further equivalence with (iv) will be needed for our proof of Proposition 3.5 below.

Proposition 3.3. *Let G be a graph product of the groups G_i for $1 \le i \le n$, Let $X = \cup_{i=1}^n X_i$ where X_i is a finite inverse-closed generating set for G_i and $\Gamma_i := \Gamma(G_i, X_i)$ for each i, and let $y \in X^*$. The following are equivalent.*

- **(i):** *y is a geodesic word for G with respect to X.*
- **(ii):** *y is a trimmed word over X.*
- **(iii):** *$y \in \cap_{i=1}^n \pi_i^{-1}(\Gamma_i(\$\Gamma_i)^*)$.*
- **(iv):** *$y \overset{x1-2*}{\to} \Theta(y)$.*

Proof. (i) $\Rightarrow$ (ii): If y is not trimmed, then y can be rewritten using length-preserving operations $y \overset{x1-2*}{\to} z$ to a word z that can be further rewritten using a length-reducing operation of type (x0). Hence y cannot be a geodesic.

(ii) $\Rightarrow$ (iii): Suppose that $\pi_i(y) \in (X_i \cup \{\$\})^* \setminus (\Gamma_i(\$\Gamma_i)^*)$ for some i. Then $\pi_i(y) = y_0\$y_1 \cdots \$y_k$ for some $k \ge 0$ and each $y_j \in X_i^*$, where for some j we have $y_j \notin \Gamma_i$. Then $y \overset{x2*}{\to} y'y_jy''$, and a local reduction (operation of type (x0)) may be applied to the latter. Hence y is not trimmed.

(iii) $\Rightarrow$ (iv): Suppose that $y \in \cap_{i=1}^n \pi_i^{-1}(\Gamma_i(\$\Gamma_i)^*)$.

Each rule (Rj) of the rewriting system R gives rise to a commutative diagram via the map h with a sequence of operations on words over X. In particular, for any words $w, x \in T^*$ with $w \overset{R0-2}{\to} x$, we have

$$\begin{array}{ccc} w & \xrightarrow{R0} & x \\ h\downarrow & & \downarrow h \\ h(w) & \overset{x2*}{\to}\overset{x0}{\to}\overset{x2*}{\to} & h(x) \end{array} \qquad \begin{array}{ccc} w & \xrightarrow{R1} & x \\ h\downarrow & & \downarrow h \\ h(w) & \overset{x2*}{\to}\overset{x1}{\to}\overset{x2*}{\to} & h(x) \end{array} \qquad \begin{array}{ccc} w & \xrightarrow{R2} & x \\ h\downarrow & & \downarrow h \\ h(w) & \xrightarrow{x2*} & h(x) \end{array}$$

For $1 \le i \le n$ and $0 \le j \le 1$ we can refine the operation (xj) by defining the rewriting operation (xji) to denote an operation of type (xj) in which a subword over X_i is rewritten, and similarly we let rewriting rule (Rji) denote a rule of type (Rj) in which effectively an X_i^* subword is rewritten.

Using these refined operations and the maps π_i, we can extend the above diagrams to words over $X_i \cup \{\$\}$ as follows. We say that a rewriting operation $yuz \to yvz$, for $y, z \in (X_i \cup \{\$\})^*$ and $u, v \in X_i$ with $u =_{G_i} v$, is of *type* **(s0i)** if $l(u) > l(v)$ and of *type* **(s1i)** if $l(u) = l(v)$. Then for any words $w, x \in X^*$ with $w \overset{x0-2}{\to} x$, and any $j \in \{0, 1\}$ and $1 \le i, k \le n$ with $i \ne k$, we have

$$\begin{array}{ccc} w & \xrightarrow{xji} & x \\ \pi_i\downarrow & & \downarrow \pi_i \\ \pi_i(w) & \xrightarrow{sji} & \pi_i(x) \end{array} \qquad \begin{array}{ccc} w & \xrightarrow{x1k} & x \\ \pi_i\downarrow & & \downarrow \pi_i \\ \pi_i(w) & \xrightarrow{id} & \pi_i(x) \end{array} \qquad \begin{array}{ccc} w & \xrightarrow{x2} & x \\ \pi_i\downarrow & & \downarrow \pi_i \\ \pi_i(w) & \xrightarrow{id} & \pi_i(x) \end{array}$$

where id denotes the identity map on $(X_i \cup \{\$\})^*$. Hence for any words $w, x \in T^*$ with $w \stackrel{R0-2}{\to} x$ and any $j \in \{0,1\}$ and $1 \leq i, k \leq n$ with $i \neq k$, we have

$$\begin{array}{ccc} w & \xrightarrow{Rji} & x \\ \pi_i \circ h \downarrow & & \downarrow \pi_i \circ h \\ \pi_i(h(w)) & \xrightarrow{sji} & \pi_i(h(x)) \end{array} \qquad \begin{array}{ccc} w & \xrightarrow{R1k} & x \\ \pi_i \circ h \downarrow & & \downarrow \pi_i \circ h \\ \pi_i(h(w)) & \xrightarrow{id} & \pi_i(h(x)) \end{array} \qquad \begin{array}{ccc} w & \xrightarrow{R2} & x \\ \pi_i \circ h \downarrow & & \downarrow \pi_i \circ h \\ \pi_i(h(w)) & \xrightarrow{id} & \pi_i(h(x)) \end{array} .$$

For our word $y \in \cap_{i=1}^n \pi_i^{-1}(\Gamma_i(\$\Gamma_i)^*)$, the inclusion map $\iota : X^* \to T^*$ allows us to consider $y = \iota(y)$ as a word over the alphabet T, where $h(\iota(y)) = h(y) = y$. Since R is a terminating and confluent rewriting system, we have $y \stackrel{R0-2*}{\to} irr(y)$, and so by the commutative diagrams above, $y \stackrel{x0-2*}{\to} \Theta(y)$.

Suppose that an operation of type (x0) appears in this sequence of rewritings. Then $y \stackrel{x1-2*}{\to} y' \stackrel{x0i}{\to} z \stackrel{x0-2*}{\to} \Theta(y)$ for some $y', z \in X^*$ and some $1 \leq i \leq n$. Again applying the commutative diagrams above, then $\pi_i(y) \stackrel{s1i*}{\to} \pi_i(y') \stackrel{s0i}{\to} \pi_i(z)$. However, operations of type (s1i) map elements of $\Gamma_i(\$\Gamma_i)^*$ to $\Gamma_i(\$\Gamma_i)^*$, so $\pi_i(y') =\in \Gamma_i(\$\Gamma_i)^*$ and no operation of type (s0i) can be applied to $\pi_i(y')$, giving a contradiction.

(iv) $\Rightarrow$ (i): Suppose that $y \stackrel{x1-2*}{\to} \Theta(y)$. Then $l_X(y) = l_X(\Theta(y))$. If z is the shortlex least representative over X of the same element of G as y, then since the set $\{\Theta(w)\}$ is a set of normal forms, we have $\Theta(z) = \Theta(y)$. Now $z = \iota(z) \stackrel{R0-2*}{\to} irr(z)$, and so by the argument above we have $z = h(z) \stackrel{x0-2*}{\to} h(irr(z)) = \Theta(z)$. However, since z is geodesic, no length-decreasing operations can apply, so we have $l_X(z) = l_X(\Theta(z)) = l_X(y)$. Therefore y is also geodesic. □

In the next Corollary we collect for later use two other results that follow from the proof of Proposition 3.3.

Corollary 3.4. *Using the notation above:*

(1) *The subset $h(irr(R)) = \{\Theta(w) \mid w \in X^*\} \subseteq X^*$ is a set of geodesic normal forms for G over X.*
(2) *For any word $w \in X^*$ there is a sequence of rewriting operations $w \stackrel{x0-2*}{\to} \Theta(w)$.*

Proof. Statement (1) is shown in the proof of (iv) $\Rightarrow$ (i) above. For (2), let w be any element of X^*. Using the inclusion map $\iota : X^* \to T^*$, we have $w = \iota(w) \stackrel{R0-2*}{\to} irr(w)$, and so from the first set of commutative diagrams in the proof of (iii) $\Rightarrow$ (iv) in Proposition 3.3 above, we have $w = h(\iota(w)) \stackrel{x0-2*}{\to} \Theta(w)$. □

In the following Proposition we show that a result similar to Proposition 3.3 holds for conjugacy geodesics.

Proposition 3.5. *Let G be a graph product of the groups G_i for $1 \leq i \leq n$ and let $X = \cup_{i=1}^n X_i$ where X_i is a finite inverse-closed generating set for G_i for each i.*

Also for each i let $\Gamma_i := \Gamma(G_i, X_i)$, $\widetilde{\Gamma}_i := \widetilde{\Gamma}(G_i, X_i)$, and

$$\widetilde{U}_i := \{u_0\$u_1 \cdots \$u_m \mid m \geq 1 \text{ and } u_1, ..., u_{m-1}, u_m u_0 \in \Gamma_i\},$$

and let $y \in X^*$. The following are equivalent.

(i): *y is a conjugacy geodesic for G with respect to X.*

(ii): *y is a conjugationally trimmed word over X.*

(iii): *$y \in \cap_{i=1}^n \pi_i^{-1}(\widetilde{\Gamma}_i \cup \widetilde{U}_i)$.*

Proof. (i) $\Rightarrow$ (ii): Suppose that $y \in X^*$ is not conjugationally trimmed. Then $y \overset{x1-4*}{\to} z$ for some word z that can be rewritten using a length-reducing operation of type (x0) or (x5) to a word representing an element of the same conjugacy class. Since $l(y) = l(z)$, then y cannot be a conjugacy geodesic.

(ii) $\Rightarrow$ (iii): Suppose that $y \notin \cap_{i=1}^n \pi_i^{-1}(\widetilde{\Gamma}_i \cup \widetilde{U}_i)$. Then there is an index i such that $\pi_i(y) \in (X_i \cup \{\$\})^* \setminus (\widetilde{\Gamma}_i \cup \widetilde{U}_i)$. If $\pi_i(y) \in X_i^*$, then all letters of y lie in $X_i \cup C_i$, and so $y \overset{x2*}{\to} \pi_i(y)y'$ for some $y' \in C_i^*$. Now $\pi_i(y) \notin \widetilde{\Gamma}_i$ implies that a conjugacy reduction (x5) can be applied to the word $\pi_i(y)y'$, and so y cannot be conjugationally trimmed in this case. On the other hand, if $\pi_i(y) \notin X_i^*$, then $\pi_i(y) = u_0\$ \cdots \$u_m$ such that $m \geq 1$, each $u_i \in X_i^*$, and at least one of $u_1, ..., u_{m-1}$, or $u_m u_0$ does not lie in Γ_i. A similar argument shows that in this case $y \overset{x2-3*}{\to} z$ for a word z to which a local reduction (x0) can be applied, implying again that y cannot be conjugationally trimmed.

(iii) $\Rightarrow$ (i): Suppose that $y \in \cap_{i=1}^n \pi_i^{-1}(\widetilde{\Gamma}_i \cup \widetilde{U}_i)$, but that y is not a conjugacy geodesic. (Note that since $\widetilde{\Gamma}_i \subseteq \Gamma_i$ and $\widetilde{U}_i \subseteq \Gamma_i(\$\Gamma_i)^*$, Proposition 3.3 implies that the word y is a geodesic for G over X.) Then there is a geodesic word $w \in X^*$ such that the element wyw^{-1} of G is represented by a word over X that is shorter than y. In particular, the result in Corollary 3.4(1) that the Θ normal forms are geodesics shows that the word $\Theta(wyw^{-1})$ must then be shorter than y. Choose such a pair of words $y \in \cap_{i=1}^n \pi_i^{-1}(\widetilde{\Gamma}_i \cup \widetilde{U}_i)$ and $w \in \Gamma = \Gamma(G, X)$ with $l_X(\Theta(wyw^{-1})) < l_X(y)$ such that $l_X(y) + 2l_X(w)$ is least possible among all pairs with these properties.

Using Corollary 3.4(2) we have $wyw^{-1} \overset{x0-2*}{\to} \Theta(wyw^{-1})$ (where w^{-1} is the symbolic inverse of w over X). Since $l_X(\Theta(wyw^{-1})) < l_X(y)$, at least one operation of type (x0) must apply in this sequence of rewriting operations.

Since property (iii) of Proposition 3.5 holds for y, for each $1 \leq i \leq n$ we can write $\pi_i(y) = y_i u_i y_i'$ with either $y_i \in \widetilde{\Gamma}_i$ and $u_i = y_i' = \lambda$ (where λ denotes the empty word), or $u_i \in \$(\Gamma_i\$)^*$ and $y_i' y_i \in \Gamma_i$. Since w is a geodesic, then for each $1 \leq i \leq n$, using Proposition 3.3 we can write $\pi_i(w) = v_i w_i$ with $v_i \in (\Gamma_i\$)^*$ and $w_i \in \Gamma_i$. Then $\pi_i(wyw^{-1}) = v_i w_i y_i u_i y_i' w_i^{-1} v_i^{-1}$ (where the formal inverse of a word $b_1\$ \cdots b_k\$$ with each $b_j \in X_i^*$ is defined to be $\$b_k^{-1} \cdots \$b_1^{-1}$).

In our commutative diagrams in the proof above, we did not consider the interaction of rewriting operations of type (x0i) with the map π_k when $k \neq i$, but we need to do so now. For any word $s \in (X_j \cup \$)^*$, we write $s \overset{t*}{\to} s'$ if s' can be obtained from s by finitely many (possibly zero) replacements $\$\$ \to \$$; i.e. by shortening but not eliminating subwords that are strings of $ signs. Suppose that $z \in X^*$ and $\pi_i(z) = abc$ with $a \in (X_i^*\$)^*$, $b \in X_i^*$, and $c \in (\$X_i^*)^*$. Suppose that the operation $b \stackrel{x0i}{\to} b'$ induces the operation $z \stackrel{x0i}{\to} z' := ab'c$, and that $\lambda \neq b' \in X_i^*$. If the vertex v_j is adjacent to v_i in the graph Λ, and the vertex v_k is not adjacent to v_i, then

$$\begin{array}{ccc} z & \xrightarrow{x0i} & z' \\ \pi_j \downarrow & & \downarrow \pi_j \\ \pi_j(z) & \xrightarrow{id} & \pi_j(z') \end{array} \qquad\qquad \begin{array}{ccc} z & \xrightarrow{x0i} & z' \\ \pi_k \downarrow & & \downarrow \pi_k \\ \pi_k(z) & \xrightarrow{t*} & \pi_k(z') \end{array}$$

In the following claim, we use these commutative diagrams to show that in the process of rewriting $wyw^{-1} \stackrel{x0-2*}{\to} \Theta(wyw^{-1})$, each time an (x0$i$) rewriting operation is applied, then on the level of images under the π_j maps, when $j \neq i$ the effect is the application of a $\stackrel{t*}{\to}$ operation, and when $j = i$, effectively either a rewrite of the $w_iy_iw_i^{-1}$ subword of wyw^{-1} is replaced by a word of length at least $l(y_i)$ (in the case that $\pi_i(y) \in \widetilde{\Gamma}_i$) or rewrites of the w_iy_i and $y_i'w_i^{-1}$ subwords of wyw^{-1} are replaced by words of length at least $l(y_i) + l(y_i')$ (otherwise).

Claim: Suppose that $wyw^{-1} \stackrel{x0-2*}{\to} z \stackrel{x0i}{\to} x \stackrel{x0-2*}{\to} \Theta(wyw^{-1})$ and for each $1 \leq j \leq n$ the word $z \in X^*$ satisfies either:

(a): In the case that $\pi_j(y) = y_j \in \widetilde{\Gamma}_j$:
$\pi_j(z) = \tilde{v}_j z_j \tilde{v}_j'$ for some
$\tilde{v}_j \in (\Gamma_j\$)^*$, $\tilde{v}_j' \in (\$\Gamma_j)^*$, and $z_j \in X_j^*$ such that
$v_j \stackrel{t*}{\to} \tilde{v}_j$, $v_j^{-1} \stackrel{t*}{\to} \tilde{v}_j'$, $z_j =_{G_j} w_jy_jw_j^{-1}$ and
$l(y_j) \leq l(z_j) \leq l(y_j) + 2l(w_j)$,
or

(b): In the case that $\pi_j(y) \in \widetilde{U}_j$:
$\pi_j(z) = \tilde{v}_j z_j \tilde{u}_j z_j' \tilde{v}_j'$ for some
$\tilde{u}_j \in \$(\Gamma_j\$)^*$, $\tilde{v}_j \in (\Gamma_j\$)^*$, $\tilde{v}_j' \in (\$\Gamma_j)^*$, and $z_j, z_j' \in X_j^*$ such that
$u_j \stackrel{t*}{\to} \tilde{u}_j$, $v_j \stackrel{t*}{\to} \tilde{v}_j$, $v_j^{-1} \stackrel{t*}{\to} \tilde{v}_j'$, $z_j =_{G_j} w_jy_j$, $z_j' =_{G_j} y_j'w_j^{-1}$, and $l(y_j) + l(y_j') \leq l(z_j) + l(z_j') \leq l(y_j) + l(y_j') + 2l(w_j)$.

Then for each $1 \leq j \leq n$ the word $x \in X^*$ also satisfies either:

(a'): In the case that $\pi_j(y) = y_j \in \widetilde{\Gamma}_j$:
$\pi_j(x) = \hat{v}_j x_j \hat{v}_j'$ for some
$\hat{v}_j \in (\Gamma_j\$)^*$, $\hat{v}_j' \in (\$\Gamma_j)^*$, and $x_j \in X_j^*$ such that
$v_j \stackrel{t*}{\to} \hat{v}_j$, $v_j^{-1} \stackrel{t*}{\to} \hat{v}_j'$, $x_j =_{G_j} w_jy_jw_j^{-1}$ and
$l(y_j) \leq l(x_j) \leq l(y_j) + 2l(w_j)$, or

(b'): In the case that $\pi_j(y) \in \widetilde{U}_j$:
$\pi_j(x) = \hat{v}_j x_j \hat{u}_j x_j' \hat{v}_j'$ for some
$\hat{u}_j \in \$(\Gamma_j\$)^*$, $\hat{v}_j \in (\Gamma_j\$)^*$, $\hat{v}_j' \in (\$\Gamma_j)^*$, and $x_j, x_j' \in X_j^*$ such that

$u_j \stackrel{t*}{\to} \hat{u}_j$, $v_j \stackrel{t*}{\to} \hat{v}_j$, $v_j^{-1} \stackrel{t*}{\to} \hat{v}_j'$, $x_j =_{G_j} w_j y_j$, $x_j' =_{G_j} y_j' w_j^{-1}$, and $l(y_j)+l(y_j') \leq l(x_j) + l(x_j') \leq l(y_j) + l(y_j') + 2l(w_j)$.

Proof of claim. Since no length reducing operation $z \stackrel{x0i}{\to} x$ can be performed on a subword a of z satisfying $\pi_i(a) \in \Gamma_i(\$\Gamma_i)^*$, the associated operation $\pi_i(z) \stackrel{s0i}{\to} \pi_i(x)$ must apply to a subword of $\pi_i(z)$ disjoint from the $\tilde{v}_i$, $\tilde{v}_i^{-1}$, and $\tilde{u}_i$ subwords.

Case 1. Suppose that $\pi_i(y) = y_i \in \widetilde{\Gamma}_i$. Then the associated (s0i) operation must have the form $\tilde{v}_i z_i \tilde{v}_i' \to \hat{v}_i x_i \hat{v}_i'$ for some $x_i \in X_i^*$ where $\hat{v}_i = \tilde{v}_i$, $\hat{v}_i' = \tilde{v}_i'$, $x_i =_{G_i} z_i =_{G_i} w_i y_i w_i^{-1}$, and $l(x_i) < l(z_i) \leq l(y_i) + 2l(w_i)$. Now since $x_i \sim_i y_i$ and the word $y_i = \pi_i(y) \in \widetilde{\Gamma}_i$ is a conjugacy geodesic for the group G_i over the generating set X_i in this case, we must have $l(y_i) \leq l(x_i)$.

If x_i were the empty word λ, then since $x_i \sim_i y_i$ and $y_i \in \widetilde{\Gamma}_i$, we have $y_i = \lambda$ as well. Then $w_i w_i^{-1} \stackrel{s0-1i*}{\to} z_i \stackrel{s0i}{\to} x_i$, so we have $l(w_i) > 0$. Now $w \stackrel{x2*}{\to} \tilde{w} w_i$ for a word $\tilde{w} \in X^*$, $wyw^{-1} =_G \tilde{w} y \tilde{w}^{-1}$, and $l_X(w) > l_X(\tilde{w})$. But then replacing w with $\tilde{w}$ contradicts our choice of words $y \in \cap_{i=1}^n \pi_i^{-1}(\widetilde{\Gamma}_i \cup \widetilde{U}_i)$ and $w \in \Gamma$ with $l_X(\Theta(wyw^{-1})) < l_X(y)$ and $l_X(y) + 2l_X(w)$ minimal with respect to this property. Hence $l(x_i) \geq 1$.

Then the commutative diagrams above the Claim show that for all $j \neq i$, $\pi_j(z) \stackrel{t*}{\to} \pi_j(x)$. Hence $v_j \stackrel{t*}{\to} \tilde{v}_j \stackrel{t*}{\to} \hat{v}_j$ and since $v_j, \tilde{v}_j \in (\Gamma_i\$)^*$, then $\hat{v}_j \in (\Gamma_i\$)^*$. The proofs that $u_j \stackrel{t*}{\to} \hat{u}_j \in \$(\Gamma_i\$)^*$ and $v_j' \stackrel{t*}{\to} \hat{v}_j' \in (\$\Gamma_i)^*$ are identical. Since the subwords z_j and z_j' of $\pi_j(z)$ lie in X_j^*, the operation $\pi_j(z) \stackrel{t*}{\to} \pi_j(x)$ can't affect these subwords, and so $z_j = x_j$ and $z_j' = x_j'$ for all indices $j \neq i$. Therefore for the word x, conditions (a') or (b') hold for all indices j, completing Case 1.

Case 2. Suppose that $\pi_i(y) \in \widetilde{U}_i$. Similar to the argument in the previous case, the (s0i) operation associated to the rewriting operation $z \stackrel{x0i}{\to} x$ has the form $\pi_i(z) = \tilde{v}_i z_i \tilde{u}_i z_i' \tilde{v}_i' \to \hat{v}_i x_i \hat{u}_i x_i' \hat{v}_i' = \pi_i(x)$ where $\tilde{v}_i = \hat{v}_i$, $\tilde{u}_i = \hat{u}_i$, $\hat{v}_i' = \tilde{v}_i'$, and either $x_i =_{G_i} z_i$ with $l(x_i) < l(z_i)$ and $x_i' = z_i'$, or else $x_i = z_i$ and $x_i' =_{G_i} z_i'$ with $l(x_i') < l(z_i')$. We consider the first of these two forms of the rewriting operation; the proof in Case 2 for the second form is similar. Now $x_i =_{G_i} z_i =_{G_i} w_i y_i$ and $x_i' = z_i' =_{G_i} y_i' w_i^{-1}$ with $l(x_i) + l(x_i') < l(z_i) + l(z_i') \leq l(y_i) + l(y_i') + 2l(w_i)$. Moreover, $x_i' x_i =_{G_i} y_i' w_i^{-1} w_i y_i =_{G_i} y_i' y_i$. The fact that $y_i u_i y_i' = \pi_i(y) \in \widetilde{U}_i$ in this case implies that $y_i' y_i \in \Gamma_i$, i.e., $y_i' y_i$ is a geodesic word. Hence $l(y_i)+l(y_i') \leq l(x_i) + l(x_i')$.

If x_i were the empty word, then since $x_i =_{G_i} w_i y_i$ and $w_i \in \Gamma_i$, the word w_i^{-1} is another geodesic representative of y_i, and so $y_i' w_i^{-1}$ is also geodesic. As in the previous case, $w \stackrel{x2*}{\to} w' w_i$ and $y \stackrel{x2*}{\to} y_i y'' y_i'$. But then replacing w with w' and y with $y'' y_i' w_i^{-1}$ again contradicts our choice of words $y \in \cap_{i=1}^n \pi_i^{-1}(\widetilde{\Gamma}_i \cup \widetilde{U}_i)$ and

$w \in \Gamma$ with $l_X(\Theta(wyw^{-1})) < l_X(y)$ and $l_X(y) + 2l_X(w)$ minimal with respect to this property. Hence $l(x_i) \geq 1$. Similarly $l(x_i') \geq 1$.

Then for all $j \neq i$, $\pi_j(z) \xrightarrow{t*} \pi_j(x)$. Therefore as in the prior case, for the word x, condition (a') or (b') holds for all indices j, completing Case 2 and the proof of the Claim.

Since the word wyw^{-1} satisfies the property that one of (a) or (b) holds for all $1 \leq j \leq n$, iteratively applying this claim shows that for each j, one of (a') or (b') must hold for the normal form word $x := \Theta(wyw^{-1})$. Denoting the number of occurrences of X_i letters in a word u over $X_i \cup \$$ by $l_i(u)$, then whenever $u \xrightarrow{t*} \tilde{u}$ we have $l_i(u) = l_i(\tilde{u})$. Therefore $l_X(\Theta(wyw^{-1})) = \sum_{i=1}^n l_i(\pi_i(\Theta(wyw^{-1}))) \geq \sum_{i=1}^n l(y_i) + l_i(u_i) + l(y_i') = l_X(y)$. But our initial assumption on the pair y, w included the inequality $l_X(\Theta(wyw^{-1})) < l_X(y)$, resulting in the required contradiction. $\square$

We are now ready to prove Theorem 3.1.

Proof. The class of regular languages is closed under finitely many operations of union, intersection, concatenation, and Kleene star (i.e. $()^*$), and is closed under inverse images of monoid homomorphisms (see, for example, [13, Chapter 3]).

Proposition 3.3 shows that $\Gamma = \cap_{i=1}^n \pi_i^{-1}(\Gamma_i(\$\Gamma_i)^*)$. Then applying these closure properties yields a new proof of the result of Loeffler, Meier, and Worthington [15, Theorem 1] that whenever the sets Γ_i of geodesics for the vertex groups G_i are regular languages, then the language Γ of geodesics for the graph product group G over X is also regular.

From Proposition 3.5 we have that the language of conjugacy geodesics for the graph product group G over the generating set X satisfies $\widetilde{\Gamma} = \cap_{i=1}^n \pi_i^{-1}(\widetilde{\Gamma}_i \cup \widetilde{U}_i)$. By hypothesis the language $\widetilde{\Gamma}_i$ is regular for each i, and so the closure properties above imply that it suffices to show that the language $\widetilde{U}_i$ over $X_i \cup \{\$\}$ is regular, given that the language Γ_i over the alphabet X_i is regular. The set Γ_i is recognized by a finite state automaton with a set $Q = \{q_0, \dots q_m\}$ of $m+1$ states, where q_0 is the start state, and with transition function $\delta : Q \times X_i \to Q$. Then Γ_i can be written as $\Gamma_i = L_0\overline{L_0} \cup \cdots \cup L_m\overline{L_m}$, where for each $0 \leq j \leq m$, the set L_j is the language of all the words w over X_i such that $\delta(q_0, w) = q_j$ and $\overline{L_j}$ is the set of all words z such that $\delta(q_j, z)$ is an accept state. Then L_j and $\overline{L_j}$ are regular languages. Now $\widetilde{U}_i = \cup_{j=0}^m \overline{L_j}\$(\Gamma_i\$)^* L_j$, and therefore $\widetilde{U}_i$ is also a regular language. $\square$

Example 3.6. A *right-angled Coxeter group* is a graph product of cyclic groups $G_i = \langle a_i \mid a_i^2 = 1\rangle$ of order 2. Then the graph product group $G = G\Lambda$ has generating set $X = \{a_i\}$, and there is a bijection between T and the set of cliques of the graph Λ. Theorem 3.1 shows that the geodesic language $\Gamma(G, X)$ and the geodesic conjugacy language $\widetilde{\Gamma}(G, X)$ are both regular, and so the geodesic growth series $\gamma(G, X)$ and geodesic conjugacy growth series $\widetilde{\gamma}(G, X)$ are both rational functions.

In this example we provide the details for both spherical and geodesic languages and series for a specific right-angled Coxeter group, namely a graph product $G =$

$G\Lambda$ of three groups $G_i = \langle a_i \mid a_i^2 = 1\rangle$ $(1 \le i \le 3)$ such that in the graph Λ the vertex pairs v_1, v_2 and v_2, v_3 are adjacent, but the vertices v_2 and v_3 are not adjacent. That is, $G = G_1 \times (G_2 * G_3) = \mathbb{Z}_2 \times (\mathbb{Z}_2 * \mathbb{Z}_2)$. Here $X = \{a_1, a_2, a_3\}$ and $T = \{\{a_1\}, \{a_2\}, \{a_3\}, \{a_1, a_2\}, \{a_1, a_3\}\}$. The rewriting system

$$\begin{array}{llll} R := \{ & \{a_1\}^2 \xrightarrow{R0} 1, & \{a_1\}\{a_1,a_2\} \xrightarrow{R0} \{a_2\}, & \{a_1\}\{a_1,a_3\} \xrightarrow{R0} \{a_3\}, \\ & \{a_2\}^2 \xrightarrow{R0} 1, & \{a_2\}\{a_1,a_2\} \xrightarrow{R0} \{a_1\}, & \\ & \{a_3\}^2 \xrightarrow{R0} 1, & \{a_3\}\{a_1,a_3\} \xrightarrow{R0} \{a_1\}, & \\ & \{a_1,a_2\}\{a_1\} \xrightarrow{R0} \{a_2\}, & \{a_1,a_2\}\{a_2\} \xrightarrow{R0} \{a_1\}, & \\ & \{a_1,a_2\}^2 \xrightarrow{R0} \{a_2\}\{a_2\}, & \{a_1,a_2\}\{a_1,a_3\} \xrightarrow{R0} \{a_2\}\{a_3\}, & \\ & \{a_1,a_3\}\{a_1\} \xrightarrow{R0} \{a_3\}, & \{a_1,a_3\}\{a_3\} \xrightarrow{R0} \{a_1\}, & \\ & \{a_1,a_3\}^2 \xrightarrow{R0} \{a_3\}\{a_3\}, & \{a_1,a_3\}\{a_1,a_2\} \xrightarrow{R0} \{a_3\}\{a_2\}, & \\ & \{a_1\}\{a_2\} \xrightarrow{R2} \{a_1,a_2\}, & \{a_1\}\{a_3\} \xrightarrow{R2} \{a_1,a_3\}, & \\ & \{a_2\}\{a_1\} \xrightarrow{R2} \{a_1,a_2\}, & \{a_2\}\{a_1,a_3\} \xrightarrow{R2} \{a_1,a_2\}\{a_3\}, & \\ & \{a_3\}\{a_1\} \xrightarrow{R2} \{a_1,a_3\}, & \{a_3\}\{a_1,a_2\} \xrightarrow{R2} \{a_1,a_3\}\{a_2\} & \} \end{array}$$

is a subset of the set of all rewriting rules of types (R0)-(R2), but is already sufficient to be a complete rewriting system for G (see [4, Appendix]). Then the set $irr(R) = T^* \setminus T^*LT^*$, where L is the finite set of words on the left hand sides of the rules in R, is a regular language of normal forms for G over T. Since the class of regular languages is closed under images of monoid homomorphisms ([13, Chapter 3]), then the language $h(irr(R))$ is also regular. Since the language $H := h(irr(R))$ is a set of geodesic normal forms for G over X, then the strict growth series satisfies $f_H = \sigma(G,X)$; hence the spherical growth series $\sigma(G,X)$ is rational. The combination of Theorem 2.4 and Proposition 2.1 shows that the spherical conjugacy language $\widetilde{\Sigma}(G,X)$ is regular and the spherical conjugacy growth series $\widetilde{\sigma}(G,X)$ is rational as well.

The homomorphisms $\pi_i : X^* \to (X_i \cup \{\$\})^*$ are defined by $\pi_1(a_1) = a_1$, $\pi_1(a_2) = \lambda$, $\pi_1(a_3) = \lambda$, $\pi_2(a_1) = \lambda$, $\pi_2(a_2) = a_2$, $\pi_2(a_3) = \$$, $\pi_3(a_1) = \lambda$, $\pi_3(a_2) = \$$, and $\pi_3(a_3) = a_3$. Proposition 3.3 implies that the geodesic language $\Gamma := \Gamma(G,X) = \cap_{i=1}^3 \pi_i^{-1}(\{\lambda, a_i\}(\$\{\lambda, a_i\})^*)$. Since the symbol \$ does not appear in the image of the map π_1, then a geodesic word contains at most one occurrence of the letter a_1. Moreover, the preimage sets under π_2 and π_3 imply that any two occurrences of a_2 must have an a_3 between them, and vice-versa. Then

$$\Gamma = \{\lambda, a_3\}(a_2a_3)^*\{\lambda, a_1\}(a_2a_3)^*\{\lambda, a_2\} \cup \{\lambda, a_2\}(a_3a_2)^*\{\lambda, a_1\}(a_3a_2)^*\{\lambda, a_3\} .$$

The strict growth function for this language satisfies $\phi_\Gamma(0) = 1$, $\phi_\Gamma(1) = 3$, and $\phi_\Gamma(n) = 2n + 2$ for all $n \ge 2$. The geodesic growth series for this group satisfies $\gamma(G,X)(z) = (1 + z + z^2 - z^3)/(1 - z)^2$.

Applying Proposition 3.5, the geodesic conjugacy language is $\widetilde{\Gamma} := \widetilde{\Gamma}(G,X) = \cap_{i=1}^3 \pi_i^{-1}(\{\lambda, a_i\} \cup \{\lambda, a_i\}(\$\{\lambda, a_i\})^*\$ \cup (\$\{\lambda, a_i\})^*\$a_i)$. Analyzing this in the same way, then

$$\begin{aligned}\widetilde{\Gamma} = \;& \{a_2, a_3, a_1a_2, a_1a_3, a_2a_1, a_3a_1\} \cup (a_2a_3)^*\{\lambda, a_1, a_2a_1a_3\}(a_2a_3)^* \\ & \cup(a_3a_2)^*\{\lambda, a_1, a_3a_1a_2\}(a_3a_2)^* \;.\end{aligned}$$

The strict growth function for this language satisfies $\phi_{\widetilde{\Gamma}}(0) = 1$, $\phi_{\widetilde{\Gamma}}(1) = 3$, $\phi_{\widetilde{\Gamma}}(2) = 6$, $\phi_{\widetilde{\Gamma}}(3) = 6$, $\phi_{\widetilde{\Gamma}}(2k) = 2$, and $\phi_{\widetilde{\Gamma}}(2k+1) = 4k+2$ for all $k \geq 2$. Then the geodesic conjugacy growth series is given by $\widetilde{\gamma}(G,X)(z) = (1+3z+4z^2-9z^4+z^5+4z^6)/(1-z^2)^2$.

Example 3.7. Let G be the projective special linear group $G := PSL_2(\mathbb{Z}) = \mathbb{Z}_2 * \mathbb{Z}_3 = \langle a, b, c \mid a^2 = 1, b^2 = c, bc = 1\rangle$ with the generating set $X = \{a, b, c\}$. Theorem 2.4 shows that the spherical conjugacy language $\widetilde{\Sigma}(G,X)$ is not regular and the corresponding spherical conjugacy growth series is not rational. However, from Theorem 3.1 we have that the geodesic conjugacy language $\widetilde{\Gamma}(G,X)$ is regular and the geodesic conjugacy growth series $\widetilde{\gamma}(G,X)$ is a rational function. Indeed, it follows from Theorem 3.1 that for any graph product of finite groups, the geodesic conjugacy language is regular and the geodesic conjugacy growth series is rational, with respect to a union of finite generating sets of the vertex groups.

Theorem 3.8. *Let G and H be groups with finite inverse-closed generating sets A and B, respectively. Let $\widetilde{\gamma}_G(z) := \widetilde{\gamma}(G,A)(z) = \sum_{i=0}^\infty r_iz^i$ and $\widetilde{\gamma}_H(z) := \widetilde{\gamma}(H,B)(z) = \sum_{i=0}^\infty s_iz^i$ be the geodesic conjugacy growth series, and let $\gamma_G := \gamma(G,A)$ and $\gamma_H := \gamma(H,B)$ be the geodesic growth series for these pairs.*

(i) The geodesic conjugacy growth series $\widetilde{\gamma}_\times$ of the direct product $G \times H = \langle G, H \mid [G,H]\rangle$ of groups G and H with respect to the generating set $A \cup B$ is given by $\widetilde{\gamma}_\times = \sum_{i=0}^\infty \delta_i z^i$ where $\delta_i := \sum_{j=0}^i \binom{i}{j} r_j s_{i-j}$.

(ii) The geodesic conjugacy growth series $\widetilde{\gamma}_*$ of the free product $G*H$ of the groups G and H with respect to the generating set $A \cup B$ is given by

$$\widetilde{\gamma}_* - 1 = (\widetilde{\gamma}_G - 1) + (\widetilde{\gamma}_H - 1) - z\frac{d}{dz}\ln\left[1 - (\gamma_G - 1)(\gamma_H - 1)\right].$$

Proof. Denote the geodesic languages by $\widetilde{\Gamma}_G := \widetilde{\Gamma}(G,A)$, $\widetilde{\Gamma}_H := \widetilde{\Gamma}(H,B)$, $\Gamma_G := \Gamma(G,A)$, and $\Gamma_H := \Gamma(H,B)$.

(i) The proof in this case follows the same argument as the proof of the formula for the geodesic growth series of $G \times H$ in terms of the geodesic growth series of G and H in [15, Proposition 1]. In particular, in $G \times H$ each conjugacy geodesic word w of length i can be obtained by taking a word y in $\widetilde{\Gamma}_G$ of length $0 \leq j \leq i$ and a word z in $\widetilde{\Gamma}_H$ of length $i - j$, and "shuffling" the letters so that the letters of y and z appear in the same order, but not necessarily contiguously, in w.

(ii) The geodesic conjugacy language $\widetilde{\Gamma}_* := \widetilde{\Gamma}(G*H, A\cup B)$ can be written as a disjoint union

$$\widetilde{\Gamma}_* = \{\lambda\} \cup (\widetilde{\Gamma}_G \setminus \{\lambda\}) \cup (\widetilde{\Gamma}_H \setminus \{\lambda\}) \cup \widetilde{\Gamma}_{A\bullet} \cup \widetilde{\Gamma}_{B\bullet}$$

where λ is the empty word, $\widetilde{\Gamma}_{A\bullet}$ is the set of all conjugacy geodesic words beginning with a letter in A and containing at least one letter in B, and similarly $\widetilde{\Gamma}_{B\bullet}$ is the set of all conjugacy geodesic words beginning with a letter in B and containing at least one letter in A. As a consequence,

$$\widetilde{\gamma}_* = 1 + (\widetilde{\gamma}_G - 1) + (\widetilde{\gamma}_H - 1) + f_{\widetilde{\Gamma}_{A\bullet}} + f_{\widetilde{\Gamma}_{B\bullet}},$$

where as usual f_L denotes the strict growth series of the language L. Now $\widetilde{\Gamma}_{A\bullet}$ can also be decomposed as a disjoint union

$$\begin{aligned}\widetilde{\Gamma}_{A\bullet} \quad = \quad & \cup_{n=1}^{\infty}\{y_1z_1\cdots y_nz_ny_{n+1} \mid y_2,...,y_n,y_{n+1}y_1 \in \Gamma_G \setminus \{\lambda\}, y_1 \in A^+, \\ & \qquad z_1,...,z_n \in \Gamma_H \setminus \{\lambda\}\}.\end{aligned}$$

As a consequence, the growth series of this language is

$$f_{\widetilde{\Gamma}_{A\bullet}} = \sum_{n=1}^{\infty}(\gamma_H - 1)[(\gamma_G - 1)(\gamma_H - 1)]^{n-1}\left(z\frac{d}{dz}\gamma_G\right),$$

where the i-th coefficient in the series $z\frac{d}{dz}\gamma_G = \sum_{i=0}^{\infty}\alpha_i z^i$, given by

$$\alpha_i = i \cdot (\#\text{ of geodesics in } (G, A) \text{ of length } i),$$

counts the number of pairs of words y_1, y_n with $y_1 \in A^+$, $y_n \in A^*$, and y_ny_1 a geodesic for the pair (G, A) of length i. The formula for $f_{\widetilde{\Gamma}_{B\bullet}}$ is obtained in the same way. Putting these together, then

$$\begin{aligned}\widetilde{\gamma}_* \quad &= \quad 1 + (\widetilde{\gamma}_G - 1) + (\widetilde{\gamma}_H - 1) + \sum_{n=1}^{\infty}(\gamma_H - 1)[(\gamma_G - 1)(\gamma_H - 1)]^{n-1}\left(z\frac{d}{dz}\gamma_G\right) + \\ &\qquad \sum_{n=1}^{\infty}(\gamma_G - 1)[(\gamma_H - 1)(\gamma_G - 1)]^{n-1}\left(z\frac{d}{dz}\gamma_H\right) \\ &= \quad \widetilde{\gamma}_G + \widetilde{\gamma}_H - 1 + z\left(\frac{d}{dz}\gamma_G\right)(\gamma_H - 1)\frac{1}{1 - (\gamma_G - 1)(\gamma_H - 1)} + \\ &\qquad z\left(\frac{d}{dz}\gamma_H\right)(\gamma_G - 1)\frac{1}{1 - (\gamma_G - 1)(\gamma_H - 1)} \\ &= \quad \widetilde{\gamma}_G + \widetilde{\gamma}_H - 1 + z\frac{\left(\frac{d}{dz}\gamma_G\right)(\gamma_H - 1) + \left(\frac{d}{dz}\gamma_H\right)(\gamma_G - 1)}{1 - (\gamma_G - 1)(\gamma_H - 1)},\end{aligned}$$

resulting in the required formula. □

Example 3.9. In the free group $F_2 = \mathbb{Z} * \mathbb{Z}$, associate $G = \langle a \mid \rangle$ with the first copy of the integers and $H = \langle b \mid \rangle$ with the second, and let $A = \{a, a^{-1}\}$, $B = \{b, b^{-1}\}$. We have

$$\widetilde{\gamma}_G = \widetilde{\gamma}_H = \gamma_G = \gamma_H = \frac{1+z}{1-z},$$

and so $1-(1-\gamma_G)(1-\gamma_H) = \frac{1-2z-3z^2}{(1-z)^2}$. Plugging these into the formula in Theorem 3.8(ii), we obtain

$$\widetilde{\gamma}(F_2, \{a^{\pm 1}, b^{\pm 1}\}) = \widetilde{\gamma}_* = \frac{1+z-z^2-9z^3}{1-3z-z^2+3z^3}.$$

(We note that in [20, Corollary 14.1] Rivin has computed the equality conjugacy growth series for this group (see Section 4 for the definition of the equality conjugacy language and equality conjugacy growth series), and that in this example, the equality conjugacy language and the geodesic conjugacy language are the same set, and so the corresponding growth series are also equal. The rational function above differs from Rivin's formula by adding 1, because Rivin's growth series does not count the constant term corresponding to the empty word λ in the equality conjugacy language.)

Example 3.10. Let G and H be finite groups with generating sets $A := G \setminus \{1_G\}$ and $B := H \setminus \{1_H\}$, and let $m := |A| = |G| - 1$ and $n := |B| = |H| - 1$. The corresponding languages are $\widetilde{\Gamma}_G = \Gamma(G, A) = A \cup \{\lambda\}$ and $\widetilde{\Gamma}_H = \Gamma(H, B) = B \cup \{\lambda\}$, where λ is the empty word, and hence we have growth series $\widetilde{\gamma}_G = \gamma(G, A) = mz+1$ and $\widetilde{\gamma}_H = \gamma(H, B) = nz+1$. For the free product $G * H$, with generating set $A \cup B$, the formula in Theorem 3.8 shows that the geodesic conjugacy growth series is

$$\widetilde{\gamma}_*(z) = \widetilde{\gamma}(G * H, A \cup B)(z) = \frac{1+(m+n)z+mnz^2-mn(m+n)z^3}{1-mnz^2}.$$

In particular, for the group $P := PSL_2(\mathbb{Z})$ with the generating set X from Example 3.7, the geodesic conjugacy growth series is given by the rational function

$$\widetilde{\gamma}(P, X)(z) = \frac{1+3z+2z^2-6z^3}{1-2z^2}.$$

Proposition 3.11. *If G and H are finite groups with a common subgroup K, then the free product $G *_K H$ of G and H amalgamated over K, with respect to the generating set $X := G \cup H \cup K - \{1\}$, has regular geodesic conjugacy language $\widetilde{\Gamma}(G *_K H, X)$ and rational geodesic conjugacy growth series $\widetilde{\gamma}(G *_K H, X)$.*

Proof. Let $X_G := G \setminus K$, $X_H := H \setminus K$, and $X_K := K \setminus \{1\}$; then $X = X_K \cup X_G \cup X_H$. Given a sequence $x_1, ..., x_n$ of elements of X, this sequence is called *cyclically reduced* if either $n = 1$ and $x_1 \in X$ or else $n > 1$ and for each $1 \le i \le n$ the elements x_i and $x_{i+1(\text{mod } n)}$ lie in $X_G \cup X_H$ and are from different factors (i.e. if $x_i \in X_G$ then $x_{i+1} \in X_H$ and vice versa), and the product $x_1 \cdots x_n$ in $G *_K H$ is the associated *cyclically reduced product*. Every element of $G *_K H$ is conjugate to a cyclically reduced product. Moreover, by [16, Theorem IV.2.8], given any cyclically reduced sequence $x_1, ..., x_n$ with $n \ge 2$ and any $g \in G *_K H$, every product of a cyclically reduced sequence $x'_1, ..., x'_k$ satisfying $gx_1 \cdots x_k g^{-1} =_{G *_K H} x'_1 \cdots x'_k$ can be obtained by cyclically permuting the original sequence $x_1, ..., x_n$ and conjugating the resulting product by an element of K. Now every conjugacy geodesic word over X must be a cyclically reduced product, and this theorem implies also that every cyclically reduced product is a conjugacy geodesic. That is,

$$\widetilde{\Gamma}(G *_K H, X) = \{\lambda\} \cup X \cup (X_G X_H)^* \cup (X_G X_H)^*,$$

giving a regular expression for the language $\widetilde{\Gamma}(G *_K H, X)$. □

4. Open questions

We remark that intermediate between the two conjugacy languages defined in Section 1 is a third set of words given by the subset of $\Sigma(G, X)$ defined by

$$\widetilde{\mathcal{E}} = \widetilde{\mathcal{E}}(G, X) := \{y_g \mid g \in G, |g| = |g|_c\} \ ,$$

which we refer to as the *equality conjugacy language* for G over X. It is immediate from the definitions that $\widetilde{\Sigma} \subseteq \widetilde{\mathcal{E}} \subseteq \widetilde{\Gamma}$. Moreover, the equality language can be expressed as the intersection of the geodesic conjugacy language and the spherical language; that is, $\widetilde{\mathcal{E}} = \widetilde{\Gamma} \cap \Sigma$. We denote the strict growth series for this language by

$$\widetilde{\epsilon} = \widetilde{\epsilon}(G, X) := f_{\widetilde{\mathcal{E}}(G,X)},$$

called the *equality conjugacy growth series*. In [20, Corollary 14.1] Rivin gives a rational function formula for the equality conjugacy growth series (which he denotes by $\mathcal{F}[C_{F_k}](z)$) for a finitely generated free group with respect to a free basis (see also Example 3.9 for the rank 2 case), and so the free group does not give an obstruction to rationality being preserved by free products. More generally, for any right-angled Artin group G (i.e., graph product of infinite cyclic groups) with canonical generating set X (the union of the cyclic generators of the vertex groups), it follows from [11, Corollary 3.4 and proof of Theorem B] that $\Sigma(G, X)$ is regular, and from Theorem 3.1 that $\widetilde{\Gamma}(G, X)$ is regular, and hence $\widetilde{\mathcal{E}}(G, X)$ is regular and $\widetilde{\epsilon}(G, X)$ is rational.

Question 4.1. *If G is a graph product of finitely many groups G_i and each group G_i has a finite inverse-closed generating set X_i such that $\widetilde{\mathcal{E}}(G_i, X_i)$ is a regular language, is the language $\widetilde{\mathcal{E}}(G, \cup_i X_i)$ regular? Is rationality of the equality conjugacy growth series $\widetilde{\epsilon}(G, X)$ preserved by direct and free products?*

In some cases, regularity of languages and rationality of growth series associated to groups are known to depend upon the generating set chosen. For example, Stoll [22] has shown that rationality of the usual (cumulative) growth series $b_{\Sigma(G,X)}$ (and hence also of the spherical growth series $\sigma(G, X) = f_{\Sigma(G,X)}$) depends upon the generating set for the higher Heisenberg groups, and Cannon [18, p. 268] has shown that regularity of the geodesic language $\Gamma(\mathbb{Z}^2 \rtimes \mathbb{Z}_2, X)$ depends on the generating set X for a semidirect product of $\mathbb{Z}^2$ by the cyclic group of order 2. In the case of conjugacy growth, Hull and Osin [14, Theorem 1.3] have shown an example of a finitely generated group G with a finite index subgroup H such that the conjugacy growth function $\beta_{\widetilde{\Sigma}(G,X)}$ grows exponentially, but H has only two conjugacy classes.

Then the spherical and geodesic conjugacy growth series for G are both infinite series, but these two series for H are both polynomials.

Question 4.2. *Does there exist a finite inverse-closed generating set X for the free group F on two generators such that $\widetilde{\Sigma}(F, X)$ is regular?*

For the free basis which gives a non-regular spherical conjugacy language (Proposition 2.2), we also consider formal language theoretic classes that are less restrictive than context-free languages.

Question 4.3. *Let $F = F(a, b)$ be the free group on generators a and b. Is $\widetilde{\Sigma}(F, \{a^{\pm 1}, b^{\pm 1}\})$ an indexed language? A context-sensitive language?*

In Corollary 2.3 we show that the spherical conjugacy language cannot be regular (or indeed context-free) in any group that contains a free subgroup as a direct or free factor, with respect to a generating set that is the union of the free basis and the generators of the other factor. This leads us to wonder whether free groups are "poison subgroups" from the viewpoint of regular spherical conjugacy languages.

Question 4.4. *Let G be a group with $F = F(a, b)$ as subgroup, and let X be an inverse-closed generating set for G. Can $\widetilde{\Sigma}(G, X)$ be a regular language?*

As we remarked in Section 1, Cannon has shown that for word hyperbolic groups, the geodesic language for every finite generating set is regular [8, Chapter 3].

Question 4.5. *Let G be a word hyperbolic group and let X be a finite generating set for G. Is the geodesic conjugacy language $\widetilde{\Gamma}(G, X)$ necessarily regular?*

In particular, is the geodesic conjugacy language for a (compact, finite genus) surface group regular? What about the spherical conjugacy language? Since infinite index subgroups of surface groups are free, answering Question 4.4 would shed some light on the behavior of the spherical conjugacy language in surface groups.

In [9] Grigorchuk and Nagnibeda generalized the notion of growth functions for finitely generated groups by considering the *complete growth series* for G over X given by

$$\rho(z) := \sum_{i=0}^{\infty} (\sum_{|g|=i} g) z^i \in \mathbb{Z}[G][[z]]$$

with coefficients in the integral group ring $\mathbb{Z}[G]$. They define a concept of rationality for such series, and show that for any word hyperbolic group with respect to any generating set, the complete growth series is rational. Viewing the function ρ as a generalization of the spherical growth series $\sigma(G, X)$, one can analogously define the *complete conjugacy growth series* to be

$$\tilde{\rho}(z) := \sum_{i=0}^{\infty} (\sum_{|g|=|g|_c=i} g) z^i$$

in $\mathbb{Z}[G][[z]]$.

Question 4.6. *Is the complete conjugacy growth series $\tilde{\rho}$ rational for word hyperbolic groups?*

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L. Ciobanu, Mathematics Department, University of Neuchâtel, Rue Emile-Argand 11, CH-2000 Neuchâtel, Switzerland
E-mail address: laura.ciobanu@unine.ch

S. Hermiller, Department of Mathematics, University of Nebraska, Lincoln, NE 68588-0130, USA
E-mail address: smh@math.unl.edu