Variable-Length Limited Feedback for Amplify-and-Forward Relay Networks

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Abstract—We study the channel quantization problem for amplify-and-forward (AF) relay networks with a sum power constraint for the relay nodes. Our target is to design a quantizer to minimize the outage probability. It is priorly known that any fixed-length quantizer with a finite-cardinality codebook cannot attain the same minimum outage probability as the case where all nodes in the AF relay networks have access to perfect channel state information (CSI). We propose a variable-length quantizer with a random infinite-cardinality codebook, and we prove that the proposed quantizer is able to achieve the full-CSI outage probability with a finite average feedback rate. Numerical simulations validate our theoretical analysis.

Keywords—amplify-and-forward, variable-length quantizer, outage probability, feedback rate

I. INTRODUCTION

Cooperative diversity techniques have received significant attention, since they can greatly enhance the spectral efficiency and extend the network coverage [1]. In a wireless relay network, the destination node receives signals from the source node with the help of relay nodes in the form of “distributed antennas”. Several cooperation strategies, such as amplify-and-forward (AF), decode-and-forward, compress-and-forward, have been proposed in the literature. Among these, AF is an attractive solution with very low complexity that requires no decoding at relay nodes.

In the case of point-to-point wireless communication, the performance of the system depends on the availability of channel state information (CSI) at the transmitter and the design of the corresponding finite-rate feedback [2]–[4]. Similarly, the performance of wireless relay networks depends on the availability of CSI at the relay nodes and the destination node [5]–[8]. The destination node can acquire the entire CSI through training sequences from the source node and relay nodes. Meanwhile, although each relay node can have the knowledge of its own receiving channel via training sequences from the source node, it does not have a direct access to the channel from itself to the destination node. Thus the relay nodes rely on the feedback information from the destination node [9]. Perfect CSI at the relay nodes requires an “infinite” number of feedback bits from the destination node, which is unrealistic due to the limitations of the feedback links. Hence, in practice, it is desired to design efficient transmission schemes based on quantized CSI for wireless relay networks.

There has been a lot of work on quantized channel feedback in wireless relay networks. In particular, in a cooperative network with a single AF relay in [5], power control methods have been analyzed to minimize the outage probability with limited feedback available at the transmitter. In a cooperative network with multiple AF relays, the capacity loss and bit error probability with quantized feedback have been studied in [6], when each relay node is subject to an individual power constraint. Also, [7] has investigated the optimal beamforming vector for relay nodes in the full-CSI scenario and the outage probability in the limited feedback scenario when the sum power constraint is imposed on the relay nodes. Compared to the full-CSI scenario where all relay nodes know the perfect CSI, the schemes in [6] and [7] always suffer from performance loss.

All of these previous schemes have relied on fixed-length quantizers (FLQs), in which the receiver feeds back the same number of bits for every channel state. In general, the receiver can send a different number of feedback bits for different channel states, resulting in a variable-length quantizer (VLQ). Recently, a VLQ has been proposed to achieve the full-CSI outage probability with a finite feedback rate for the non-cooperative setting of a multiple-input single-output (MISO) system [10]. One can thus expect that a VLQ structure will similarly offer high performance gains in cooperative networks. On the other hand, the results of [10] for MISO systems are not directly applicable to the VLQ design problem in AF relay networks. In fact, in a MISO system, the instantaneous signal-to-noise ratio (SNR) is simply given by the inner product of the beamforming and channel vectors. However, in AF networks, the SNR exhibits a highly-nonlinear dependence not only on the relay beamforming vector, but also on the channel values themselves. This leads to great difficulties in the design and performance analysis of VLQs.

We overcome these difficulties by considering a random quantizer codebook instead of the structured codebooks presented in [10]. We first prove that the outage probability of our proposed VLQ is the same as that of the full-CSI scenario. Then, we derive an upper bound on the average feedback rate of our proposed VLQ to show it is finite and small for any value of the sum power constraint. In addition, we perform numerical simulations to verify the effectiveness of our proposed VLQ.

The rest of this paper is organized as follows. The system model and problem formulation are described in Section II. The proposed VLQ with the encoding rules and an infinite-
cardinality random codebook is introduced in Section III. In Section IV, we prove the proposed VLQ achieves the same minimum outage probability as the full-CSI scenario does, and we provide an upper bound on the average feedback rate. Numerical simulations are presented in Section V, and conclusions are drawn in Section VI. We also provide some technical proofs in the appendices.

**Notation:** Bold-face letters refer to vectors or matrices. For a vector or matrix $x, x^\dagger$ represents its transpose, $x^\dagger$ represents its conjugate transpose, $||x||$ is the $l^2$-norm, and $[x]_i$ denotes its $i$-th element. The sets of complex, real, and natural numbers are denoted by $\mathbb{C}, \mathbb{R},$ and $\mathbb{N}$ respectively. $\Pr \{ \cdot \}$ and $\mathbb{E} \{ \cdot \}$ represent the probability and expectation, respectively. We use the notation $\mathcal{CN}(\mathbf{a}, \mathbf{b})$ to stand for a circularly-symmetric complex Gaussian random vector with mean of $\mathbf{a}$ and variance of $\mathbf{b}$. Similarly, $\mathbb{N}(\mathbf{a}, \mathbf{b})$ is for a real Gaussian random vector. For any $x \in \mathbb{R}$, $|x|$ is the largest integer that is less than or equal to $x$. For any $x \in \mathbb{C}$, $\text{Re}(x)$ is the conjugate, $\text{Real}(x)$ is the real part, and $\text{Imag}(x)$ is the imaginary part. For a logical statement $\text{ST}$, we let $\mathbf{1}\{\text{ST}\} = 1$ when ST is true, and $\mathbf{1}\{\text{ST}\} = 0$ otherwise. Finally, $[x_1; x_2]$ is a column vector formed by stacking two column vectors $x_1$ and $x_2$ together.

## II. Problem Formulation

In the AF relay network depicted in Fig. 1, a source node $S$ is transmitting to a destination node $D$ with the aid of $N$ AF relay nodes $R_1, \ldots, R_N$, where $N \geq 2$. Each node is equipped with only a single antenna. We assume that there is no direct link between $S$ and $D$. Denote the channel from $S$ to $R_n$ by $f_n \sim \mathcal{CN}(0,1)$ and the channel from $R_n$ to $D$ by $g_n \sim \mathcal{CN}(0,1)$. The entire channel state is represented by $\mathbf{H} = [f_1, \ldots, f_N; g_1, \ldots, g_N]^\top \in \mathbb{C}^{2N \times 1}$. We assume a quasi-static channel model, in which the channels vary independently from one block to another, while within each block the channels remain constant.

In Phase I, the received signal at the relay node $R_n$ is $y_{R_n} = \sqrt{P_n} f_n x + v_{R_n}$, where $x$ is the information bearing symbol sent by $S$ with $\mathbb{E} \{ |x|^2 \} = 1$ for each channel state, where the expectation is over all transmitted symbols, and $P_n$ is the average power at $S$. The background noises $v_{R_n}$ are independent and modeled as $\mathcal{CN}(0,1)$.

In Phase II, each relay node normalizes and retransmits its received signal. The normalized signal to be re-transmitted with unit power at $R_n$ is $x_{R_n} \equiv \frac{y_{R_n}}{\sqrt{\mathbb{E}_{\mathbb{H}} \{ |y_{R_n}|^2 \}}} = \frac{\sqrt{P_n} f_n x + v_{R_n}}{\sqrt{P_n} |f_n|^2 + 1}$.

Thereafter, $R_n$ sends $\sqrt{P_n} w_n^\top x_{R_n}$, where $P_n$ is the total transmit power shared at all relay nodes. To meet this sum power constraint, $w_n$ should satisfy $\sum_{n=1}^N |w_n|^2 = 1$, or equivalently, $||w||^2 = 1$ for the beamforming vector $w = [w_1, \ldots, w_N]^\top$. Without loss of generality, we assume $P_S = P_K = P$, while results for other values of $P_S$ and $P_K$ can be obtained similarly.

The received signal at the destination node $D$ is

$$y_{D} = \sum_{n=1}^N \sqrt{P_n} g_n w_n^\top x_{R_n} + v_{D}$$

$$= \sum_{n=1}^N \frac{P_n w_n^\top f_n x_n}{\sqrt{|f_n|^2 + \frac{1}{P}}} + \frac{\sqrt{P_n} w_n^\top g_n v_{R_n}}{\sqrt{|f_n|^2 + \frac{1}{P}}} + v_{D}$$

$$= \sqrt{P} \sum_{n=1}^N w_n^\top f_n g_n x + \tilde{v}_{D},$$

where $\tilde{v}_{D} = \sum_{n=1}^N \frac{w_n^\top g_n v_{R_n}}{\sqrt{|f_n|^2 + \frac{1}{P}}}$ and $v_{D} \sim \mathcal{CN}(0,1)$ is the background noise at $D$. For given $f_n$ and $g_n$, the noise term $\tilde{v}_{D}$ is distributed as $\tilde{v}_{D} \sim \mathcal{CN} \left(0, 1 + \sum_{n=1}^N \frac{\sqrt{P_n} |g_n|^2}{|f_n|^2 + \frac{1}{P}}\right)$. Then, the signal-to-noise ratio (SNR) at $D$ is given by

$$\Gamma(\mathbf{w}, \mathbf{H}) = \frac{1}{1 + \sum_{n=1}^N \frac{\sqrt{P_n} |g_n|^2}{|f_n|^2 + \frac{1}{P}}} = \frac{\mathbf{w}^\top \mathbf{h}^\top \mathbf{w}}{\mathbf{w}^\top (\mathbf{I} + \mathbf{D}) \mathbf{w}},$$

where $\mathbf{h} = \left[\frac{f_1}{|f_1|^2 + \frac{1}{P}}, \ldots, \frac{f_N}{|f_N|^2 + \frac{1}{P}}\right]^\top$, and $\mathbf{D}$ is a $N \times N$ diagonal matrix with the $n$-th diagonal element as $\frac{|g_n|^2}{|f_n|^2 + \frac{1}{P}}$.

We consider outage probability as the performance measure. For a target data rate $\tau$, outage occurs if the end-to-end rate at $D$ is below $\tau$, i.e., $\frac{1}{2} \log_2 (1 + \Gamma(\mathbf{w}, \mathbf{H})) < \tau$. Without loss of generality, we assume $\tau = \frac{1}{2}$ throughout this paper. Results for other values of $\tau$ can be obtained straightforwardly.

In the full-CSI scenario where the perfect knowledge of $\mathbf{H}$ is known at all relay nodes in Fig. 1, the optimal beamforming vector $\mathbf{w}^\star$ that maximizes $\Gamma(\mathbf{w}, \mathbf{H})$ is given in [7] as $\mathbf{w}^\star = \frac{\mathbf{h}}{||\mathbf{h}||^2 + \frac{1}{P}}$, and the maximum SNR is

$$\Gamma(\mathbf{w}^\star, \mathbf{H}) = \frac{N}{P} \sum_{n=1}^N \frac{|f_n|^2 |g_n|^2}{|f_n|^2 + |g_n|^2 + \frac{1}{P}} = \frac{N}{P} \sum_{n=1}^N \Gamma_n,$$
where $\Gamma_n = \frac{|f_n|^2 + |g_n|^2}{|f_n|^2 + |g_n|^2 + \eta}$. Then, the minimum outage probability is given as

$$\text{Out}(\text{Full}) = \Pr\left\{ \frac{1}{2} \log_2 (1 + \Gamma(w^*,H)) < \frac{1}{2} \right\}$$

$$= \Pr\{ \Gamma(w^*,H) < 1 \} = \Pr\left\{ \sum_{n=1}^{N} \Gamma_n < \frac{1}{P} \right\}. \quad (3)$$

In the limited-feedback scenario, we assume each relay node $R_b$ has the perfect knowledge of $[f_n]$, and the destination node $D$ knows $H$ [6], [7]. Define $\mathcal{W} \triangleq \{ w : w \in C^{N\times 1}, |w| = 1 \}$. For an arbitrary quantizer $Q : C^{2N\times 1} \rightarrow \mathcal{W}$, the destination node $D$ maps $H$ into $Q(H) \in \mathcal{W}$, and feeds back the index of $Q(H)$ to the relay nodes. The relay nodes decode the index and use $Q(H)$ as the beamforming vector. The resulting SNR is $\Gamma(Q(H),H)$, and the corresponding outage probability is

$$\text{Out}(Q) = \Pr\{ \Gamma(Q(H),H) < 1 \}. \quad (4)$$

In the subsequent sections, we will show the full-CSI outage probability $\text{Out}(\text{Full})$ in (3) can be achieved by our proposed VLQ, and the corresponding average feedback rate is finite.

III. VARIABLE-LENGTH LIMITED FEEDBACK BASED ON RANDOM CODEBOOKS

For a given channel state $H$, we propose a VLQ associated with the random codebook $\{ w_i \}_N$, where $w_i \in \mathcal{W}$ is independent and identically distributed with a uniform distribution on $\mathcal{W}$ for $i \in \mathbb{N}$ [11]. The random codebook provides a performance benchmark since if certain performance is attained on average, one deterministic codebook can be found to surpass this performance.

For a realization of $\{ w_i \}_N$, the proposed VLQ is represented by

$$Q_{vl} = \{ w_i, \mathcal{I}_i, b_i \}, \quad (4)$$

where $\mathcal{I}_i$ denotes the channel partition region of $w_i$ for $i \in \mathbb{N}$, $w_i$ is used as the relay beamforming vector when $H \in \mathcal{I}_i$, and $b_i$ is the feedback binary string representing the index of $w_i$. Different from the channel partition regions of FLOs which consist of channel states that achieve the best performance with the "centroid" codeword, $\mathcal{I}_0$ in $Q_{vl}$ is set as

$$\mathcal{I}_0 = \{ H : \Gamma(w_0,H) \geq 1 \} \cup \bigcap_{i \in \mathbb{N}} \{ H : \Gamma(w_i,H) < 1 \}, \quad (5)$$

and $\mathcal{I}_i$ for $i \in \mathbb{N} - \{0\}$ is set as

$$\mathcal{I}_i = \{ H : \Gamma(w_i,H) \geq 1 \} \cap \bigcap_{k=0}^{i-1} \{ H : \Gamma(w_k,H) < 1 \}. \quad (6)$$

For $i \in \mathbb{N}$, $\{ H : \Gamma(w_i,H) \geq 1 \}$ is the set of channels that are in non-outage when $w_i$ is adopted as the beamforming vector, and $\{ H : \Gamma(w_i,H) < 1 \}$ is its complementary set. For a channel state with $\Gamma(w^*,H) < 1$, any beamforming vector will lead to outage. Thus $Q_{vl}$ simply selects the first codeword $w_0$ as the beamforming vector. For a channel state with $\Gamma(w^*,H) \geq 1$, $Q_{vl}$ checks each codeword in $\{ w_i \}_N$ sequentially until it finds some $w_i$ with $\Gamma(w_i,H) \geq 1$. In terms of outage probability, the contribution of such $w_i$ will be identical to that of the optimal beamforming vector $w^*$.

Variable-length coding is applied to encode the indices of $w_i$ for $i \in \mathbb{N}$. Concretely, we let $b_0 = \{0\}$, $b_1 = \{1\}$, $b_2 = \{00\}$, $b_3 = \{01\}$ and sequentially so on for all binary strings in the set $\{0,1,00,01,10,11,\ldots\}$. The length of $b_i$ is $\log_2(i+2)$.

With the random codebook $\{ w_i \}_N$, the outage probability and average feedback rate of $Q_{vl}$ are given by

$$\text{Out}(Q_{vl}) = \mathbb{E}_{H} \mathbb{E}_{w_i} \Pr\{ \Gamma(w_i,H) < 1, \forall i \in \mathbb{N} \}$$

$$= E_H E_{\{w_i\}_N} \left[ \left\{ \Gamma(w_i,H) < 1, \forall i \in \mathbb{N} \right\} \right], \quad (7)$$

$$\text{FR}(Q_{vl}) = \sum_{i=0}^{\infty} |\log_2(i+2)| \times \Pr\{ H \in \mathcal{I}_i \}$$

$$= \sum_{i=0}^{\infty} |\log_2(i+2)| \times E_H E_{\{w_i\}_N} \left[ \left\{ 1 \{ H \in \mathcal{I}_i \} \right\} \right]. \quad (8)$$

In the following section, we will show that the outage probability of $Q_{vl}$ in (7) is equal to the full-CSI outage probability in (3), and the average feedback rate in (8) is finite.

IV. MAIN RESULTS

A. Outage Optimality

To prove the proposed VLQ achieves the full-CSI outage probability in (3), we need the following two lemmas. Lemmas 1 and 2 are proved in Appendices A and B, respectively.

**Lemma 1.** If $\Gamma(w^*,H) > 1$, there exists $\Pi \in (0,1)$ such that for any $w \in \mathcal{W}$ with $|w - w^*| \leq \Pi$, $\Gamma(w,H) \geq 1$ holds. The value of $\Pi$ can be

$$\Pi = \frac{\Gamma(w^*,H) - 1}{2\sqrt{N}P \left( \sum_{n=1}^{N} |g_n|^2 \right)^2 \left( 1 + \sum_{n=1}^{N} \frac{|g_n|^2}{|f_n|^2 + \eta} \right)}. \quad (9)$$

**Lemma 2.** Let $\mathcal{W}_R \triangleq \{ w_R : w_R \in \mathbb{R}^{2N\times 1}, |w_R| = 1 \}$. For a fixed real vector $u \in \mathcal{W}_R$, a real number $0 \leq t \leq 1$, and a random real vector $v$ which is uniformly distributed on the real unit sphere $\mathbb{S}_R$, we have

$$\Pr\{ \frac{u^\top v}{t} \geq 1 \} = \frac{1}{2} \Gamma_1 - \frac{1}{2} \Gamma_2$$

where $\Gamma_1(a,b) = \frac{1}{\beta(a,b)} \int_0^1 x^{a-1} (1-x)^{b-1} \, dx$ is the regularized incomplete beta function, $\beta(a,b) = \int_0^1 x^{a-1} (1-x)^{b-1} \, dx$ is the beta function [12].

**Theorem 1.** For any $P > 0$, we have

$$\text{Out}(Q_{vl}) = \text{Out}(\text{Full}). \quad (10)$$

[1] One possible procedure of revealing the knowledge of $H$ to the destination node $D$ can be found in [6].

[2] The proposed VLQ in (4) can be extended to the scenario with prefix-free coding. We do not provide those results in this paper due to space limitations.
In the following, we provide an intuitive explanation of the result in Theorem 1. For a channel state \( \mathbf{H} \) with \( \Gamma(\mathbf{w}^*,\mathbf{H}) > 1 \), the full-CSI performance is achieved by \( \mathbf{w}^* \). To achieve the same performance, one should use a unit-normal vector \( \mathbf{w} \in \mathbb{W} \), which is “close” enough to \( \mathbf{w}^* \) such that \( \Gamma(\mathbf{w},\mathbf{H}) \geq 1 \). We show that there exists a non-zero probability region in the unit sphere, where all the unit-normal vectors in the region result in non-outage. However, to “closely” represent \( \mathbf{w}^* \) for any such \( \mathbf{H} \), we need infinitely many codewords in \( \{\mathbf{w}\}_\mathcal{N} \) to ensure at least one efficient vector in that region is available to make \( \mathbf{H} \) non-outage. Obviously, a FLQ with a finite feedback rate cannot include infinitely many codewords. Whereas our proposed VLQ in (4) includes infinitely many codewords to achieve the full-CSI outage probability while providing a finite average feedback rate.

B. Average Feedback Rate

Theorem 2 presents an upper bound on the average feedback rate of our proposed quantizer \( Q_{\mathcal{A}} \). A proof sketch is given here due to space limitations.

**Theorem 2.** For any \( P > 0 \), we have

\[
\text{FR}(Q_{\mathcal{A}}) \leq C_0 + C_1 e^{-\frac{\beta}{P}} \left( \frac{1}{P} + \frac{1}{P^{N+1}} \right),
\]

where \( C_0, C_1 > 0 \) are constants independent of \( P \).

**Sketch of Proof:** Noticing that \( \log_2(i+2) \leq \log_2(i+1) \leq \log_2(2P+2) = 1 + \log_2(P+1) \), FR(\( Q_{\mathcal{A}} \)) in (8) can be upper-bounded by

\[
\text{FR}(Q_{\mathcal{A}}) \leq \sum_{i=0}^{\infty} \text{Pr}\{ \mathbf{H} \in \mathcal{S}_i \} + \sum_{i=0}^{\infty} \log_2(i+1) \times \text{Pr}\{ \mathbf{H} \in \mathcal{S}_i \}
= 1 + \sum_{i=1}^{\infty} \log_2(i+1) \times \text{Pr}\{ \mathbf{H} \in \mathcal{S}_i \}. \tag{12}
\]

For any given \( \mathbf{H} \) and the random beamforming vector \( \mathbf{w} \), let

\[ p = \text{Pr}\{ \Gamma(\mathbf{w},\mathbf{H}) < 1 \}. \]

From the encoding rules in (6), \( \text{Pr}\{ \mathbf{H} \in \mathcal{S}_i \} \) is equal to \( p^i \times (1-p) \) for \( i \in \mathbb{N} - \{0\} \) and given \( \mathbf{H} \) with \( \Gamma(\mathbf{w}^*,\mathbf{H}) \geq 1 \). Then, it follows from (12) that

\[
\text{FR}(Q_{\mathcal{A}}) \leq 1 + \int_{\mathcal{H}} \Psi f_H(\mathbf{H}) \, d\mathbf{H}, \tag{13}
\]

where

\[
\mathcal{H} = \{ \mathbf{H} : \mathbf{H} \in \mathbb{C}^{2N\times1}, \Gamma(\mathbf{w}^*,\mathbf{H}) \geq 1 \},
\]

\[
\Psi = \sum_{i=1}^{\infty} p^i (1-p) \times \log_2(i+1).
\]

Using [13, Lemma 1], we obtain

\[
\Psi \leq \frac{p(1-p)}{\log_2 2} + \left( \frac{6}{\log_2 2} + 2 \right) \frac{p^2}{\log_2 2} \times \frac{p^2 \log 1 - p}{\log_2 2} \leq C_2 + C_3 \log \frac{1}{1-p}, \tag{14}
\]

where \( C_2 = \frac{6}{\log_2 2} + 3 \) and \( C_3 = \frac{2}{\log_2 2} \). Substituting (14) into (13) yields

\[
\text{FR}(Q_{\mathcal{A}}) \leq 1 + \int_{\mathcal{H}} \Psi f_H(\mathbf{H}) \, d\mathbf{H}

= 1 + C_2 \int_{\mathcal{H}} f_H(\mathbf{H}) \, d\mathbf{H} + C_3 \int_{\mathcal{H}} \log \frac{1}{1-p} f_H(\mathbf{H}) \, d\mathbf{H}

\leq C_4 + C_3 \int_{\mathcal{H}} \log \frac{1}{1-p} f_H(\mathbf{H}) \, d\mathbf{H}, \tag{15}
\]

where \( C_4 = 1 + C_2 \).

The following lemma provides an upper bound on \( p \).

**Lemma 3.** **We have**

\[
p \leq 1 - \frac{\left(1 - \left(1 - \frac{1}{2N}\right)^2\right)^{2N-1}}{(2N-1) \times \beta(\frac{2}{2N} + 1)},
\]

where \( \Pi \) is given in (9), and \( \beta(a,b) = \int_{0}^{1} x^{a-1}(1-x)^{b-1} \, dx \) is the beta function.

We can obtain the upper bound on FR(\( Q_{\mathcal{A}} \)) in (11) by substituting the upper bound in Lemma 3 into (15), doing mathematical calculations, and applying the following lemma (the proof is omitted due to space limitations).

**Lemma 4.** For \( N \geq 2 \), the pdf of \( \frac{\Gamma(\mathbf{w},\mathbf{H})}{p} = \sum_{n=1}^{N} \frac{f_{\mathbb{D}}(\mathbf{H})}{f_{\mathbb{D}}(\mathbf{H}) + \Gamma(\mathbf{w},\mathbf{H})} \) is upper-bounded by

\[
f_{x_{\mathbb{D}}}(x) \leq e^{-x} \left[ E_0 x^{N-1} + E_1 \left( \frac{1}{p^{N-1}} + \frac{p}{p^N} \right) + 1 \right]_{\{N \geq 3\}} \times E_2 \sum_{m=1}^{N-2} \left( \frac{x^{m}}{p^{N-m-1} + p^{N-m}} \right),
\]

where \( E_0, E_1, E_2 > 0 \) are constants independent of \( P \).

Since \( e^{-\frac{\beta}{P}} \left( \frac{1}{p} + \frac{1}{p^{N+1}} \right) \) in (11) is bounded for any \( P > 0 \), the average feedback rate of our proposed quantizer \( Q_{\mathcal{A}} \) is finite. As shown in the numerical simulations, the average feedback rate of \( Q_{\mathcal{A}} \) can actually be very small.

V. Numerical Simulations

In this section, we present numerical simulations to verify the theoretical results for the outage probability and the average feedback rate of our proposed quantizer \( Q_{\mathcal{A}} \).

For each value of the sum power constraint \( P \), a sufficiently large number of channel realizations are generated to observe at least 1,000 outage events. For each channel realization with non-outage in the full-CSI case, a random unit-normal vector \( \mathbf{w} \) is generated repeatedly until one that helps the channel realization avoid outage is found. With such settings, the simulated feedback rate in Fig. 2 is computed as the average number of feedback bits; the simulated outage probability in Fig. 3 is computed as the number of outage incidents divided by the number of all channel realizations. In the simulations, we observe no endless iteration when generating \( \mathbf{w} \) in any channel realization.
Fig. 2. Simulated average feedback rates of \( Q_{\text{FL}} \) when \( N = 2, 3, 4 \) (\( N \) is the number of relay nodes).

Fig. 2 shows the simulated average feedback rates when \( N = 2, 3, 4 \). It can be seen that the average feedback rate is finite and small for any \( P \); it is no larger than 2, 2 or 3 bits when \( N = 2, 3 \) or 4, respectively.

In Fig. 3, we compare the outage probabilities of \( Q_{\text{FL}} \) and the FLQ in [7] denoted by \( Q_{\text{FL}} \). For a given \( H \), \( Q_{\text{FL}} \) utilizes \( B \) bits for quantization based on the random codebook \( \{w_i\}_{i=0, \ldots, 2^B - 1} \), and the chosen beamforming vector is \( Q_{\text{FL}}(H) = \text{argmax}_{w \in \{w_i\}_{i=0, \ldots, 2^B - 1}} \Gamma(w, H) \). Thus the feedback rate of \( Q_{\text{FL}} \) is \( B \) bits per channel state. In the simulations, we let \( B = 2, 3, 4 \) for \( N = 2, 3, 4 \), which are larger than the average feedback rates of \( Q_{\text{FL}} \) shown in Fig. 2. It can be seen that \( Q_{\text{FL}} \) has shown great improvement in outage probability compared with \( Q_{\text{FL}} \), which validates that \( Q_{\text{FL}} \) is superior to \( Q_{\text{FL}} \).

\[ \Gamma(H) = \sum_{k=1}^{N} \Gamma_k(H) = \sum_{k=1}^{N} \left[ \Gamma(w^{(k-1)}, H) - \Gamma(w^{(k)}, H) \right], \]

where \( w^{(0)} = w^* \), \( w^{(k)} = [\hat{w}_1, \ldots, \hat{w}_k, |w|_{k+1}, \ldots, |w|_N]^T \) and \( w^{(N)} = w \). Let \( f_n = \frac{\ln 2}{|w|^2} \) and

\[ A_k = \sum_{n=1}^{N} \left[ w^{(k-1)} \right]_n^* f_n g_n \sqrt{f_n}, \]

\[ B_k = 1 + \sum_{n=1}^{N} \left| w^{(k-1)} \right|_n^2 |g_n|^2 f_n, \]

\[ \hat{A}_k = \sum_{n=1}^{N} \left| w^{(k)} \right|_n^* f_n g_n \sqrt{f_n} = A_k - \left| \left[ w^* \right]_k - \left| w \right|_k \right| f_k g_k \sqrt{f_k}, \]

\[ \hat{B}_k = 1 + \sum_{n=1}^{N} \left| w^{(k)} \right|_n^2 |g_n|^2 f_n = B_k - \left( \left| \left[ w^* \right]_k \right|^2 - \left| \left| w \right|_k \right|^2 \right) g_k^2 f_k. \]

From (1), \( \Gamma_k(H) = P \frac{|A_k|^2}{B_k} - P \frac{|A_k^*-|w_k|^2/|g_k|^2 f_k|^2}{B_k} \) is expanded as

\[
\Gamma_k(H) = \frac{P |A_k|^2}{B_k} - \frac{P |A_k^*-|w_k|^2/|g_k|^2 f_k|^2}{B_k}.
\]

Since \( P \frac{|\left| w \right|_k|^2 - |w_k|^2/|g_k|^2 f_k|^2}{B_k} \geq 0 \), we obtain

\[
\Gamma_k(H) \leq \frac{P |A_k|^2}{B_k} - \frac{P |A_k^*-|w_k|^2/|g_k|^2 f_k|^2}{B_k} + 2P \frac{\text{Re} \left\{ A_k^* (\left| w \right|_k^2 - |w_k|^2/|g_k|^2 f_k) \right\}}{B_k}.
\]

VI. Conclusion

In this paper, we have proposed a VLQ for the AF relay networks subject to the sum power constraint for relay nodes. We have proved that the proposed VLQ can achieve the full-CI outage probability with a finite average feedback rate. In the future, we intend to work on a VLQ for the AF relay networks with an individual power constraint for each relay node to achieve the full-CI outage probability with a finite average feedback rate.

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Appendix A: Proof of Lemma 1

For any \( w \in \mathcal{W} \), in order to bound the difference between \( \Gamma(w^*, H) \) and \( \Gamma(w, H) \), we use the idea of successively altering each component of \( w^* \) until we reach \( w \), while keeping track of the SNR variation at each step of the alteration [6, Appendix B]. Thus \( \Gamma(H) = \Gamma(w^*, H) - \Gamma(w, H) \) can be decomposed as

\[ \hat{A}_k = \sum_{n=1}^{N} \left| w^{(k)} \right|_n^* f_n g_n \sqrt{f_n} = A_k - \left| \left[ w^* \right]_k - \left| w \right|_k \right| f_k g_k \sqrt{f_k}, \]

\[ \hat{B}_k = 1 + \sum_{n=1}^{N} \left| w^{(k)} \right|_n^2 |g_n|^2 f_n = B_k - \left( \left| \left[ w^* \right]_k \right|^2 - \left| \left| w \right|_k \right|^2 \right) g_k^2 f_k. \]

From (1), \( \Gamma_k(H) = P \frac{|A_k|^2}{B_k} - P \frac{|A_k^*-|w_k|^2/|g_k|^2 f_k|^2}{B_k} \) is expanded as

\[ \Gamma_k(H) = \frac{P |A_k|^2}{B_k} - \frac{P |A_k^*-|w_k|^2/|g_k|^2 f_k|^2}{B_k}.
\]

Since \( P \frac{|\left| w \right|_k|^2 - |w_k|^2/|g_k|^2 f_k|^2}{B_k} \geq 0 \), we obtain

\[
\Gamma_k(H) \leq P \frac{|A_k|^2}{B_k} - P \frac{|A_k^*-|w_k|^2/|g_k|^2 f_k|^2}{B_k} + 2P \frac{\text{Re} \left\{ A_k^* (\left| w \right|_k^2 - |w_k|^2/|g_k|^2 f_k) \right\}}{B_k}.
\]
\[+ 2P|A_k|\|w^*\|_k - |w^*\|_k \cdot |f_k| |g_k| \sqrt{f_k} \]

where (a) is from the inequality \( |c_1|^2 - |c_2|^2 | \leq |c_1 - c_2|^2 \) for \( c_1, c_2 \in \mathbb{C} \) (proof is omitted); (b) is from \( B_k \geq 1 \) and \( \hat{B}_k \geq 1 \). Furthermore, since \( \|w^*\|_k + |w^*|_k \leq \|w^*\|_k + |w^*|_k \leq 2, \|f_k\| \sqrt{f_k} \leq 1 \) and \( |A_k| \leq \sum_{n=1}^{N} \|w^{(k-1)}\|_n \times |f_n||g_n| \sqrt{f_n} \leq \sum_{n=1}^{N} |f_n||g_n| \sqrt{f_n} \), it follows from (16) that

\[
\tilde{\Gamma}(H) \leq 2P \left( \sum_{n=1}^{N} |g_n| \right)^2 \left( 1 + \sum_{n=1}^{N} |g_n|^2 \right) \sum_{n=1}^{N} \|w^*\|_k - |w^*\|_k \times |f_k| |g_k| \]

Hence, \( \tilde{\Gamma}(H) \) is upper-bounded by

\[
\Gamma(H) \leq 2P \left( \sum_{n=1}^{N} |g_n| \right)^2 \left( 1 + \sum_{n=1}^{N} |g_n|^2 \right) \sum_{n=1}^{N} \|w^*\|_k - |w^*\|_k \times |f_k| |g_k| \]

When \( \Gamma(w^*, H) - 1 > 0 \), letting \( \Pi = \Gamma(w^*, H) - 1 \). If \( \|w^* - w\| \leq \Pi \), we obtain \( \Gamma(w, H) = \Gamma(w^*, H) - \tilde{\Gamma}(H) \geq \Gamma(w^*, H) - \Xi \times |w^* - w| \geq \Gamma(w^*, H) - \Xi \times \Pi = 1 \).

To complete the proof, let us verify \( 0 < \Pi < 1 \): (i) since \( \Gamma(w^*, H) - 1 > 0 \) and \( \Xi > 0 \), \( \Pi > 0 \); (ii) since \( \Gamma(w^*, H) = P \sum_{n=1}^{N} \|f_n\|^2 |g_n|^2 \frac{1}{|f_n|^2 + |g_n|^2 + \frac{1}{P}} < P \sum_{n=1}^{N} |g_n|^2 \) \( \frac{1}{|f_n|^2 + |g_n|^2 + \frac{1}{P}} < P \sum_{n=1}^{N} |g_n|^2 \). \( \Pi < \frac{\Gamma(w^*, H)}{\Xi} < \frac{1}{\Xi} \frac{\sum_{n=1}^{N} |g_n|^2 \left( 1 + \sum_{n=1}^{N} |g_n|^2 \right) + \frac{1}{P}}{2\sqrt{N} \left( 1 + \sum_{n=1}^{N} |g_n|^2 \right)} \times |w^* - w|. \)

**APPENDIX B: PROOF OF LEMMA 2**

Similar to [14, Eqs.(23)-(24)], \( \Pr \{u^\top v \geq t\} = \frac{S_{2N,t,\text{cap}}}{S_{2N}} \), where \( S_{2N,t,\text{cap}} \) is the surface area of the spherical cap formed by the intersection of the subspace \( u^\top v \geq t \) and the real unit hyper-sphere \( \mathbb{H}^N_R \). Moreover, from [15], we have \( S_{2N} = \frac{2N^N}{(N-1)!} \) and \( S_{2N,t,\text{cap}} = \frac{N^N}{(N-1)!} \ight( \frac{2N-1}{2} \right) \left( \frac{1}{2} \right) \). Thus Lemma 2 is obtained.