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Weakly Connected Total Domination in Graphs

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Abstract

Let $G = (V(G), E(G))$ be a connected undirected graph. The *closed neighborhood* of any vertex $v \in V(G)$ is $N_G[v] = \{u \in V(G) : uv \in E(G)\} \cup \{v\}$. For $C \subseteq V(G)$, the *closed neighborhood* of C is $N[C] = \cup_{v \in C} N_G[v]$. A set $S \subseteq V(G)$ is a *total dominating set* of G if for each $x \in V(G)$, there exists $y \in S$ such that $xy \in E(G)$, that is, $N(S) = V(G)$. A total dominating set $S \subseteq V(G)$ is a *weakly connected total dominating set* of a connected graph G if the subgraph $\langle S \rangle_w = (N_G(S), E_w)$ weakly induced by S is connected, where E_w is the set of all edges with at least one vertex in S. The *weakly connected total domination number* of G, denoted by $\gamma_{wt}(G)$, is the minimum cardinality among all weakly connected total dominating sets of G.

In this paper, the weakly connected total dominating sets in graphs resulting from some binary operations are characterized. As consequences, the weakly connected total domination number of the aforementioned graphs are determined.

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1 Introduction

Let $G = (V(G), E(G))$ be a connected undirected graph. For any vertex $v \in V(G)$,the *open neighborhood* $N(v)$ of v is $\{u \in V : uv \in E(G)\}$. The *closed neighborhood* $N[v]$ of v is that set $N(v) \cup \{v\}$. For a set $X \subseteq V(G)$, the *open neighborhood* $N(X)$ of X is $\cup_{v \in X} N(v)$ and the *closed neighborhood* $N[X]$ is $\cup_{v \in X} N[v]$.

A set $S \subseteq V(G)$ is a *total dominating set* of G if for each $x \in V(G)$, there exists $y \in S$ such that $xy \in E(G)$, that is, $N(S) = V(G)$. The *total domination number* of G, denoted by $\gamma_t(G)$, is the minimum cardinality among all total dominating sets of G. A total dominating set $S \subseteq V(G)$ is a *weakly connected total dominating set* of a connected graph G if the subgraph $\langle S \rangle_w = (N_G(S), E_w)$ weakly induced by S is connected, where E_w is the set of all edges with at least one vertex in S. The *weakly connected total domination number* of G, denoted by $\gamma_{wt}(G)$, is the minimum cardinality among all weakly connected total dominating sets of G.

A subset C of $V(G)$ is *dominating* in G if $N[C] = V$, or equivalently, if every vertex $u \in V \setminus D$, there exists $v \in D$ such that $uv \in E$. The minimum cardinality among all dominating sets in G is denoted by $\gamma(G)$. A dominating set $C \subseteq V(G)$ is a *weakly connected dominating* set in G if the subgraph $\langle C \rangle_w = (N_G[C], E_W)$ weakly induced by C is connected, where E_W is the set of all edges with at least one vertex in C. The *weakly connected domination number* $\gamma_w(G)$ of a graph G is the minimum cardinality among all weakly connected dominating sets of G.

Let G_1 and G_2 be two graphs and r a natural number less than or equal to the minimum of $\omega(G_1)$ and $\omega(G_2)$. These graphs have a copy of K_r as a subgraph and the graph obtained from G_1 and G_2 by identifying these two copies of K_r is called a *K_r-gluing* of G_1 and G_2. When $r = 0$, the gluing is just the disjoint union of the two graphs.

Let G and H be graphs of order m and n, respectively. The *corona* $G \circ H$ of G and H is the graph obtained by taking one copy of G and m copies of H, and then joining the *ith* vertex of G to every vertex of the *ith* copy of H.

The *join* of two graphs G and H, denoted by $G + H$, is the graph with vertex-set $V(G+H) = V(G) \cup V(H)$ and edge-set $E(G+H) = E(G) \cup E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

Dunbar et al. in [2], investigated the concept of weakly connected domination of graphs. The weakly connected total domination number of a connected graph has been introduced in [4].

In this paper, the weakly connected total domination number of some graphs are determined. Also, the weakly connected total dominating sets in graphs resulting from some binary operations are characterized and subsequently, the weakly connected total domination number of these graphs are obtained. This study can provide a better understanding on the topic of weakly connected total domination in graphs. Moreover, the results that will be generated in this study can contribute to the fast growing development of the theory of domination in graphs.

2 Weakly Connected Total Dominating Sets in Some Graphs

Remark 2.1 $2 \leq \gamma_t(G) \leq \gamma_{wt}$ *for any connected graph* G *with* $|V(G)| \geq 2$.

Remark 2.2 *For a complete graph* K_n , $\gamma_{wt}(K_n) = 2$ *for all* $n \geq 2$.

Lemma 2.3 *Let* G *be a connected graph of order* $n \geq 2$. *If* G *has a minimum total dominating set* S *such that* $\langle S \rangle$ *is connected, then* $\gamma_{wt}(G) = \gamma_t(G)$.

Proof. Let $x, y \in N_G[S] = V(G)$, where $x \neq y$. If $x, y \in S$, then there is a path in $\langle S \rangle$ connecting x and y because $\langle S \rangle$ is connected. If $x \in S$ and $y \notin S$, let $z \in S$ such that $yz \in E(G)$. If $x = z$, then x and y is connected by an edge in $\langle S \rangle_w$. If $x \neq z$, then x and z is connected by a path in $\langle S \rangle$. Thus, x and y is connected by a path in $\langle S \rangle_w$. Finally, if $x, y \notin S$, then choose $u, v \in S$ such that $xu, yv \in E(G)$. If $u = v$, then x and y is connected by the path $[x, u, y]$ in $\langle S \rangle_w$. If $u \neq v$, then u and v are connected by a path in $\langle S \rangle$. Hence, x and y are connected by a path in $\langle S \rangle_w$. Thus, S is a weakly connected total dominating set in G and $\gamma_{wt}(G) \leq |S|$. Since $|S| = \gamma_t(G) \leq \gamma_{wt}(G)$, it follows that $\gamma_{wt}(G) = \gamma_t(G)$. ■

The following lemma is found in [5].

Lemma 2.4 *Let* G *be a connected graph. A subset* S *of* $V(G)$ *is weakly disconnected (i.e.* $\langle S \rangle_w$ *is not connected) if and only if the following property is satisfied: (N) There exist* $x, y \in S$ *with* $x \neq y$ *such that* $N_G[x] \cap N_G[y] = \emptyset$ *and for every* x*-*y *path* $P = [x, a_1, a_2, ..., a_k, y]$ *in* G*, there exists* $i \in \{1, 2, ..., k-1\}$ *with* $a_i, a_{i+1} \in V(P) \backslash S$.

Theorem 2.5 *Let* G *be a connected graph of order* $n \geq 2$. *Then* $\gamma_{wt}(G) = 2$ *if and only if* $\gamma_t(G) = 2$.

Proof. If $\gamma_{wt}(G) = 2$, then $\gamma_t(G) = 2$ since $2 \leq \gamma_t(G) \leq \gamma_{wt}(G)$. Suppose $\gamma_t(G) = 2$ and let $S = \{a, b\}$ be a total dominating set of G. Then $\langle S \rangle$ is connected. By Lemma 2.3, $\gamma_{wt}(G) = \gamma_t(G) = 2$. ■

Theorem 2.6 *Let G be a connected graph of order $n \geq 2$ such that $\gamma_t(G) \neq 2$. Then $\gamma_{wt}(G) = 3$ if and only if $\gamma_t(G) = 3$.*

Proof. Suppose $\gamma_{wt}(G) = 3$. Since $3 \leq \gamma_t(G) \leq \gamma_{wt}(G)$, $\gamma_t(G) = 3$.

Next, suppose $\gamma_t(G) = 3$. Let $S = \{a, b, c\}$ be a total dominating set in G. Then $\langle S \rangle$ is connected. By Lemma 2.3, $\gamma_{wt}(G) = \gamma_t(G) = 3$. ■

Corollary 2.7 *If G is the K_r-gluing of K_p and K_q, $2 \leq r \leq p \leq q$, then $\gamma_{wt}(G) = 2$.*

Proof. Pick $x, y \in V(K_r)$ and let $S = \{x, y\}$. Then S is a total dominating set in G. Thus $\gamma_t(G) = 2$. By Theorem 2.5, $\gamma_{wt}(G) = 2$. ■

Corollary 2.8 *Let G be a graph of order $n \geq 3$ obtained from the complete graph K_n by deleting an edge. Then $\gamma_{wt}(G) = 2$.*

Proof. Suppose that G is a graph obtained from the complete graph K_n by deleting an edge, say $e = \{x, y\}$. Pick $v \in V(G) \setminus \{x, y\}$ and let $S = \{x, v\}$. Then S is a total dominating set in G; hence $\gamma_t(G) = 2$. Therefore, by Theorem 2.5 $\gamma_{wt}(G) = 2$. ■

Theorem 2.9 *Let Ω be the set of independent edges of K_n and let $G = K_n \setminus \Omega$. Then $\gamma_{wt}(G) = 2$.*

Proof. Pick $x, y \in V(G)$ with $xy \in E(G)$ and let $S = \{x, y\}$. Let $z \in V(G)$. Since Ω is an independent set, $xz \in E(G)$ or $yz \in E(G)$. Thus, S is a total dominating set in G. By Theorem 2.5, $\gamma_{wt}(G) = \gamma_t(G) = 2$. ■

Corollary 2.10 *Let p and q be positive integers such that $2 \leq p \leq q$. If G is a graph obtained from the complete graph K_q by deleting edges of K_p, then $\gamma_{wt}(G) = 2$.*

Proof. Let $x, y \in V(G)$ with $xy \in E(G)$ and put $S = \{x, y\}$. Then S is a total dominating set G and $\gamma_t(G) = 2$. By Theorem 2.5, $\gamma_{wt} = 2$. ■

3 Weakly Connected Total Domination in the Join of Graphs

Theorem 3.1 *Let G and H be graphs. A subset S of $V(G+H)$ is a weakly connected total dominating set if and only if one of the following holds:*

(*i*) *$S \subseteq V(G)$ and S is a weakly connected total dominating set in G.*

(*ii*) *$S \subseteq V(H)$ and S is a weakly connected total dominating set in H.*

(*iii*) *$S \cap V(G) \neq \emptyset$ and $S \cap V(H) \neq \emptyset$.*

Proof. Suppose S is a weakly connected total dominating set in $G+H$. Suppose $S \cap V(G) = \emptyset$ or $S \cap V(H) = \emptyset$. Then we have the following cases:

Case 1: Suppose $S \cap V(H) = \emptyset$. Then $S \subseteq V(G)$ and $\langle S \rangle_w$ is connected in G. Let $x \in V(G)$. Since S is a total dominating set in $G+H$, there exists $u \in S$ such that $ux \in E(G+H)$. Since $S \cap V(H) = \emptyset$, it follows that $u \in V(G)$ and $ux \in E(G)$. Hence, S is a weakly connected total dominating set in G.

Case 2: Suppose $S \cap V(G) = \emptyset$. Then $S \subseteq V(H)$ and $\langle S \rangle_w$ is connected in H. Let $x \in V(H)$. Since S is a total dominating set in $G+H$, there exists $y \in S$ such that $xy \in E(G+H)$. Then $y \in V(H)$ since $S \cap V(G) = \emptyset$. Hence, $xy \in E(H)$. Thus, S is a weakly connected total dominating set in H.

If (i) and (ii) do not hold, then $S \cap V(G) \neq \emptyset$ and $S \cap V(H) \neq \emptyset$.

The converse is an immediate consequence of the definition of $G+H$. ■

Corollary 3.2 *Let G and H be graphs. Then $\gamma_{wt}(G+H) = 2$.*

Proof. Pick $x \in V(G)$ and $y \in V(H)$ and set $S = \{x, y\}$. By Theorem 3.1 (iii), S is a weakly connected total dominating set in $G+H$. Hence, by Remark 2.1 $\gamma_{wt}(G+H) = |S| = 2$. ■

The next two results follow directly from Theorem 3.1.

Corollary 3.3 *Let G be a graph and $n \geq 1$. Then $\gamma_{wt}(K_n + G) = 2$, for all $n \geq 1$.*

Corollary 3.4 *Let m and n be positive integers. Then*

(*i*) $\gamma_{wt}(F_n) = 2$, $n \geq 2$.

(*ii*) $\gamma_{wt}(F_{m,n}) = 2$, $m, n \geq 2$.

(*iii*) $\gamma_{wt}(W_n) = 2$, $n \geq 3$.

(*iv*) $\gamma_{wt}(W_{m,n}) = 2$, $m \geq 2, n \geq 3$.

(*v*) $\gamma_{wt}(K_{m,n}) = 2$, $n \geq 2$.

4 Weakly Connected Total Domination in the Corona of Graphs

Theorem 4.1 *Let G be a connected graph of order $m \geq 2$ and let H be any graph of order $n \geq 1$. Then $C \subseteq V(G \circ H)$ is a weakly connected total dominating set in $G \circ H$ if and only if $V(G) \cap C$ is a weakly connected dominating set in G and for every $v \in V(G)$ one of the following holds:*

(i) $v \notin C$, $N_G(v) \cap C \neq \emptyset$ *and* $V(H^v) \cap C$ *is a total dominating set in* H^v*;*

(ii) $\{v\} = V(v + H^v) \cap C$ *and* $N_G(v) \cap C \neq \emptyset$*;*

(iii) $v \in C$ *and* $V(H^v) \cap C \neq \emptyset$.

Proof. Suppose C is a weakly connected total dominating set in $G \circ H$. Suppose $V(G) \cap C = \emptyset$. Then $\langle C \rangle_w$ will have $|V(G)|$ components contrary to the fact that $\langle C \rangle_w$ is connected. Thus $V(G) \cap C \neq \emptyset$. Suppose there exists $v \in V(G) \backslash C$ such that $uv \notin E(G \circ H)$ for all $u \in V(G) \cap C$. Then $v + H^v$ contains a component of $\langle C \rangle_w$. Since $m \geq 2$, $\langle C \rangle_w$ has at least two components, contrary to our assumption. This implies that $V(G) \cap C$ is a dominating set in G. If $\langle V(G) \cap C \rangle_w$ is not connected, then, by Lemma 2.4, there exist $x, y \in V(G) \cap C$ with $x \neq y$ such that $N_G[x] \cap N_G[y] = \emptyset$ and for any x-y path $P = [x, x_1, x_2, ..., x_k, y]$ in G, there exists $i \in \{1, 2, ..., k-1\}$ such that $x_i, x_{i+1} \in V(P) \backslash (V(G) \cap C)$. Since the x-y paths in G are exactly the x-y paths in $G \circ H$, it follows that $\langle C \rangle_w$ is not connected. Again, this contradicts our assumption. Therefore $\langle V(G) \cap C \rangle_w$ is connected.

Now let $v \in V(G)$. Suppose $v \notin C$. Since C is a total dominating set, $V(H^v) \cap C$ is a total dominating set in H^v. Also, since $\langle C \rangle_w$ is connected, $N_G(v) \cap C \neq \emptyset$. Suppose $v \in C$. Suppose further that $V(H^v) \cap C = \emptyset$. Since C is a total dominating set, there exists $u \in N_G(v) \cap C$ (hence, $N_G(v) \cap C \neq \emptyset$).

For the converse, suppose that $V(G) \cap C$ is a weakly connected dominating set in G and let $x \in V(G \circ H)$. Let $v \in V(G)$ such that $x \in V(v + H^v)$. If $x \neq v$, then by (i), (ii), or (iii), there exists $w \in C$ such that $xw \in E(G \circ H)$. Suppose $x = v$. If (i) holds, then there exists $z \in V(H^v) \cap C$ such that $xz \in E(G \circ H)$. If (ii) or (iii) holds, then there exists $p \in C$ such that $xp \in E(G \circ H)$. This shows that C is total dominating set in $G \circ H$. By (i) and the assumption that $\langle V(G) \cap C \rangle_w$ is connected in G, it follows that $\langle C \rangle_w$ is connected in $G \circ H$. ■

Corollary 4.2 *Let G be a connected graph of order $m \geq 2$ and let H be any graph of order $n \geq 1$. Then $\gamma_{wt}(G \circ H) = m$.*

Proof. Let $C = V(G)$. By Theorem 4.1, C is a weakly connected total dominating set in $G \circ H$. Let C^* be a weakly connected total dominating set in $G \circ H$. Then

$$|C^*| = \sum_{v \in V(G)} |V(v + H^v) \cap C^*|.$$

By (i), (ii), and (iii), it follows that $|V(v + H^v) \cap C^*| \geq 1$ for each $v \in V(G)$. Thus $|C^*| \geq |V(G)| = m = |C|$. This implies that C is a minimum weakly connected total dominating set in $G \circ H$. The desired result now follows. ■

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