

Mixed Convection Boundary Layer Flow over a Moving Vertical Flat Plate in an External Fluid Flow with Viscous Dissipation Effect

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Abstract

The steady boundary layer flow of a viscous and incompressible fluid over a moving vertical flat plate in an external moving fluid with viscous dissipation is theoretically investigated. Using appropriate similarity variables, the governing system of partial differential equations is transformed into a system of ordinary (similarity) differential equations, which is then solved numerically using a Maple software. Results for the skin friction or shear stress coefficient, local Nusselt number, velocity and temperature profiles are presented for different values of the governing parameters. It is found that the set of the similarity equations has unique solutions, dual solutions or no solutions, depending on the values of the mixed convection parameter, the velocity ratio parameter and the Eckert number. The Eckert number significantly affects the surface shear stress as well as the heat transfer rate at the surface.

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Introduction

Mixed convection flows, or combined forced and free convection flows, arise in many transport processes both naturally and in engineering applications. They play an important role, for example, in atmospheric boundary-layer flows, heat exchangers, solar collectors, nuclear reactors and in electronic equipment. Such processes occur when the effects of buoyancy forces in forced convection or the effects of forced flow in free convection become significant. The interaction of forced and free convection is especially pronounced in situations where the forced flow velocity is low and/or the temperature differences are large. This flow is also a relevant type of flow appearing in many industrial processes, such as manufacture and extraction of polymer and rubber sheets, paper production, wire drawing and glass-fiber production, melt spinning, continuous casting, etc. (Tadmor and Klein [1]). This flow has also many industrial applications such as heat treatment of material traveling between a feed roll and wind-up roll or conveyer belts, extrusion of steel, cooling of a large metallic plate in a bath, liquid films in condensation process and in aerodynamics, etc. As per standard texts books by Bejan [2], Kays and Crawford [3], Bergman et al. [4] and other literatures the free and mixed convection flow occur in atmospheric and oceanic circulation, electronic machinery, heated or cooled enclosures, electronic power supplies, etc. This topic has also many applications such as its influence on operating temperatures of power generating and electronic devices. In addition it should be mentioned that this type of flow plays a great role in thermal

manufacturing applications and is important in establishing the temperature distribution within buildings as well as heat losses or heat loads for heating, ventilating and air conditioning systems (see Abraham and Sparrow [5], and Sparrow and Abraham [6]). As it is well-known, the difference between convective heat transfer and forced convection problems is thermodynamic and mathematical, as well, the convective flows being driven by buoyancy effect due to the presence of gravitational acceleration and density variations from one fluid layer to another (Bejan [2]). A considerable amount of research has been reported on this topic (Jaluria [7], Karve and Jaluria [8], etc.). Similarity solutions for moving plates were investigated also by many authors. Among them, Afzal et al. [9], Afzal [10,11], Fang [12], Fang and Lee [13], Weidman et al. [14], Ishak et al. [15] studied the boundary layer flow on a moving permeable plate parallel to a moving stream and concluded that dual solutions exist if the plate and the free stream move in the opposite directions. The steady mixed convection boundary layer flow on a vertical surface without the effect of viscous dissipation was studied by Dey and Nath [16], while Hieber [17], Schneider [18], Afzal and Hussain [19], Ishak [20] and Ishak et al. [21] studied the mixed convection flow above a heated horizontal surface. Schneider [18] reported that solutions do not exist if the buoyancy parameter is smaller than a certain critical value. Afzal and Hussain [19] reinvestigated this problem and reported the existence of dual solutions in the neighborhood of the separation region.

Different from Dey and Nath [16], the present paper considers the case of a moving plate and the viscous dissipation term in the

energy equation is taken into consideration. Like the forced convection, the problem of free convection boundary layer flow near a continuously moving surface has also attracted considerable interest of many authors, such as Ingham [22], because it has many practical applications in manufacturing processes. Ingham [22] showed that the problem of free convection boundary layer flow near a continuously moving vertical plate has non-unique solutions. Merkin [23] probably is the first author who found the existence of dual (non-unique) solutions for mixed convection flow, when he investigated the boundary layer flow in a saturated porous medium. Thereafter, the existence of dual solutions has been pointed out by many researchers, for example, Afzal et al. [9], Weidman et al. [14], Xu and Liao [24], Ishak et al. [25,26] and Bachok and Ishak [27] when they investigated the boundary layer flow over a moving surface in a parallel stream. It is worth mentioning that the flow and heat transfer characteristics over a moving or stretching surface were analyzed by Chen [28] and Bataller [29]. Recently, a paper by Bachok et al. [30] investigated the flow over a moving surface in a nanofluid. Examples of practical applications of this problem include the aerodynamic extrusion of plastic sheets, the cooling of an infinite metallic plate in a cooling bath, glass blowing, continuous casting spinning fibers, etc. The design of a thermal processing station for moving surfaces requires a knowledge of heat transfer rates and corresponding surface temperature variations (Sparrow and Abraham [6]).

Problem Formulation

Consider a steady mixed convection boundary layer flow of a viscous and incompressible fluid over a moving vertical flat plate in an external moving fluid. We consider a Cartesian coordinate system (\bar{x}, \bar{y}) in which the \bar{x} -axis is measured along the plate in the upward direction and the \bar{y} -axis is measured in the direction normal to the plate. It is assumed that the velocities of the free stream (or inviscid flow) and the flat plate are U_∞ and U_w , respectively and the viscous dissipation term is $(\nu/C_p)(\partial \bar{u}/\partial \bar{y})^2$ (see Bejan [2]). It is also assumed that the temperature of the plate is $\bar{T}_w(\bar{x})$, while the uniform temperature of the free stream (inviscid flow) is T_∞ . Under these assumptions, the boundary layer equations are (Ingham [22])

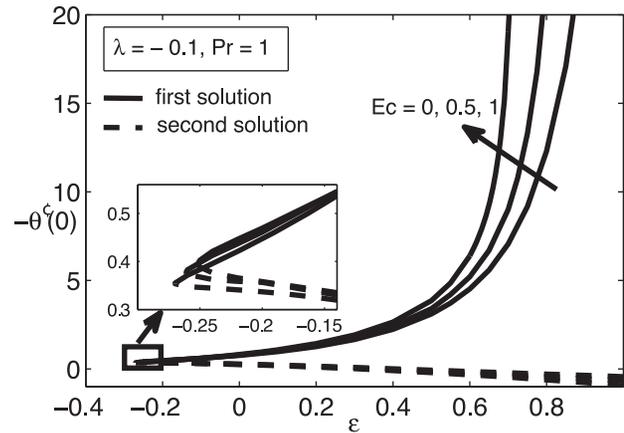


Figure 2. Variation of $-\theta'(0)$ for different values of Ec when $\lambda = -0.1$ and $Pr = 1$.
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$$\frac{\partial \bar{u}}{\partial \bar{x}} + \frac{\partial \bar{v}}{\partial \bar{y}} = 0 \tag{1}$$

$$\bar{u} \frac{\partial \bar{u}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{u}}{\partial \bar{y}} = \nu \frac{\partial^2 \bar{u}}{\partial \bar{y}^2} + g \beta (\bar{T} - T_\infty) \tag{2}$$

$$\bar{u} \frac{\partial \bar{T}}{\partial \bar{x}} + \bar{v} \frac{\partial \bar{T}}{\partial \bar{y}} = \alpha \frac{\partial^2 \bar{T}}{\partial \bar{y}^2} + \frac{\nu}{C_p} \left(\frac{\partial \bar{u}}{\partial \bar{y}} \right)^2 \tag{3}$$

subject to the boundary conditions

$$\begin{aligned} \bar{v} = 0, \quad \bar{u} = U_w, \quad \bar{T} = \bar{T}_w(\bar{x}) \quad \text{at} \quad \bar{y} = 0 \\ \bar{u} \rightarrow U_\infty, \quad \bar{T} \rightarrow T_\infty \quad \text{as} \quad \bar{y} \rightarrow \infty \end{aligned} \tag{4}$$

where U_w and U_∞ are constants with $U_\infty > 0$. In the above equations, \bar{u} and \bar{v} are the fluid velocities in the \bar{x} and \bar{y} directions, respectively, ν is the kinematic viscosity, β is the thermal expansion

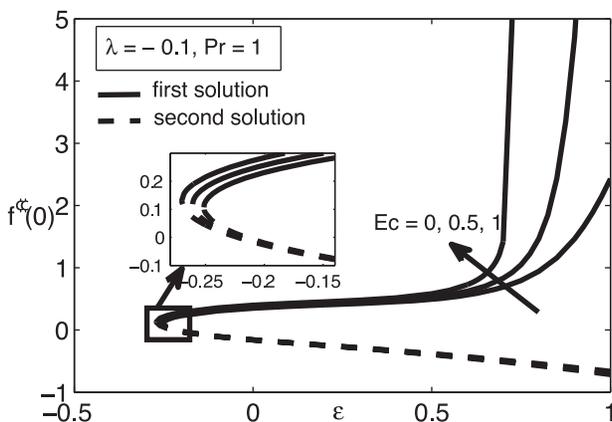


Figure 1. Skin friction coefficient $f''(0)$ for different values of Ec when $\lambda = -0.1$ and $Pr = 1$.
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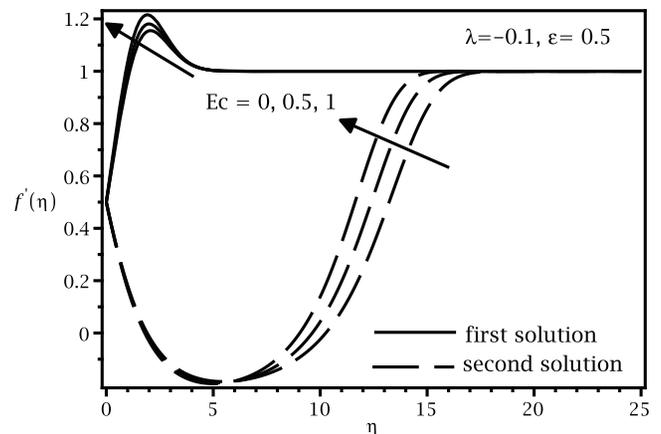


Figure 3. Velocity profiles $f'(\eta)$ for different values of Ec when $\epsilon = 0.5$, $\lambda = -0.1$ and $Pr = 1$.
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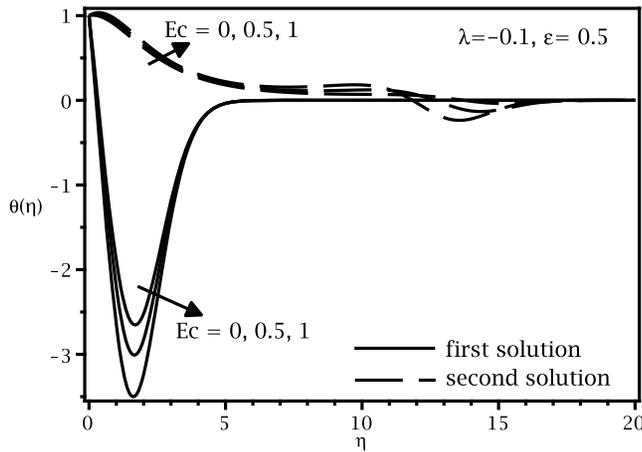


Figure 4. Temperature profiles $\theta(\eta)$ for various values of Ec when $\varepsilon=0.5$, $\lambda=-0.1$ and $Pr=1$.
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coefficient, α is the thermal diffusivity, C_p is the specific heat at constant pressure and g is the acceleration due to gravity. Different from Dey and Nath [16], the present paper considers the case of a moving plate and takes into consideration the viscous dissipation term in the energy equation. The present paper is also different from the paper by Ingham [22], who considered the free convection flow in a quiescent fluid.

We introduce now the following dimensionless variables

$$\begin{aligned} x &= \bar{x}/L, & y &= \text{Re}^{1/2}(\bar{y}/L), & u &= \bar{u}/U_\infty, \\ v &= \text{Re}^{1/2}(\bar{v}/U_\infty), & T &= \bar{T}/\Delta T \end{aligned} \quad (5)$$

where L is the characteristic length of the plate, $\Delta T > 0$ is the characteristic temperature and $\text{Re} = U_\infty L/\nu$ is the Reynolds number. Thus Eqs. (1)–(3) become

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \quad (6)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{\partial^2 u}{\partial y^2} + \lambda T \quad (7)$$

$$u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} = \frac{1}{\text{Pr}} \frac{\partial^2 T}{\partial y^2} + Ec \left(\frac{\partial u}{\partial y} \right)^2, \quad (8)$$

subject to the boundary conditions

$$\begin{aligned} v=0, & \quad u=\varepsilon, & T=T_w(x) & \text{ at } y=0, \\ u \rightarrow 1, & & T \rightarrow 0 & \text{ as } y \rightarrow \infty, \end{aligned} \quad (9)$$

where $\text{Pr} = \nu/\alpha$ is the Prandtl number, λ is the mixed convection parameter, Ec is the Eckert number and ε is the velocity ratio parameter, which are defined as [22,31,32,33]

$$\lambda = \frac{Gr}{\text{Re}^2}, \quad Ec = \frac{U_\infty^2}{C_p \Delta T}, \quad \varepsilon = \frac{U_w}{U_\infty}, \quad (10)$$

where $Gr = g\beta\Delta T L^3/\nu^2$ is the Grashof number. It should be mentioned that $\lambda > 0$ corresponds to assisting flow (heated plate), $\lambda < 0$ corresponds to opposing flow (cooled plate) and $\lambda = 0$ corresponds to forced convection flow.

To obtain similarity solutions of Eqs. (1)–(3), the wall temperature $T_w(x)$ is taken as (see Ingham [22])

$$T_w(x) = \frac{1}{x} \quad (11)$$

The idea of “similarity solution” is to simplify the governing equations by reducing the number of independent variables, by a coordinate transformation. The terminology “similarity” is used because, despite the growth of the boundary layer with distance x from the leading edge, the velocity as well as the temperature profiles remain geometrically similar. We introduce now the similarity transformation:

$$\psi = x^{1/2} f(\eta), \quad T = T_w(x) \theta(\eta) = x^{-1} \theta(\eta), \quad \eta = x^{-1/2} y \quad (12)$$

where ψ is the stream function defined as $u = \partial\psi/\partial y$ and $v = -\partial\psi/\partial x$ which identically satisfies the continuity equation (6). By this transformation, Eqs. (2) and (3) reduce to the following nonlinear ordinary differential equations:

$$f''' + \frac{1}{2} f f'' + \lambda \theta = 0 \quad (13)$$

$$\theta'' + \text{Pr} \left(\frac{1}{2} f \theta' + f' \theta + Ec f'^2 \right) = 0 \quad (14)$$

with the boundary conditions

$$\begin{aligned} f(0) &= 0, & f'(0) &= \varepsilon, & \theta(0) &= 1 \\ f'(\eta) &\rightarrow 1, & \theta(\eta) &\rightarrow 0 & \text{ as } \eta \rightarrow \infty \end{aligned} \quad (15)$$

It is worth mentioning that equations (13) and (14) were derived by Ingham [22] but without the viscous dissipation term ($Ec f'^2$) in

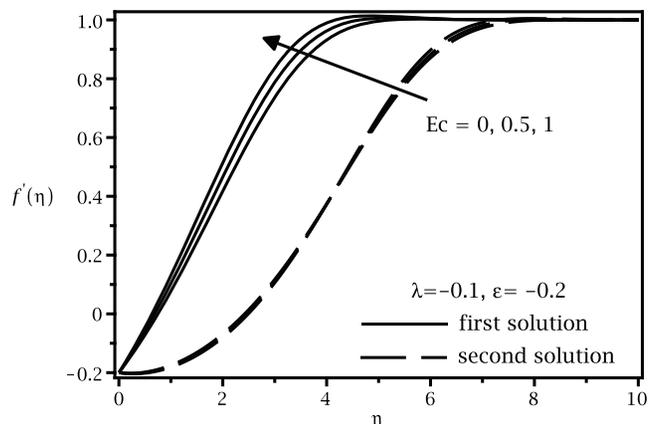


Figure 5. Velocity profiles $f'(\eta)$ for different values of Ec when $\varepsilon=-0.2$, $\lambda=-0.1$ and $Pr=1$.
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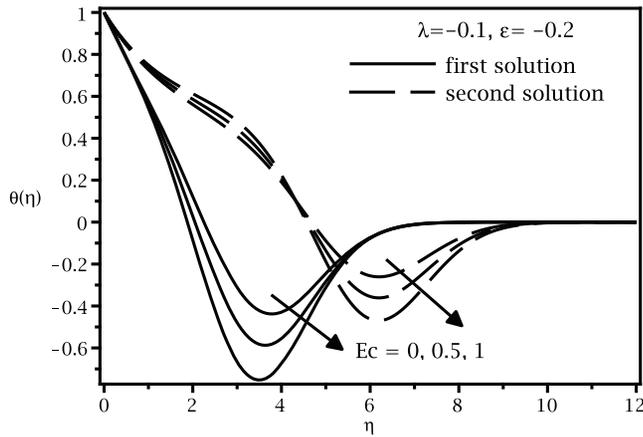


Figure 6. Temperature profiles $\theta(\eta)$ for different values of Ec when $\varepsilon = -0.2$, $\lambda = -0.1$ and $Pr = 1$.
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the energy equation (14). Moreover, the boundary conditions are different, where Ingham [22] considered the free convection flow with $f'(\eta) \rightarrow 0$ as $\eta \rightarrow \infty$.

The physical quantities of interest are the skin friction coefficient C_f and the local Nusselt number Nu_x , which are respectively defined as

$$C_f = \frac{\bar{\tau}_w}{\rho U_\infty^2}, \quad Nu_x = \frac{\bar{x} \bar{q}_w}{k \Delta T}, \quad (16)$$

Table 1. Values of $f''(0)$ and $-\theta'(0)$ for different values of λ , Ec and ε when $Pr = 1$.

λ	Ec	ε	Dey and Nath [16]		Present work		
			$f''(0)$	$-\theta'(0)$	$f''(0)$	$-\theta'(0)$	
-0.1	0	-0.1			0.307066	0.595873	
					[-0.103876]	[0.319708]	
		0	0.350971	0.760608	0.350986	0.760823	
	0.1	-0.1			[-0.158162]	[0.273451]	
			0	0.380260	0.973171	0.380260	0.973171
		0.1			[-0.202513]	[0.217419]	
0.5	-0.1	-0.1			0.330728	0.602316	
					[-0.105064]	[0.312167]	
		0	0.373730	0.785243	0.373730	0.785243	
	0.1	-0.1			[-0.159043]	[0.262877]	
			0	0.403493	1.024008	0.403493	1.024008
		0.1			[-0.203813]	[0.201039]	
0.1	0	-0.1			0.357351	0.509747	
		0	0.332920	0.593633	0.332920	0.593633	
	0.5	-0.1	-0.1			0.302131	0.691214
		0	0.344851	0.483145	0.344851	0.483145	
0.1	-0.1	-0.1			0.321280	0.577087	
	0	0.321280	0.577087	0.321280	0.577087		
0.1	-0.1	-0.1			0.287885	0.696999	
	0	0.287885	0.696999	0.287885	0.696999		

• [] second solution.
doi:10.1371/journal.pone.0060766.t001

Table 2. Values of ε_c and ε_t for several values of Ec and λ when $Pr = 1$.

λ	Ec	ε_c	ε_t
-0.1	0	-0.2513	
	0.5	-0.2607	
	1	-0.2701	
0.1	0	-0.5516	1.1992
	0.5	-0.5740	1.3130
	1	-0.5998	1.4081

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where k is the thermal conductivity of the fluid, $\bar{\tau}_w$ is the surface shear stress and \bar{q}_w is the surface heat flux, which are given by

$$\bar{\tau}_w = \mu \left(\frac{\partial \bar{u}}{\partial \bar{y}} \right)_{\bar{y}=0}, \quad \bar{q}_w = -k \left(\frac{\partial \bar{T}}{\partial \bar{y}} \right)_{\bar{y}=0}. \quad (17)$$

Substituting (5) and (12) into Eqs. (16) and (17), we get

$$\text{Re}_x^{1/2} C_f = f''(0), \quad (x \text{Re}_x^{-1/2}) Nu_x = -\theta'(0), \quad (18)$$

where $\text{Re}_x = U_\infty \bar{x}/\nu$ is the local Reynolds number.

Results and Discussion

Numerical solutions to the nonlinear ordinary (similarity) differential equations (13) and (14) with the boundary conditions (15) were obtained using a shooting method with the help of Maple software. This method is described in details in the recent paper by Aman and Ishak [34]. It was found that these equations have multiple (dual) solutions, which were obtained by setting different initial guesses for the missing values of the skin friction coefficient $f''(0)$ and the local Nusselt number (heat transfer rate) $-\theta'(0)$, where all profiles satisfy the boundary conditions (15) asymptotically but with different shapes. These two different shapes of profiles produced for a particular value of parameter show that the system of equations (13)–(15) has two solutions. In order to

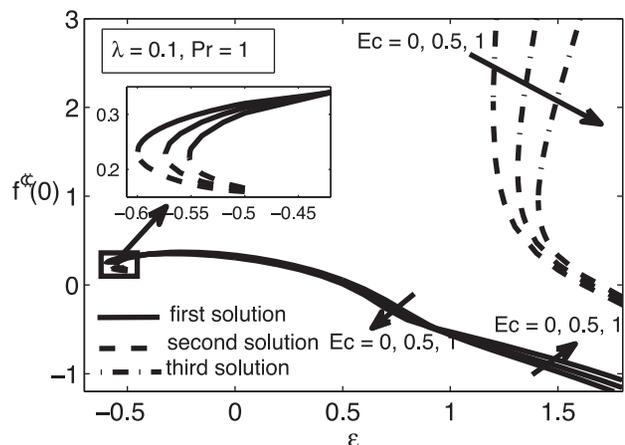


Figure 7. Skin friction coefficient $f''(0)$ for different values of Ec when $\lambda = 0.1$ and $Pr = 1$.

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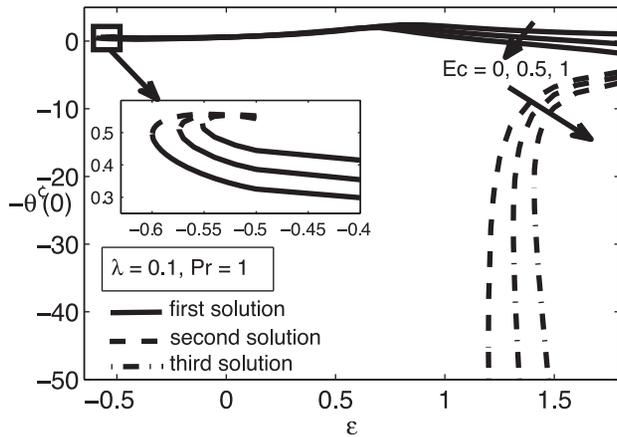


Figure 8. Variation of $-\theta'(0)$ for different values of Ec when $\lambda=0.1$ and $Pr=1$.
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compare the present results with those reported by Dey and Nath [16], we consider also the case when $Ec=0$ (viscous dissipation is absent) and $\epsilon=0$ (stationary plate). The quantitative comparison for the values of $f''(0)$ and $-\theta'(0)$ is shown in Table 1, and it is found that they are in a very good agreement. Moreover, the values of $f''(0)$ and $-\theta'(0)$ for $Ec \neq 0$ and $\epsilon \neq 0$ are also included in Table 1 for future references.

Figs. 1 and 2 present the variations of $f''(0)$ and $-\theta'(0)$ as a function of ϵ for $Ec=0, 0.5, 1$ when $\lambda = -0.1$ (opposing flow) and $Pr = 1$, while the samples of the respective velocity and temperature profiles are given in Figs. 3, 4 5, and 6. We found that there are two (dual) solutions, a first (upper branch) solution and a second (lower branch) solution for certain range of ϵ . We identify the first (upper branch) and second (lower branch) solutions by how they appear in Figs. 1 and 2, i.e. the first solution has higher values of $f''(0)$ and $-\theta'(0)$ than the second solution, for a given ϵ . For negative values of ϵ , there is a critical value $\epsilon = \epsilon_c$ (< 0) as presented in Table 2, for which the solution of equations (13) and (14) with the boundary conditions (15) exists. There is no solution of these boundary layer equations for $\epsilon < \epsilon_c$. In this case, the full Navier-Stokes and energy equations should be solved in the region $\epsilon < \epsilon_c$. As can be seen from Figs. 1 and 2, the Eckert number Ec significantly affects the skin friction coefficient $f''(0)$ as well as the

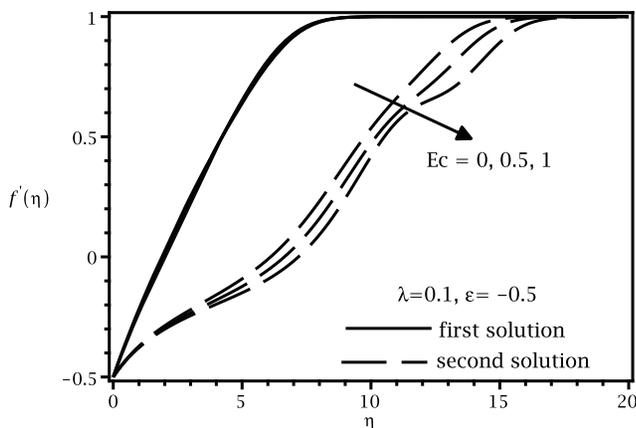


Figure 9. Velocity profiles $f'(\eta)$ for different values of Ec when $\epsilon = -0.5$, $\lambda=0.1$ and $Pr=1$.
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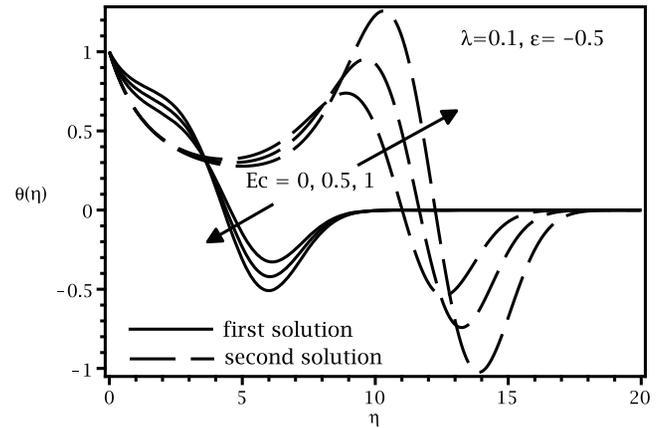


Figure 10. Temperature profiles $\theta(\eta)$ for different values of Ec when $\epsilon = -0.5$, $\lambda=0.1$ and $Pr=1$.
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heat transfer rate at the surface $-\theta'(0)$. Table 2 shows that the value of $|\epsilon_c|$ increases with the increase of the Eckert number Ec . Thus the Eckert number Ec increases the range of λ for which the solution exists. Figs. 3, 4, 5, and 6 show that the boundary conditions (15) as $\eta \rightarrow \infty$ are asymptotically satisfied, which support the validity of the present results, besides supporting the dual nature of the solution to the boundary-value problem (13)–(15).

For positive values of the mixed convection parameter λ (assisting flow), Figs. 7 and 8 show the variation of $f''(0)$ and $-\theta'(0)$ as a function of ϵ for $Ec=0, 0.5, 1$ when $\lambda = 0.1$ and $Pr = 1$, while the samples of the respective velocity and temperature profiles are given in Figs. 9, 10, 11, and 12. It is seen that there are regions of three solutions for $\epsilon \geq \epsilon_t$, unique solutions for $-0.5 < \epsilon < \epsilon_t$, dual solutions for $\epsilon_c \leq \epsilon \leq -0.5$ and no solutions for $\epsilon < \epsilon_c$, where ϵ_t and ϵ_c are the critical values of ϵ as presented in Table 2. It should be mentioned that such a behaviour near the turning point as shown in Fig. 7 has also been depicted by Weidman et al. [14] for the case of forced convection boundary layer flow over moving surfaces ($\lambda=0$) with suction effect.

For the dual solutions, we expect that the upper branch solution is stable and physically realizable while the lower branch solution

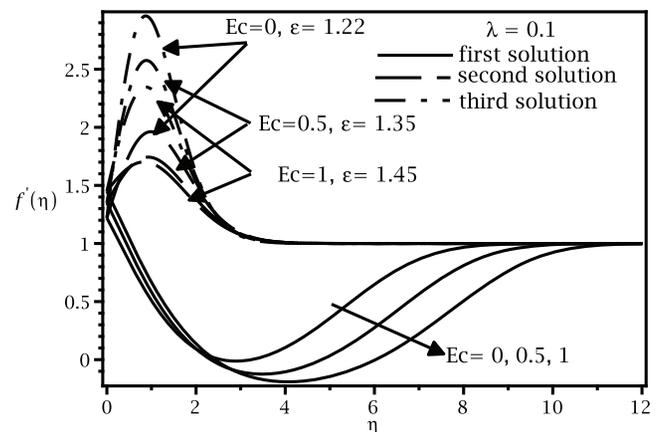


Figure 11. Velocity profiles $f'(\eta)$ for different values of Ec and ϵ when $\lambda=0.1$ and $Pr=1$.
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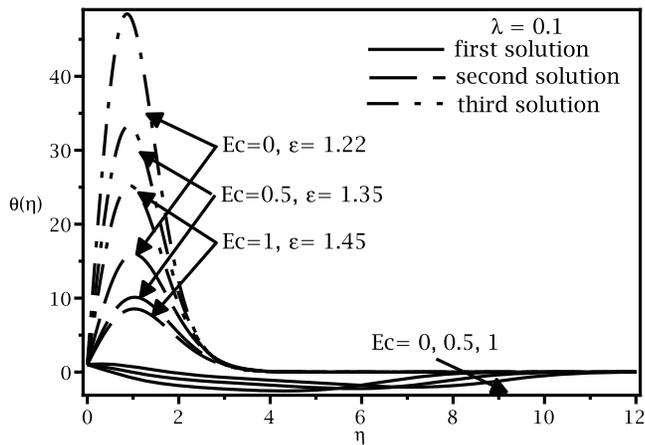


Figure 12. Temperature profiles $\theta(\eta)$ for different values of Ec and ε when $\lambda=0.1$ and $Pr=1$.
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is not. The procedure for showing this has been described by Merkin [23], Weidman et al. [14], Postelnicu and Pop [35], and very recently by Rosca and Pop [36]. It is worth mentioning to this end that this type of multiple (dual) solutions is essentially a backward flow, and it shows a physical phenomenon quite distinct from the flow with no dual solutions, being important in many practical problems (Fang et al. [37]).

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Conclusions

We have numerically studied the mixed convection boundary-layer flow over a moving vertical flat plate in the presence of an external flow. Using appropriate similarity variables, the governing system of partial differential equations is transformed into a system of ordinary differential equations. This system is then solved numerically using a shooting method. Results for the skin friction coefficient, local Nusselt number, velocity profiles as well as temperature profiles are presented for different values of the governing parameters. It is found that the set of similarity equations has three solutions, dual solutions, unique solution or no solution, depending on the value of the mixed convection parameter and the velocity ratio parameter. The Eckert number significantly affects the surface shear stress as well as the heat transfer rate at the surface.

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Author Contributions

Analyzed the data: NB AI. Contributed reagents/materials/analysis tools: AI IP. Wrote the paper: NB AI IP.