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Article

Informational Non-Differentiable Entropy and Uncertainty Relations in Complex Systems

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Abstract: Considering that the movements of complex system entities take place on continuous, but non-differentiable, curves, concepts, like non-differentiable entropy, informational non-differentiable entropy and informational non-differentiable energy, are introduced. First of all, the dynamics equations of the complex system entities (Schrödinger-type or fractal hydrodynamic-type) are obtained. The last one gives a specific fractal potential, which generates uncertainty relations through non-differentiable entropy. Next, the correlation between informational non-differentiable entropy and informational non-differentiable energy implies specific uncertainty relations through a maximization principle of the informational non-differentiable entropy and for a constant value of the informational non-differentiable energy. Finally, for a harmonic oscillator, the constant value of the informational non-differentiable energy is equivalent to a quantification condition.

Keywords: non-differentiable entropy; informational non-differentiable entropy; informational non-differentiable energy; uncertainty relations

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1. Introduction

Complex systems are large interdisciplinary research topics that have been studied by means of a mixed basic theory that mainly derives from physics and computer simulation. Such systems are made of many interacting elementary units that are called "agents".

The way in which such a system manifests itself cannot be exclusively predicted only by the behavior of individual elements. Its manifestation is also induced by the manner in which the elements relate in order to influence global behavior. The most significant properties of complex systems are emergence, self-organization, adaptability, *etc.* [1–4].

Examples of complex systems can be found in human societies, brains, the Internet, ecosystems, biological evolution, stock markets, economies and many others [1,2]. Particularly, polymers are examples of such complex systems. Their forms include a multitude of organizations starting from simple, linear chains of identical structural units and ending with very complex chains consisting of sequences of amino acids that form the building blocks of living fields. One of the most intriguing polymers in nature is DNA, which creates cells by means of a simple, but very elegant language. It is responsible for the remarkable way in which individual cells organize into complex systems, such as organs, which, in turn, form even more complex systems, such as organisms. The study of complex systems can offer a glimpse into the realistic dynamics of polymers and solve certain difficult problems (protein folding) [1–4].

Correspondingly, theoretical models that describe the dynamics of complex systems are sophisticated [1–4]. However, the situation can be standardized taking into account that the complexity of interaction processes imposes various temporal resolution scales, while pattern evolution implies different freedom degrees [5].

In order to develop new theoretical models, we must admit that complex systems displaying chaotic behavior acquire self-similarity (space-time structures seem to appear) in association with strong fluctuations at all possible space-time scales [1–4]. Then, in the case of temporal scales that are large with respect to the inverse of the highest Lyapunov exponent, the deterministic trajectories are replaced by a collection of potential trajectories, while the concept of definite positions by that of probability density. One of the most interesting examples is the collision process in complex systems, a case in which the dynamics of the particles can be described by non-differentiable curves.

Since non-differentiability appears as the universal property of complex systems, it is necessary to construct a non-differentiable physics. Thus, the complexity of the interaction processes is replaced by non-differentiability; accordingly, it is no longer necessary to use the whole classical "arsenal" of quantities from standard physics (differentiable physics).

This topic was developed within scale relativity theory (SRT) [6,7] and non-standard scale relativity theory (NSSRT) [8–22]. In this case, we assume that the movements of complex system entities take place on continuous, but non-differentiable, curves (fractal curves), so that all physical phenomena involved in the dynamics depend not only on space-time coordinates, but also on space-time scale resolution. From such a perspective, physical quantities describing the dynamics of complex systems may be considered fractal functions [6,7]. Moreover, the entities of the complex system may be reduced to and identified with their own trajectories, so that the complex system will behave as a special fluid lacking interaction (via their geodesics in a non-differentiable (fractal) space). We have called such fluid a "fractal fluid" [8–22].

In the present paper, we shall introduce new concepts, like non-differentiable entropy, informational non-differentiable entropy, informational non-differentiable energy, *etc.*, in the NSSRT approach (the scale relativity theory with an arbitrary constant fractal dimension). Based on a fractal potential, which is the "source" of the non-differentiability of trajectories of the complex system entities, we establish the relationships among non-differentiable entropy. The correlation fractal potential-non-differentiable entropy implies uncertainty relations in the hydrodynamic representation, while the correlation of informational non-differentiable entropy/informational non-differentiable energy implies specific uncertainty relations through a maximization principle of the informational non-differentiable entropy and for a constant value of the informational non-differentiable energy. The constant value of the informational non-differentiable energy made explicit for the harmonic oscillator induces a quantification condition. We note that there exists a large class of complex systems that take smooth trajectories. However, the analysis of the dynamics of these classes is reducible to the above-mentioned statements by neglecting their fractality.

2. Hallmarks of Non-Differentiability

Let us assume that the motion of complex system entities takes place on fractal curves (continuous, but non-differentiable). A manifold that is compatible with such movement defines a fractal space. The fractal nature of space generates the breaking of differential time reflection invariance. In such a context, the usual definitions of the derivative of a given function with respect to time [6,7],

$$\frac{dF}{dt} = \lim_{\Delta t \to 0_+} \frac{F(t+\Delta t) - F(t)}{\Delta t} = \lim_{\Delta t \to 0_-} \frac{F(t) - F(t-\Delta t)}{\Delta t} \tag{1}$$

are equivalent in the differentiable case. The passage from one to the other is performed via $\Delta t \to -\Delta t$ transformation (time reflection invariance at the infinitesimal level). In the non-differentiable case, $(\frac{dQ_+}{dt})$ and $(\frac{dQ_-}{dt})$ are defined as explicit functions of t and dt,

$$\frac{dQ_+}{dt} = \lim_{\Delta t \to 0_+} \frac{Q(t, t+\Delta t) - Q(t, \Delta t)}{\Delta t}$$

and:

$$\frac{dQ_-}{dt} = \lim_{\Delta t \to 0_-} \frac{Q(t, \Delta t) - Q(t, t-\Delta t)}{\Delta t} \tag{2}$$

The sign $(+)$ corresponds to the forward process, while $(-)$ corresponds to the backward process. Then, in space coordinates $d\mathbf{X}$, we can write [6,7]:

$$d\mathbf{X}_\pm = d\mathbf{x}_\pm + d\boldsymbol{\xi}_\pm = \boldsymbol{\nu}_\pm dt + d\boldsymbol{\xi}_\pm \tag{3}$$

with $\boldsymbol{\nu}_\pm$ the forward and backward mean speeds,

$$\begin{aligned} \boldsymbol{\nu}_+ &= \frac{d\mathbf{x}_+}{dt} = \lim_{\Delta t \to 0_+} \left\langle \frac{\mathbf{X}(t+\Delta t) - \mathbf{X}(t)}{\Delta t} \right\rangle \\ \boldsymbol{\nu}_- &= \frac{d\mathbf{x}_-}{dt} = \lim_{\Delta t \to 0_-} \left\langle \frac{\mathbf{X}(t) - \mathbf{X}(t-\Delta t)}{\Delta t} \right\rangle \end{aligned} \tag{4}$$

and $d\boldsymbol{\xi}_{\pm}$ a measure of non-differentiability (a fluctuation induced by the fractal properties of trajectory) having the average:

$$\langle d\boldsymbol{\xi}_{\pm}\rangle = 0, \tag{5}$$

where the symbol $\langle\ \rangle$ defines the mean value.

While the speed-concept is classically a single concept, if space is a fractal, then we must introduce two speeds ($\boldsymbol{\nu}_+$ and $\boldsymbol{\nu}_-$), instead of one. These "two-values" of the speed vector represent a specific consequence of non-differentiability that has no standard counterpart (according to differential physics).

However, we cannot favor $\boldsymbol{\nu}_+$ as compared to $\boldsymbol{\nu}_-$. The only solution is to consider both the forward $(dt > 0)$ and backward $(dt < 0)$ processes. Then, it is necessary to introduce the complex speed [6,7]:

$$\begin{aligned}\hat{\mathbf{V}} &= \frac{\boldsymbol{\nu}_+ + \boldsymbol{\nu}_-}{2} - i\frac{\boldsymbol{\nu}_+ - \boldsymbol{\nu}_-}{2} = \frac{d\mathbf{x}_+ + d\mathbf{x}_-}{2dt} - i\frac{d\mathbf{x}_+ - d\mathbf{x}_-}{2dt} \\ &= \mathbf{V}_D - i\mathbf{V}_F, \mathbf{V}_D = \frac{\boldsymbol{\nu}_+ + \boldsymbol{\nu}_-}{2}, \mathbf{V}_F = \frac{\boldsymbol{\nu}_+ - \boldsymbol{\nu}_-}{2}\end{aligned} \tag{6}$$

If $\mathbf{V}_D$ is differentiable and resolution scale (dt) speed independent, then $\mathbf{V}_F$ is non-differentiable and resolution scale (dt) speed dependent.

Using the notations $d\mathbf{x}_{\pm} = d_{\pm}\mathbf{x}$, Equation (6) becomes:

$$\hat{\mathbf{V}} = \left(\frac{d_+ + d_-}{2dt} - i\frac{d_+ - d_-}{2dt}\right)\mathbf{x} \tag{7}$$

This enables us to define the operator:

$$\frac{\hat{d}}{dt} = \frac{d_+ + d_-}{2dt} - i\frac{d_+ - d_-}{2dt} \tag{8}$$

Let us now assume that the fractal curve is immersed in a three-dimensional space and that $\mathbf{X}$ of components X^i $(i = 1, 2, 3)$ is the position vector of a point on the curve. Let us also consider a function $f(\mathbf{X}, t)$ and the following series expansion up to the second order:

$$\begin{aligned} df &= f(X^i + dX^i, t + dt) - f(X^i, dt) \\ &= \left(\frac{\partial}{\partial X^i}dX^i + \frac{\partial}{\partial t}dt\right) f(X^i, t) + \frac{1}{2}\left(\frac{\partial}{\partial X^i}dX^i + \frac{\partial}{\partial t}dt\right)^2 f(X^i, t)\end{aligned} \tag{9}$$

Using notations, $dX^i_{\pm} = d_{\pm}X^i$, the forward and backward average values of this relation take the form:

$$\begin{aligned}\langle d_{\pm}f\rangle &= \left\langle\frac{\partial f}{\partial t}dt\right\rangle + \langle\nabla f \cdot d_{\pm}\mathbf{X}\rangle + \frac{1}{2}\left\langle\frac{\partial^2 f}{\partial t^2}(dt)^2\right\rangle + \\ &+ \left\langle\frac{\partial^2 f}{\partial X^i \partial t}d_{\pm}X^i dt\right\rangle + \frac{1}{2}\left\langle\frac{\partial^2 f}{\partial X^i \partial X^l}d_{\pm}X^i d_{\pm}X^l\right\rangle\end{aligned} \tag{10}$$

We shall stipulate the following: the mean values of function f and its derivatives coincide with themselves, and the differentials $d_{\pm}X^i$ and dt are independent. Therefore, the averages of their products coincide with the product of averages. Thus, Equation (10) becomes:

$$\begin{aligned} d_{\pm}f &= \frac{\partial f}{\partial t}dt + \nabla f\,\langle d_{\pm}\mathbf{X}\rangle + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}\,\langle(dt)^2\rangle + \\ &+ \frac{\partial^2 f}{\partial X^i \partial t}\,\langle d_{\pm}X^i dt\rangle + \frac{1}{2}\frac{\partial^2 f}{\partial X^i \partial X^l}\,\langle d_{\pm}X^i d_{\pm}X^l\rangle\end{aligned} \tag{11}$$

or more, using Equation (3),

$$\begin{aligned} d_{\pm}f &= \frac{\partial f}{\partial t}dt + \nabla f d_{\pm}\mathbf{x} + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(dt)^2 + \frac{\partial^2 f}{\partial X^i \partial t}d_{\pm}\mathrm{x}^i dt + \\ &+ \frac{1}{2}\frac{\partial^2 f}{\partial X^i \partial X^l}(d_{\pm}\mathrm{x}^i d_{\pm}\mathrm{x}^l + \left\langle d\xi^i_{\pm} d\xi^l_{\pm}\right\rangle), i,l = 1,2,3, \end{aligned} \tag{12}$$

where the quantities $\langle d_{\pm}\mathrm{x}^i d_{\pm}\xi^l\rangle$, $\langle d_{\pm}\xi^i d_{\pm}\mathrm{x}^l\rangle$ are null based on the Relation (5) and also on the above property referring to a product mean.

Since $d\xi_{\pm}$ describes the fractal properties of the trajectory with the fractal dimension D_F [23], it is natural to impose that $(d\xi_{\pm})^{D_F}$ is proportional with resolution scale dt [6,7],

$$(d\xi_{\pm})^{D_F} = \sqrt{2D}dt \tag{13}$$

where D is a coefficient of proportionality (for details, see [6,7]). In Nottale's theory [6,7], D is a coefficient associated with the transition fractal-non-fractal.

Let us focus now on the mean $\left\langle d\xi^i_{\pm} d\xi^l_{\pm}\right\rangle$, which has statistical significance [6,7]. If $i \neq l$, this average is zero, due to the independence of $d\xi^i$ and $d\xi^l$. Therefore, using Equation (13), we can write:

$$\left\langle d\xi^i_{\pm} d\xi^l_{\pm}\right\rangle = \pm\delta^{il}2D(dt)^{\frac{2}{D_F}-1}dt \tag{14}$$

with:

$$\delta^{il} = \begin{cases} 1, \text{ if } i = l \\ 0, \text{ if } i \neq l \end{cases}$$

and considering that:

$$\begin{cases} \left\langle d\xi^i_{+} d\xi^l_{+}\right\rangle > 0 \text{ and } dt > 0 \\ \left\langle d\xi^i_{-} d\xi^l_{-}\right\rangle > 0 \text{ and } dt < 0 \end{cases}$$

are equivalent in differentiable case.

Then, Equation (12) may be written under the form:

$$\begin{aligned} d_{\pm}f &= \frac{\partial f}{\partial t}dt + \nabla f d_{\pm}\mathbf{x} + \frac{1}{2}\frac{\partial^2 f}{\partial t^2}(dt)^2 + \frac{\partial^2 f}{\partial X^i \partial t}d_{\pm}\mathrm{x}^i dt + \\ &+ \frac{1}{2}\frac{\partial^2 f}{\partial X^i \partial X^l}d_{\pm}\mathrm{x}^i d_{\pm}\mathrm{x}^l \pm \frac{\partial^2 f}{\partial X^i \partial X^l}\delta^{il}D(dt)^{\frac{2}{D_F}-1}dt \end{aligned} \tag{15}$$

If we divide by dt and neglect the terms that contain differential factors, Equation (15) is reduced to:

$$\frac{d_{\pm}f}{dt} = \frac{\partial f}{\partial t} + \boldsymbol{\nu}_{\pm}\nabla f_{\pm} \pm D(dt)^{\frac{2}{D_F}-1}\triangle f \tag{16}$$

(for the details on the calculus, see p. 167 and pp. 193–195 in [7] ; since $d\mathrm{x}^i$ and dt are standard infinitesimals of order one, while $d\xi^i$ is an infinitesimal of order $1/D_F$, the terms $d\mathrm{x}^i d\mathrm{x}^l/dt, dt^2/dt, d\mathrm{x}^i dt/dt$ are infinitesimals of order one and are null; the last term is finite by means of Relation (14)).

Under these circumstances, let us calculate $\frac{\hat{d}f}{dt}$. In accordance with Equation (8) and taking into account Equation (16), we obtain:

$$\begin{aligned}
\frac{\hat{d}f}{dt} &= \frac{1}{2}\left[\frac{d_+f}{dt}+\frac{d_-f}{dt}-i\left(\frac{d_+f}{dt}-\frac{d_-f}{dt}\right)\right] \\
&= \frac{1}{2}\left[\left(\frac{\partial f}{\partial t}+\boldsymbol{\nu}_+\nabla f+D(dt)^{\frac{2}{D_F}-1}\triangle f\right)+\left(\frac{\partial f}{\partial t}+\boldsymbol{\nu}_-\nabla f-D(dt)^{\frac{2}{D_F}-1}\triangle f\right)\right]- \\
&\quad -\frac{i}{2}\left[\left(\frac{\partial f}{\partial t}+\boldsymbol{\nu}_+\nabla f+D(dt)^{\frac{2}{D_F}-1}\triangle f\right)-\left(\frac{\partial f}{\partial t}+\boldsymbol{\nu}_-\nabla f-D(dt)^{\frac{2}{D_F}-1}\triangle f\right)\right] \\
&= \frac{\partial f}{\partial t}+\left(\frac{\boldsymbol{\nu}_++\boldsymbol{\nu}_-}{2}-i\frac{\boldsymbol{\nu}_+-\nu_-}{2}\right)\nabla f-iD(dt)^{\frac{2}{D_F}-1}\triangle f \\
&= \frac{\partial f}{\partial t}+(\mathbf{V}_D-i\mathbf{V}_F)\nabla f-iD(dt)^{\frac{2}{D_F}-1}\triangle f
\end{aligned} \tag{17}$$

or, using the first Equation (6):

$$\frac{\hat{d}f}{dt}=\frac{\partial f}{\partial t}+\hat{\mathbf{V}}\cdot\nabla f-iD(dt)^{\frac{2}{D_F}-1}\triangle f \tag{18}$$

This relation also allows us to give the definition of the fractal operator [8,13]:

$$\frac{\hat{d}}{dt}=\frac{\partial}{\partial t}+\hat{\mathbf{V}}\cdot\nabla-iD(dt)^{\frac{2}{D_F}-1}\triangle \tag{19}$$

We note that in Nottale's works [6,7], the fractal operator (19) for $D_F = 2$ plays the role of the "covariant derivative operator". We shall call the operator (19) the "generalized covariant derivative operator".

3. Geodesics Equation

Let us consider that the transition from classical (differentiable) physics to the "fractal" (non-differentiable) one (as it is approached here) can be implemented by replacing the standard time derivative $\frac{d}{dt}$ with the "generalized covariant derivative operator" $\frac{\hat{d}}{dt}$.

As a consequence, we are now able to write the equation of geodesics (we shall call it the "principle of scale covariance", *i.e.*, a generalization of Newton's first principle) in a fractal space under its covariant form. Applying the "generalized covariant derivative operator" $\frac{\hat{d}}{dt}$ to the complex field of velocities $\hat{\mathbf{V}}$ (the first Relation (6)), we obtain:

$$\frac{\hat{d}\hat{\mathbf{V}}}{dt}=\frac{\partial\hat{\mathbf{V}}}{\partial t}+\hat{\mathbf{V}}\cdot\nabla\hat{\mathbf{V}}-iD(dt)^{\frac{2}{D_F}-1}\triangle\hat{\mathbf{V}}=\mathbf{0} \tag{20}$$

This means that at any point on a fractal path, the local acceleration, $\partial_t\hat{\mathbf{V}}$, the non-linearly (convective) term, $(\hat{\mathbf{V}}\cdot\nabla)\hat{\mathbf{V}}$, and the dissipative one, $D(dt)^{\frac{2}{D_F}-1}\Delta\hat{\mathbf{V}}$, are in balance. Therefore, the complex system dynamics can be assimilated with a "rheological" fluid dynamics. Such a dynamics is described by the complex velocity field $\hat{\mathbf{V}}$, by the complex acceleration field $\frac{\hat{d}\hat{\mathbf{V}}}{dt}$, *etc.*, as well as by the imaginary viscosity type coefficient $iD(dt)^{\frac{2}{D_F}-1}$.

For irrotational motions of the complex system entities:

$$\nabla\times\hat{\mathbf{V}}=0,\nabla\times\mathbf{V}_D=0,\nabla\times\mathbf{V}_F=0 \tag{21}$$

$\hat{\mathbf{V}}$ can be chosen with the form:

$$\hat{\mathbf{V}} = -2iD(dt)^{\frac{2}{D_F}-1}\nabla \ln \psi \tag{22}$$

where $\phi = \ln \psi$ is the velocity scalar potential. Substituting (22) in (20), we obtain:

$$\frac{\hat{d}\hat{\mathbf{V}}}{dt} = -2iD(dt)^{(\frac{2}{D_F})-1}\left[\frac{\partial}{\partial t} - 2iD(dt)^{(\frac{2}{D_F})-1}(\nabla \ln \psi)\cdot\nabla - iD(dt)^{(\frac{2}{D_F})-1}\Delta\right](\nabla \ln \psi) = 0$$

or more:

$$\begin{aligned}\frac{\hat{d}\hat{\mathbf{V}}}{dt} &= -2iD(dt)^{(\frac{2}{D_F})-1}\left\{\frac{\partial}{\partial t}(\nabla \ln \psi)\right.\\ &\left.-i\left[2D(dt)^{(\frac{2}{D_F})-1}(\nabla \ln \psi \cdot \nabla)(\nabla \ln \psi) + D(dt)^{(\frac{2}{D_F})-1}\Delta(\nabla \ln \psi)\right]\right\} = 0\end{aligned} \tag{23}$$

Using the identities [7]:

$$(\nabla \ln \psi)^2 + \Delta \ln \psi = \frac{\Delta \psi}{\psi}$$

$$\nabla\left(\frac{\Delta\psi}{\psi}\right) = 2(\nabla \ln \psi \cdot \nabla)(\nabla \ln \psi) + \Delta(\nabla \ln \psi)$$

the Equation (23) becomes:

$$\frac{\hat{d}\hat{\mathbf{V}}}{dt} = -2iD(dt)^{(\frac{2}{D_F})-1}\nabla\left[\frac{\partial}{\partial t}\ln \psi - iD(dt)^{(\frac{2}{D_F})-1}\frac{\Delta\psi}{\psi}\right].$$

This equation can be integrated up to an arbitrary phase factor, which may be set to zero by a suitable choice of phase of ψ and this yields:

$$D^2(dt)^{(\frac{4}{D_F})-2}\Delta\psi + iD(dt)^{(2/D_F)-1}\frac{\partial \psi}{\partial t} = 0. \tag{24}$$

Relation (24) is a Schrödinger-type equation. For motions of complex system entities on Peano's curves, $D_F = 2$, Equation (24) takes the Nottale's form [6,7]. Moreover, for motions of complex system entities on Peano's curves at the Compton scale, $D = \frac{\hbar}{2m_0}$ (for details, see [6,7]), with $\hbar$ the reduced Planck constant and m_0 the rest mass of the complex system entities, Relation (24) becomes the standard Schrödinger equation.

If $\psi = \sqrt{\rho}e^{iS}$, with $\sqrt{\rho}$ the amplitude and S the phase of ψ, the complex velocity field (22) takes the explicit form:

$$\begin{aligned}\hat{\mathbf{V}} &= 2D(dt)^{\frac{2}{D_F}-1}\nabla S - iD(dt)^{\frac{2}{D_F}-1}\nabla \ln \rho\\ \mathbf{V}_D &= 2D(dt)^{\frac{2}{D_F}-1}\nabla S\\ \mathbf{V}_F &= D(dt)^{\frac{2}{D_F}-1}\nabla \ln \rho\end{aligned} \tag{25}$$

Substituting (25) into (20) and separating the real and the imaginary parts, up to an arbitrary phase factor, which may be set to zero by a suitable choice of the phase of ψ, we obtain:

$$\begin{aligned}&\frac{\partial \boldsymbol{V}_D}{\partial t} + (\boldsymbol{V}_D \cdot \nabla)\boldsymbol{V}_D = -\nabla Q\\ &\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{V}_D) = 0\end{aligned} \tag{26}$$

with Q the specific fractal potential (specific non-differentiable potential):

$$Q = -2D^2(dt)^{\frac{4}{D_F}-2}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} = -\frac{\boldsymbol{V}_F^2}{2} - D(dt)^{\frac{2}{D_F}-1}\nabla\cdot\boldsymbol{V}_F \tag{27}$$

The specific fractal potential can simultaneously work with the standard potentials (for instance, an external scalar potential).

The first Equation (26) represents the specific momentum conservation law, while the second Equation (26) exhibits the state density conservation law. Equations (26) and (27) define the fractal hydrodynamics model (FHM).

The following conclusions are obvious:

(i) Any entity of the complex system is in permanent interaction with the fractal medium through a specific fractal potential.

(ii) The fractal medium is identified with a non-relativistic fractal fluid described by the specific momentum and state density conservation laws (probability density conservation law [6,7]). For motions of complex system entities on Peano's curves at the Compton scale, the fractal medium is identified with Bohm's "subquantum level" [7].

(iii) Fractal speed $\mathbf{V}_F$ does not represent an actual mechanical motion, but contributes to the transfer of specific momentum and the energy concentration. This may be clearly noticed from the absence of $\mathbf{V}_F$ in the state density conservation law and from its role in the variation principle [6,7].

(iv) Any interpretation of Q should take cognizance of the "self" or the internal nature of the specific momentum transfer. While the energy is stored in the form of mass motion and potential energy (as it actually is), some is available elsewhere, and only the total one is conserved. It is the conservation of energy and specific momentum that ensures the reversibility and existence of eigenstates, but denies a Brownian motion-type form of interaction with an external medium.

(v) The specific fractal potential (27) generates the viscosity stress tensor [8,13]:

$$\hat{\sigma}_{il} = D^2(dt)^{\frac{4}{D_F}-2}\left(\nabla_i\nabla_l\rho - \frac{\nabla_i\rho\nabla_l\rho}{\rho}\right) = \eta\left(\frac{\partial V_{Fi}}{\partial x_l} + \frac{\partial V_{Fl}}{\partial x_i}\right) \tag{28}$$

with $\eta = \frac{\rho}{2}D(dt)^{\frac{2}{D_F}-1}$ a viscosity-type coefficient. The divergence of this tensor is equal to the usual force density associated with Q:

$$\nabla_i\hat{\sigma}_{il} = -\rho\nabla_l Q \tag{29}$$

(vi) For motions of complex system entities on Peano's curves, at spatial scales higher than the mean free path and at temporal scales higher than the oscillation periods of the pulsating velocities, which overlaps the average velocity of the complex system motion, FHM reduces to the standard hydrodynamics model [24].

(vii) Since the position vector of the complex system entity is assimilated to a Wiener-type stochastic process [6,7,23], ψ is not only the scalar potential of complex velocity (through $\ln\psi$) in the fractal hydrodynamics, but also the density of probability (through $|\psi|^2$) in the Schrödinger-type theory. Then, the equivalence between the fractal hydrodynamics formalism and the Schrödinger one results. Moreover, chaoticity, either through turbulence in the fractal hydrodynamics approach [24] or by means of stochasticization in the Schrödinger-type approach, is exclusively generated by the non-differentiability of the movement trajectories in a fractal space.

4. Non-Differentiable Entropy, Uncertainty Relations

We can rewrite the specific non-differentiable potential in the form:

$$\begin{aligned} Q(\mathbf{r},t) &= -D^2(dt)^{(\frac{4}{D_F})-2}\left[\frac{\nabla^2\rho}{\rho} - \frac{1}{2}\left(\frac{\nabla\rho}{\rho}\right)^2\right] \\ &= -D^2(dt)^{(\frac{4}{D_F})-2}\left[\frac{1}{2}(\nabla\ln\rho)^2 + \nabla\nabla\ln\rho\right] \end{aligned} \tag{30}$$

Let us define a logarithmic function:

$$S_Q(\mathbf{r},\mathbf{p},t) = \ln\rho(\mathbf{r},\mathbf{p},t) \tag{31}$$

that will be called later non-differentiable entropy. It resembles Boltzmann entropy. However, if Boltzmann entropy characterizes the disorder degree of a classical system, the non-differentiable entropy evaluates the analogous quality of the non-differentiable system mentioned above.

Substituting (31) into Equation (30), we find that the specific non-differentiable potential can be expressed in terms of this function:

$$Q(\mathbf{r},\mathbf{p},t) = -\frac{1}{2}D^2(dt)^{(\frac{4}{D_F})-2}(\nabla S_Q)^2 - D^2(dt)^{(\frac{4}{D_F})-2}\nabla^2 S_Q \tag{32}$$

In this equation, the term $-\frac{1}{2}D^2(dt)^{(\frac{4}{D_F})-2}(\nabla S_Q)^2$ relates to the kinetic energy of the complex system entity, while the term $-D^2(dt)^{(\frac{4}{D_F})-2}\nabla^2 S_Q$ relates to its potential energy.

The FHM uncertainty relations result quite naturally from the momentum perturbations associated with the non-differentiable stresses, *i.e.*, by means of non-differentiable entropy. The specific non-differentiable potential Q affects the complex system entity similar to a hydrodynamic pressure with a driving specific non-differentiable force, $-\nabla Q$. Introducing the identity:

$$-\rho\nabla Q = \nabla\cdot[\rho D^2(dt)^{\frac{4}{D_F}-2}\nabla\nabla S_Q], \tag{33}$$

Equation (27) and the momentum conservation law give:

$$\nabla\cdot[\rho(m_0\mathbf{V}_D m_0\mathbf{V}_D)] = \nabla\cdot[\rho m_0^2 D^2(dt)^{\frac{4}{D_F}-2}\nabla\nabla S_Q] + ..., \tag{34}$$

where m_0 is the rest mass of the complex system entity. Accordingly, non-differentiable stresses are, in their possible effects, potentially equivalent to momentum stresses $p_i p_j = -m_0\mathbf{V}_{Di}m_0\mathbf{V}_{Dj}$ imparted to the fractal hydrodynamic fluid associated with the entity:

$$\mathbf{pp} = -m_0^2 D^2(dt)^{\frac{4}{D_F}-2}\left[\frac{\nabla\nabla(\exp S_Q)}{\exp S_Q} - \nabla S_Q\cdot\nabla S_Q\right] \tag{35}$$

The expectation values (average values) of the momentum stresses $\langle p_i p_j\rangle$ represent the observable momentum stresses of the complex system entity. According to Equation (35),

$$\langle\mathbf{pp}\rangle = m_0^2 D^2(dt)^{\frac{4}{D_F}-2}\iiint\rho\nabla S_Q\nabla S_Q d\mathbf{r} \tag{36}$$

since:

$$\iiint\nabla\nabla(\exp S_Q)d\mathbf{r} = \oint\nabla(\exp S_Q)d\mathbf{l} = 0. \tag{37}$$

According to Nottale's works [6,7] and the previous Relations (36) and (37), the momentum stresses $p_i p_j$, Equation (35), are generated by unobservable (first term) and observable (second term) stresses. The observable momentum stresses are given by the dyad:

$$\mathbf{qq} = m_0^2 D^2 (dt)^{\frac{4}{D_F}-2} \nabla S_Q \nabla S_Q, \langle \mathbf{qq} \rangle \neq 0. \tag{38}$$

They determine the observable uncertainties (variances) $\triangle x_{ij}$ of the conjugated components of the position tensor $\mathbf{rr}$ of the complex system entity. Thus, one finds from Equation (38) the relation:

$$\langle q_i q_j \rangle (\triangle x_{ij})^2 = m_0^2 D^2 (dt)^{\frac{4}{D_F}-2} \varepsilon_{ij}^2(s) \tag{39}$$

where:

$$\langle q_i q_j \rangle = m_0^2 D^2 (dt)^{\frac{4}{D_F}-2} \iiint \rho \nabla_i S_Q \nabla_j S_Q d\mathbf{r}, \tag{40}$$

$$(\Delta x_{ij})^2 = \langle (x_i - \langle x_i \rangle)(x_j - \langle x_j \rangle) \rangle \tag{41}$$

$\varepsilon_{ij}(s)$ is a function of the set of quantum numbers specifying the state of the complex system, as we shall establish in the following.

For complex systems with a separable distribution function $\rho(\mathbf{r}, t) = \rho_1(x_1, t)\rho_2(x_2, t)\rho_3(x_3, t)$, the non-diagonal variances vanish: $\Delta x_{ij} = 0$ for $i \neq j$. In this case, Equations (39)–(41) give:

$$\left\langle q_i^2 \right\rangle^{\frac{1}{2}} \Delta x_i = m_0 D (dt)^{\frac{2}{D_F}} - 1 \varepsilon_i(s) \tag{42}$$

where:

$$\left\langle q_i^2 \right\rangle = m_0^2 D^2 (dt)^{\frac{4}{D_F}-2} \iiint \rho (\nabla_i S_Q)^2 d\mathbf{r}, \tag{43}$$

$$(\Delta x_i)^2 = \left\langle (x_i - \langle x_i \rangle)^2 \right\rangle \tag{44}$$

Equation (39) is the tensorial formulation of the uncertainty relations.

For motions of complex system entities on Peano's curves at the Compton scale, the uncertainty relations for the diagonal components, Equation (42), are formally similar to those of wave mechanics for the conjugate variables of momentum and position.

The application of the (fractal hydrodynamic) uncertainty relations to concrete complex systems and the evaluation of the state function are demonstrated in the following example. Using the solution for the test particle in the spherically symmetric Coulomb or Newton fields together with the method from [25], one verifies that:

$$\left\langle q_r^2 \right\rangle = \left(\frac{m_0 D (dt)^{\frac{2}{D_F}-1}}{na} \right)^2 \left[1 - 2\frac{l}{n}\frac{l+1}{2l+1} \right] \tag{45}$$

and:

$$\left\langle (r - \langle r \rangle)^2 \right\rangle = \left(\frac{1}{2} a \right)^2 [n^2(n^2+2) - l^2(l+1)^2], \tag{46}$$

where a are specific Coulomb's or Newton's lengths and n, l are the standard quantum numbers (n is the principal quantum number and l is the orbital quantum numbers).

According to our previous relations, for the r components of the dynamical variables of the test particle in the spherically symmetric Coulomb or Newton fields, Equation (42) becomes:

$$\left\langle q_r^2 \right\rangle^{\frac{1}{2}} \Delta r = \left(m_0 D (dt)^{\frac{2}{D_F}-1} \right) \varepsilon_r(n,l), \quad n = 1, 2, ..., l \leq n-1 \tag{47}$$

where:

$$\varepsilon_r(n,l) = \left\{ \left(1 - 2\frac{l}{n}\frac{l+1}{2l+1} \right) \left[(n^2+2) - \frac{l^2}{n^2}(l+1)^2 \right] \right\}^{\frac{1}{2}} \tag{48}$$

The function of states for this case is $\varepsilon_r(n,l) \geq \sqrt{3}$; in particular,

$$\varepsilon_r(n,l) = (2+n^2)^{\frac{1}{2}}, l = l_{\text{min.}} = 0 \tag{49}$$

$$\varepsilon_r(n,l) = (\frac{2n+1}{2n-1})^{\frac{1}{2}}, l = l_{\text{max.}} = n-1 \tag{50}$$

Equation (38) indicates that the momentum transfer responsible for the indeterminacy phenomenon is given by the fractal momentum:

$$\mathbf{q} = m_0 D (dt)^{(\frac{2}{D_F})-1} \nabla \ln \rho.$$

According to (FHM), the minimum uncertainty products result from the stresses, *i.e.*, non-differentiable entropy of the complex system.

5. Informational Non-Differentiable Entropy

Now, the mean value of the non-differentiable potential (the imaginary part of the scalar potential of the complex speed, $\phi_N = Im\Phi = D(dt)^{(\frac{2}{D_F})-1} S_Q$) can be identified, without a constant factor, with the informational non-differentiable entropy (defined by analogy with the Shannon informational entropy [26–31]):

$$I_N = \langle \phi_N \rangle = \int \exp S_Q \cdot S_Q d\mathbf{r} \tag{51}$$

Accepting a maximization principle for the informational non-differentiable entropy as follows:

$$\delta I_N = \delta \int \exp S_Q \cdot S_Q d\mathbf{r} = 0 \tag{52}$$

for constraints with radial symmetry, we get $\exp S_Q = \exp(-\frac{r}{r_0})$, with $r_0 = \text{const}$. In a fractal space, substituting this value in the expression $-\nabla Q$, with Q given by (27), the force is found:

$$F(r) = -\nabla Q(r) = -\frac{4 m_0 D^2 (dt)^{(\frac{4}{D_F})-2}}{r_0} \frac{1}{r^2} \tag{53}$$

Therefore, the informational non-differentiable entropy through a maximization principle stores and transmits interactions in the form of forces.

Let us consider the probability density in the phase space, $\exp S_Q(p,q)$ with the constraints:

$$\begin{aligned}
\iint q \exp S_Q(p,q) dpdq &= \overline{q} \\
\iint p \exp S_Q(p,q) dpdq &= \overline{p} \\
\iint (q-\overline{q}) \exp S_Q(p,q) dpdq &= (\delta q)^2 \\
\iint (p-\overline{p}) \exp S_Q(p,q) dpdq &= (\delta p)^2 \\
\iint (q-\overline{q})(p-\overline{p}) \exp S_Q(p,q) dpdq &= cov(p,q)
\end{aligned} \tag{54}$$

where $\overline{q}$ is the mean value of the position, $\overline{p}$ is the mean value of the momentum, δq is the position standard deviation, δp is the momentum standard deviation and $cov(p,q)$ is the covariance of the random variables (p,q).

We now introduce informational non-differentiable entropy:

$$I_N = \iint \exp S_Q(p,q) S_Q dpdq. \tag{55}$$

Using the principle of maximum informational non-differentiable entropy (52) with constraints (54), we obtain the normalized Gaussian distribution:

$$\exp S_Q(p-\overline{p}, q-\overline{q}) = \frac{\sqrt{ac-b^2}}{2\pi} \exp[-H(p-\overline{p}, q-\overline{q})] \tag{56}$$

with:

$$\begin{aligned}
H(p-\overline{p}, q-\overline{q}) &= \frac{1}{2}[a(p-\overline{p})^2 + 2b(p-\overline{p})(q-\overline{q}) + c(q-\overline{q})^2] \\
a &= \frac{(\delta q)^2}{\Delta}, b = -\frac{cov(p,q)}{\Delta}, c = \frac{(\delta p)^2}{\Delta} \\
\Delta &= (\delta p)^2(\delta q)^2 - cov^2(p,q).
\end{aligned} \tag{57}$$

We notice that the set of parameters (a,b,c) has statistical significance given by Relations (57).

6. Informational Non-Differentiable Energy and Uncertainty Relations

For the informational non-differentiable energy, we shall use a generalization of Onicescu's relation [32,33]:

$$E = \iint\limits_{-\infty}^{\infty} \exp 2S_Q(p,q) dpdq \tag{58}$$

In such a context, the informational non-differentiable energy corresponding to the normalized Gaussians distribution in Equation (56) becomes:

$$E(a,b,c) = \iint\limits_{-\infty}^{\infty} \exp 2S_Q(p,q) dpdq \tag{59}$$

where $H(p,q) > 0$ is a condition imposed by integral (58).

We thus get:

$$E(a,b,c) = \frac{\sqrt{a\,c - b^2}}{2\pi} \tag{60}$$

Therefore, if H has energetic significance, it results that:

(i) The informational non-differentiable energy is an indication of the dispersion distribution (56), since the quantity:

$$A = \frac{2\pi}{\sqrt{a\,c - b^2}} \tag{61}$$

is a measure of ellipse areas of equal probability (or of equal non-differentiable entropy) $\exp S_Q = \text{const}$. Then, the normalized Gaussian becomes even more clustered, so that their informational non-differentiable energy will be higher.

(ii) The class of statistical hypotheses is specific to the Gaussians having the same mean given by the constant value of the informational non-differentiable energy.

(iii) If the informational non-differentiable energy is constant, then Relations (57) and (58) give the egalitarian uncertainty relation:

$$(\delta p)^2(\delta q)^2 = \frac{1}{4\pi^2 E^2(a,b,c)} + cov^2(p,q) \tag{62}$$

or the non-egalitarian one:

$$\delta p \delta q \geq \frac{1}{2\pi E(a,b,c)} \tag{63}$$

Let us exemplify the above results for the linear oscillator. In the phase space (p,q), the energy $H(p,q)$,

$$H(p,q) = \frac{p^2}{2m} + \frac{m 4\pi^2 \nu^2 q^2}{2} \tag{64}$$

with the oscillator's mass m and its frequency ν representing the ellipse:

$$\frac{p^2}{{a_0}^2} + \frac{q^2}{{b_0}^2} = 1 \tag{65}$$

of semiaxes:

$$a_0 = \sqrt{2mH}, b_0 = \sqrt{\frac{2H}{4\pi^2\nu^2 m}} \tag{66}$$

The correspondences:

$$a = \frac{1}{mH}, b = 0, c = \frac{4\pi^2\nu^2 m}{H} \tag{67}$$

result, in which case, the informational non-differentiable energy (60) becomes:

$$E = \frac{\nu}{H}. \tag{68}$$

If $E(a,b,c) = \text{const.}$, then:

$$\frac{H}{\nu} = \text{const.} \tag{69}$$

However, H/ν satisfies the quantification condition

$$\frac{H}{\nu} = nh, n = 1, 2, ... \tag{70}$$

We get:

(i) The informational non-differentiable energy is quantified:

$$E(a, b, c) = \frac{1}{nh} \tag{71}$$

(ii) (63) implies the uncertainty relation:

$$\delta p \delta q \geq n\hbar \tag{72}$$

or, for $n = 1$, the standard relation:

$$\delta p \delta q \geq \hbar \tag{73}$$

7. Conclusions

The main conclusions of the present paper are the following:

(i) Any complex structure implies test particles, field sources, *etc.*, correlated with various types of forces, together with the non-differentiable medium in which they evolve. The non-differentiable (fractal) medium is assimilated to a fractal fluid, whose particles are moving on continuous, but non-differentiable, curves. Moreover, the non-differentiable medium that cannot be separated from test particles and field sources is described either by a Schrödinger-type equation or by non-differentiable hydrodynamics with non-differentiable potential, which works simultaneously with standard potentials. The non-differentiable potential is induced by the non-differentiability of the movement curves of fractal fluid entities.

(ii) The dynamics of a complex system is described by motion equations for a complex speed field and exhibit rheological properties (memory).

(iii) Separation movements on the interaction scales imply non-differentiable hydrodynamics, which, at the differentiable scale, contains the law of momentum conservation and, at the non-differentiable scale, the law of probability density (states density) conservation.

(iv) The correlation fractal potential-non-differentiable entropy provides uncertainty relations in the fractal hydrodynamic approach. These relations are explained for the case of a test particle motion in spherically symmetric Coulomb or Newton fields.

(v) The correlation informational non-differentiable entropy-informational non-differentiable energy provides specific uncertainty relations through a maximization principle of the informational non-differentiable entropy and for a constant value of the informational non-differentiable energy. For a linear harmonic oscillator, the constant value of the informational non-differentiable energy is equivalent to a quantification condition.

Concepts, such as non-differentiable entropy, informational non-differentiable entropy, informational non-differentiable energy, *etc.*, can prove to be essential in defining wave-corpuscle duality and, moreover, in the formulation of some fundamental equations in physics, such as the Klein–Gordon equation, the Dirac equation, *etc.*

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Author Contributions

In this paper, Maricel Agop provided the original idea and constructed its framework. Together with Alina Gavriluţ, they conducted the detailed calculation and were responsible for drafting and revising the whole paper. Gabriel Crumpei devoted efforts to some valuable comments on revising the paper. Bogdan Doroftei devoted efforts to revising the paper. All authors have read and approved the final manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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