Asymptotic resolution of a “perfect lens” with negative index of refraction

by David Benjamin, University of Michigan - Ann Arbor

in collaboration with Roberto Merlin, University of Michigan - Ann Arbor

an undergraduate thesis submitted

in partial fulfillment of

the Honors Bachelor of Science in Physics

April 6, 2007

David Benjamin

Roberto Merlin
Acknowledgment

I am indebted to my advisor, Professor Roberto Merlin, for his unfaltering patience, for his confidence in me, and for his wisdom, which is responsible for whatever lack of foolishness I have achieved thus far as a physicist. If, hopefully, I now know how to make some small contribution to science, it is thanks to his guidance.
Abstract

We estimate the resolution of a lossless slab of negative refractive index with permittivity and permeability $\epsilon = -1 + \Delta \epsilon$, $\mu = -1 + \Delta \mu$, where $|\Delta \epsilon|, |\Delta \mu| << 1$, in three dimensions, i.e. with a point dipole as a source of radiation. The range of evanescent Fourier components preserved by the lens grows logarithmically as $\Delta \epsilon$ and $\Delta \mu$ approach zero. Hence we show that the resolution varies in inverse proportion to the logarithms of $\Delta \epsilon$ and $\Delta \mu$. As a consequence of the logarithmic dependence, we see that sub-wavelength resolution, while attained, improves very slowly as the optical parameters approach their ideal values and attain impedance matching with the surrounding medium.
## Contents

1 Introduction 5
   1.1 Brief Summary of Negative Refraction ............................ 5
   1.2 Negative \( \epsilon \) and \( \mu \) ........................................ 5
   1.3 A Perfect Lens? ...................................................... 6

2 Theoretical Results 8
   2.1 Geometry, Definitions, and other Preliminaries .................... 9
   2.2 Transverse Decomposition of the Source Field ...................... 10
   2.3 Applying Transmission Coefficients ................................ 11
   2.4 Analysis ............................................................. 13
       2.4.1 Approximating the Transmission Coefficients ................. 14
       2.4.2 An Estimate for the Resolution .............................. 16
       2.4.3 Conclusion ...................................................... 18

3 Appendices 19
   3.1 Appendix 1: Derivation of the Transmission Coefficients .......... 19
   3.2 Appendix 2: Integral Identities .................................... 21
   3.3 Appendix 3: Location of the Transmission Singularity ............. 22
   3.4 Appendix 4: A Note on the Numerical Computations ................ 24
   3.5 Appendix 5: Graphs of the Transmission Coefficient for Different \( \epsilon \) .... 27
   3.6 Appendix 6: Plot of the Transmitted Field over \( r \) and \( z \) ......... 29
   3.7 Appendix 7: Graphs of the Transmitted Field Integrand at Varying \( r \) .... 30
   3.8 Appendix 8: A Dipole Source Resolved into Rings ................. 34
   3.9 Appendix 9: Calculated Values of the Resolution Width .......... 35
1 Introduction

1.1 Brief Summary of Negative Refraction

In recent years there has been considerable interest in the use of materials with a negative index of refraction $n$. In 1968, Victor Veselago [1] showed that it is necessary to take the negative square root of the equation $n = \pm \sqrt{\varepsilon \mu}$, where $\varepsilon$ and $\mu$ are the relative permittivity and permeability of an electromagnetic medium, when $\varepsilon$ and $\mu$ are both negative. That $n$ is negative rather than positive has no bearing on its most common role, as the speed of light in the medium relative to the speed of light in a vacuum. Rather, it affects vector properties: in a medium with negative $n$, the electric field $E$, the magnetic field $H$, and the wave vector $k$ form a left-handed, instead of a right-handed, set, and the group velocity’s direction is opposite that of the phase velocity. The derivation of these facts follows straightforwardly from Maxwell’s equations. Some consequences of a negative index of refraction are a reversal of direction in the Doppler effect and Cherenkov radiation, as well as refraction on the same side of the normal line as the incident radiation in Snell’s Law. It is the last of these that matters presently.

1.2 Negative $\varepsilon$ and $\mu$

If only to dispel doubts that may prove a distraction to the ensuing discussion, it seems helpful to mention some reasons why negative refraction exists as more than a theoretical curiosity. As Veselago showed, a negative index of refraction is equivalent to simultaneously negative permittivity and permeability. Negative permittivity is actually not a particularly exotic phenomenon and occurs in many common metals, such as silver, for some frequencies. Since permittivity is the relationship between an applied field (at
some frequency $\omega$) and an induced polarization, a negative value of $\epsilon$ can occur for frequencies $\omega$ just above electronic resonances $\omega_0$. This is analogous to the mechanical phenomenon in which driving a system just faster than its resonant frequency yields a response that is opposite in phase to the driving force. Such an explanation is, of course, cursory, since it ignores absorption, the presence of multiple resonances, and plasma modes, but it is the fundamental idea behind schemes to make a material with a negative index of refraction.

Negative permeability is, similarly, the result of magnetic dipole resonances. It occurs in natural materials more rarely than negative permittivity, and at much lower frequencies. Because $\epsilon(\omega)$ and $\mu(\omega)$ are never simultaneously negative in natural materials, it is necessary to engineer a “metamaterial” to have magnetic and electric dipole resonances at the same frequency. In 1999, John Pendry [2] proposed embedded artificial structures inside a material so that it would inherit their resonances. Note that as long as the periodicity of the artificial structures is insignificant compared to some wavelength, the material can essentially be considered homogeneous and described by single values of $\epsilon$ and $\mu$ for light of that wavelength. Pendry’s metamaterial elements were wires with periodic gaps and incomplete ‘C’-shaped circuits, which produce electric and magnetic dipole resonances, respectively. Clearly, these have the advantage over nature that their resonant frequencies can be tuned by adjusting size.

### 1.3 A Perfect Lens?

In Veselago’s original paper [1], he noted that a planar slab with an index of refraction $n = -n_0$, where $n_0$ is the refractive index of the surrounding medium, would focus a point source of light at a single point. This is a consequence of simple ray-
tracing, using Snell’s Law for \( n = -n_0 \) at two consecutive boundaries. We do not have to worry about reflection because impedance matching ensures perfect transmission. In 2000, Pendry [3] proposed that such a lens could circumvent the diffraction limit and resolve features smaller than the wavelength. Let us see briefly how this could occur.

A plane wave can be written as:

\[
E = e^{i(k \cdot r - \omega t)} = e^{i(k_x x + k_y y + k_z z - \omega t)} \quad \text{where} \quad k_x^2 + k_y^2 + k_z^2 = k_0^2 = \frac{\omega^2}{c^2} \tag{1}
\]

In unbounded space, we must have \( k_x, k_y, \) and \( k_z \) all real, since otherwise the exponential grows without bound in some direction. However, plane waves with imaginary wave vector components are physical solutions to Maxwell’s equations in regions of space where the exponential cannot grow without bound. For example, in the calculations below we come across waves that decay in the \( z \)-direction away from their source with the following form:

\[
H_{\text{spatial}} = e^{i(k_x x + k_y y)} e^{-\kappa |z|} \quad \text{where} \quad \kappa_z = \sqrt{k_x^2 + k_y^2 - k_0^2} = ik_z \tag{2}
\]

In ordinary physical situations we usually choose the sign of the square root so as to express decay. Therefore, waves with \( k_x^2 + k_y^2 > k_0^2 = \omega^2/c^2 \), for example, cannot propagate in the \( z \) direction. The diffraction limit arises from the fact that the wave vector is the Fourier transform of the spatial waveform. It is a basic theorem of Fourier analysis that the variance of a distribution and the variance of its Fourier transform are inversely proportional. We therefore expect that the spatial spread of a wave once its evanescent components have decayed is at least of the order \( 1/k_0 \), which is proportional to the wavelength \( \lambda = 2\pi/k_0 \). Hence the diffraction limit is a consequence of the loss of plane wave solutions to Maxwell’s equations for which some wave vector components are large.
Pendry realized that the Veselago lens made out of material with \( n = -1 \) in the region \( 0 \leq z \leq \ell \) (we henceforward assume without loss of generality that \( n = 1 \) in the surrounding medium) could preserve evanescent waves, yielding a full spectrum in k-space, hence a narrow focus in physical space. He showed that another consequence of a negative index of refraction is that waves of the form \( e^{i(k_xx + k_yy)}e^{-\kappa z} \) outside of the lens couple to waves \( e^{i(k_xx + k_yy)}e^{\kappa z} \) inside the lens. Thus waves that decay outside of the lens are amplified inside the lens, and, as Pendry showed, do so in such a way as to exactly recover an image on the opposite side of the lens from its source. Since this process in theory preserves all evanescent wave components, such a lens could resolve arbitrarily small features, hence the terms “perfect lens” and “superlens.”

2 Theoretical Results

Pendry’s paper is, of course, not the final word on the subject. Obtaining \( n(\omega) = -1 \) for any wavelength is extremely difficult; for an engineered material there will necessarily be limits the precision of \( \epsilon \) and \( \mu \). Furthermore, a real light source is not monochromatic, and contains a spread of frequencies. This is especially problematic because, by the nature of the resonances needed to obtain a negative index of refraction, \( n(\omega) \) is most dispersive, ie, sensitive to small variations in \( \omega \), when \( n(\omega) < 0 \). Thus any wave packet contains frequencies for which \( n \neq -1 \). It is therefore of theoretical and practical interest to determine how “perfect” a Veselago lens remains when its optical parameters \( \epsilon \) and \( \mu \) are slightly imperfect. Furthermore, even if there were a monochromatic light source of frequency \( \omega \) such that \( n(\omega) = -1 \) inside the lens, the solution in which evanescent modes couple perfectly with amplified modes is non-physical. It turns
out that the condition for perfect transmission also determines the resonant frequency for surface polaritons of the lens [4]; the transmitted electromagnetic field diverges in many places when one tries to calculate it. A solution of the almost-perfect lens problem, in which the electromagnetic field transmitted by the lens is calculated asymptotically as $n \to -1$, is therefore not only useful for judging the feasibility of such a lens; it is also the necessary approach to achieve mathematical legitimacy.

2.1 Geometry, Definitions, and other Preliminaries

We express the Veselago lens geometrically as follows: the region $0 \leq z \leq d$ is occupied by a material with $\epsilon = -1 + \Delta \epsilon$ and $\mu = -1 + \Delta \mu$. The regions $z \leq 0$ and $z \geq d$ are vacuum. At the point $z = -d/2$ there is an electric dipole with current $j = P_0 \hat{n} \delta(x) \delta(y) \delta(z + d/2)e^{-i\omega t}$. Without loss of generality, the polarization can be taken to oscillate in the $x-z$ plane. There is a focal point inside the slab at $z = d/2$, and the focal plane outside of the slab is $z = 3d/2$, according to Snell’s law for negative $n$ and ray-tracing. The Hertz potential $\Pi$ of this dipole, defined by $E = \nabla \times (\nabla \times \Pi)$ and $H = \frac{1}{c} \nabla \times \frac{\partial}{\partial t} \Pi$ is given by [5]:

$$\Pi = e^{-i\omega t}(P_0/2\pi)\hat{n} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dq_x dq_y}{\kappa_0} e^{i(q_x x + q_y y)} e^{-\kappa_0 |z + d/2|}$$

(3)

where $\hat{n}$ is the polarization unit vector, $q^2 := q_x^2 + q_y^2$, and

$$\kappa = \begin{cases} 
-i \sqrt{\epsilon \mu \omega^2/c^2 - q^2} & \text{for } q^2 \leq \epsilon \mu \omega^2/c^2 \\
\sqrt{q^2 - \epsilon \mu \omega^2/c^2} & \text{for } q^2 \geq \epsilon \mu \omega^2/c^2
\end{cases}$$

(4)
We take the necessary derivatives of the Hertz potential to obtain the magnetic field due to the dipole in the region $-d/2 < z \leq 0$:

$$
H_{\text{source}} = f(t) \int \frac{dq_x dq_y}{\kappa_0} e^{i(q_x x + q_y y)} e^{-\kappa_0 (z + d/2)} \mathbf{v}(q_x, q_y),
$$

(5)

$$
\mathbf{v}(q_x, q_y) := in_x q_y \mathbf{\hat{x}} - (n_x \kappa_0 + in_z q_x) \mathbf{\hat{y}} - in_z q_y \mathbf{\hat{z}},
$$

(6)

$$
f(t) := -i \omega e^{-i\omega t} \left( P_0 / 2\pi \right)
$$

(7)

### 2.2 Transverse Decomposition of the Source Field

Later in the calculation we will use transmission coefficients for plane waves polarized parallel to the surfaces of the lens. Therefore, it is useful to write each Fourier component of the source wave as the sum of two plane waves: one (TE, for transverse electric) with its electric field polarized parallel to the lens and one (TM, for transverse magnetic) with its magnetic field polarized parallel to the lens. That is, we need to write for each $q_x$ and $q_y$:

$$
-i \omega e^{-i\omega t} \left( P_0 / 2\pi \right) \int \frac{dq_x dq_y}{\kappa_0} e^{i(q_x x + q_y y)} e^{-\kappa_0 (z + d/2)} \mathbf{v}(q_x, q_y) = H_{\text{TM}} + H_{\text{TE}}
$$

(8)

where $H_{\text{TE}}$ is derived from the electric field of a TE wave by Faraday’s Law $-\frac{\partial}{\partial t} H_{\text{TE}} = \nabla \times E_{\text{TE}}$. Using the fact that the only time dependence is $e^{-i\omega t}$, we see that

$$
H_{\text{TE}} = (ic/\omega) \nabla \times E_{\text{TE}}
$$

(9)

To simplify the calculation, we shall work in rotated coordinates $(x', y')$ in which $q x' = q_x x + q_y y$, $q y' = -q_y x + q_x y$ and $q_x = q \cos \phi$, $q_y = q \sin \phi$. The angle $\phi$ is therefore the same angle that appears below when we integrate using polar coordinates in two-dimensional $q$-space. This allows us to write $H_{\text{TM}}$ and $E_{\text{TE}}$ as parallel to the $y'$ axis.
The decomposition into TE and TM components proceeds as follows:

\[
\begin{align*}
H^{TM} &= h_0 \hat{y} e^{iq'x} e^{\kappa_0(z+d/2)} \\
E^{TE} &= e_0 \hat{y} e^{iq'x} e^{\kappa_0(z+d/2)} \\
H^{TE} &= (ic/\omega)[\kappa_0 \hat{x} + iq\hat{z}] e^{iq'x} e^{\kappa_0(z+d/2)}
\end{align*}
\]

(10)

If we now apply the same rotation of coordinates by the angle \(\phi\) in the \(x\)-\(y\) plane to the magnetic field and let

\[
\Psi := e^{i(q_x x + q_y y)} e^{-\kappa_0(z+d/2)} e^{-i\omega t} = e^{iq'x} e^{\kappa_0(z+d/2)}
\]

(11)
denote the exponential dependence of a plane wave, we find that the above decomposition gives:

\[
\begin{align*}
H_{x'} &= H_x \cos \phi + H_y \sin \phi = (ic\epsilon_0\kappa_0/\omega)\Psi \\
H_{y'} &= -H_x \sin \phi + H_y \cos \phi = h_0\Psi \\
H_{z'} &= H_z = (-qce_0/\omega)\Psi
\end{align*}
\]

(12)

From the earlier determination of the source magnetic field (Equation 5) from the Hertz potential of a dipole, we had seen that:

\[
\begin{align*}
H_x &= \Psi(\alpha/\kappa_0)in_q dq_x dq_y \\
H_y &= \Psi(\alpha/\kappa_0)(-in_q x - n_x \kappa_0) dq_x dq_y \\
H_z &= -\Psi(\alpha/\kappa_0)in_q dq_x dq_y
\end{align*}
\]

(13)

where we have defined \(\alpha \equiv i\omega P_0/2\pi\kappa_0\). Substituting (12) into (11), we find that the coefficients of the decomposed TM and TE fields are:

\[
\begin{align*}
e_0 &= (1/c)\alpha in_x \omega \sin \phi dq_x dq_y \\
h_0 &= -\alpha (in_q + n_x \kappa_0 \cos \phi) dq_x dq_y
\end{align*}
\]

(14)

2.3 Applying Transmission Coefficients

In the appendices, we derive transmission coefficients for plane waves polarized parallel to the surface of the slab. These transmission coefficients \(T_M(q, \epsilon, \mu)\) and
$T_E(q, \epsilon, \mu)$ are defined so that an incident plane wave $\mathbf{H}_s = \hat{y}' e^{iqx'} e^{-\kappa_0(z + d/2)}$ yields a transmitted plane wave $\mathbf{H}_t = T_M(q) \hat{y}' e^{iqx'} e^{-\kappa_0(z + d/2)}$. The function $T_E(\epsilon, \mu, q)$ for plane waves with parallel-polarized electric fields is defined analogously: an incident plane wave $\mathbf{E}_s = \hat{y}' e^{iqx'} e^{-\kappa_0(z + d/2)}$ yields a transmitted plane wave $\mathbf{E}_t = T_E(q) \hat{y}' e^{iqx'} e^{-\kappa_0(z + d/2)}$. The results of these calculations are:

$$T_M(q) = \frac{4\kappa_0 \kappa e^{\kappa d/2}}{e^{\kappa d}(\kappa + \epsilon \kappa_0)^2 - e^{-\kappa d}(\kappa - \epsilon \kappa_0)^2}$$  \hspace{1cm} (15)

$$T_E(q) = \frac{4\mu_0 \kappa e^{\kappa d/2}}{e^{\kappa d}(\kappa + \mu \kappa_0)^2 - e^{-\kappa d}(\kappa - \mu \kappa_0)^2}$$  \hspace{1cm} (16)

In order to apply these transmission coefficients to obtain a transmitted field from the dipole source field, we perform the aforementioned decomposition $\mathbf{H} = \mathbf{H}^{TM} + \mathbf{H}^{TE}$ for each Fourier component. We then substitute the values $e_0$ and $h_0$ (Equation 14) into the expression (Equation 12) for the source magnetic field in $x'$ - $y'$ coordinates. If we multiply terms with $e_0$ and $h_0$ by the corresponding transmission coefficients $T_E$ and $T_M$, we obtain the transmitted field due to a single Fourier component in $x'$ - $y'$ coordinates:

$$\begin{align*}
\mathbf{H}_{x'} &= -\alpha n_x \sin \phi e^{iqx'} e^{\kappa_0(z + d/2)} dq_x dq_y \\
\mathbf{H}_{y'} &= \left[\frac{i\alpha n_x}{\kappa_0} + \alpha n_x \cos \phi\right] e^{iqx'} e^{\kappa_0(z + d/2)} dq_x dq_y \\
\mathbf{H}_{z'} &= -\frac{i\alpha n_x}{\kappa_0} \sin \phi e^{iqx'} e^{\kappa_0(z + d/2)} dq_x dq_y
\end{align*}$$  \hspace{1cm} (17)

We rotate back with the inverse transformation:

$$\begin{pmatrix}
\mathbf{H}_x \\
\mathbf{H}_y \\
\mathbf{H}_z
\end{pmatrix} = \begin{pmatrix}
\cos \phi & -\sin \phi & 0 \\
\sin \phi & \cos \phi & 0 \\
0 & 0 & 1
\end{pmatrix}\begin{pmatrix}
\mathbf{H}_{x'} \\
\mathbf{H}_{y'} \\
\mathbf{H}_{z'}
\end{pmatrix}$$  \hspace{1cm} (18)
to obtain the following expression for the transmitted magnetic field in the original unprimed coordinate system:

\[
\begin{align*}
\mathbf{H}_x &= \int \int \alpha dq_x dq_y e^{i(q_x x + q_y y)} e^{-\kappa_0 z} \left[(in_x \omega \sin \phi/c)(ic/\omega)T_E \cos \phi + (in_z (q/\kappa_0) + n_x \cos \phi) \sin \phi T_M\right] \\
\mathbf{H}_y &= \int \int \alpha dq_x dq_y e^{i(q_x x + q_y y)} e^{-\kappa_0 z} \left[(in_z \omega \sin \phi/c)(ic/\omega)T_E \sin \phi - (in_z (q/\kappa_0) + n_x \cos \phi) \cos \phi T_M\right] \\
\mathbf{H}_z &= \int \int \frac{\alpha}{\kappa_0} dq_x dq_y e^{i(q_x x + q_y y)} e^{-\kappa_0 z} \left[(in_z \omega \sin \phi/c)(-qc/\omega)T_E\right]
\end{align*}
\]

We obtain the final exact expression for the transmitted magnetic field by using the definite integrals involving Bessel functions found in the appendices. We obtain the next set of expressions by transforming to the polar coordinates (in \(q\)-space) \(q_x = q \cos \phi, q_y = q \sin \phi\) and simplifying slightly. We use polar coordinates for the geometric variables as well: \(x = r \cos \psi, y = r \sin \psi\).

\[
\begin{align*}
\mathbf{H}_x &= \int_0^\infty dq \alpha e^{-\kappa_0 z} \int_0^{2\pi} d\phi \left[\left(n_x \sin \phi \cos (T_M - T_E) + in_z (q/\kappa_0) \sin \phi T_M\right) e^{iqr \cos (\phi - \psi)}\right] \\
\mathbf{H}_y &= \int_0^\infty dq \alpha e^{-\kappa_0 z} \int_0^{2\pi} d\phi \left[-n_x (\sin^2 \phi T_E - \cos^2 \phi T_M) + in_z (q/\kappa_0) \cos \phi T_M\right] e^{iqr \cos (\phi - \psi)} \\
\mathbf{H}_z &= \int_0^\infty dq (-i\alpha(n_x/\kappa_0))q^2 T_E e^{-\kappa_0 z} \int_0^{2\pi} d\phi \sin \phi e^{iqr \cos (\phi - \psi)}
\end{align*}
\]

We obtain the final exact expression for the transmitted magnetic field by using the definite integrals involving Bessel functions found in the appendices.

\[
\begin{align*}
\mathbf{H}_x &= \int_0^\infty -\alpha e^{-\kappa_0 z} \left[n_x \sin(2\psi)J_2(qr)(T_M - T_E) + 2q(n_z/\kappa_0) \sin(\psi)J_1(qr)T_M\right] dq \\
\mathbf{H}_y &= \int_0^\infty -\alpha e^{-\kappa_0 z} \left[n_x (J_0(qr) + (\cos(2\psi)/\kappa_0)J_2(qr)) T_E + n_x (J_0(qr) - (\cos(2\psi)/\kappa_0)J_2(qr)) + 2\pi q(n_z/\kappa_0) \cos(\psi)J_1(qr)\right] T_M dq \\
\mathbf{H}_z &= \int_0^\infty \alpha q^2 e^{-\kappa_0 z}(n_x/\kappa_0) \sin(\psi)J_1(qr)T_E dq
\end{align*}
\]

2.4 Analysis

We first note an obvious geometric corollary: \(\mathbf{H}_x\) and \(\mathbf{H}_z\) vanish at \(r = 0\) since \(J_m(0) = 0\) unless \(m = 0\), and \(\mathbf{H}_y\) vanishes at \(r = 0\) if \(n_x = 0\). Thus the field vanishes (the
electric field vanishes as well by Maxwell’s equations) at the “focal point” for a dipole polarized perpendicular to the surface of the lens.

For simplicity, consider the case $n_x = 0$ (and hence $n_z = 1$). This restriction yields particularly simple estimates for the resolution of the lens, although it will become apparent below that the arguments used apply more generally. By symmetry, the magnitude of the transmitted magnetic field only depends on $z$ and the distance $r$ from the $z$-axis. Thus we can choose any value of the polar angle $\psi$ to carry out the calculation. The combination $n_x = 0, \psi = 0$ immediately gives (Equation 21) $H_x = H_z = 0$ and,

$$
H_y = -2\pi \alpha \int_0^\infty \frac{q^2}{\kappa_0} e^{-\kappa_0 z} J_1(qr) T_M(q) dq
$$

(22)

We can define the resolution width of the lens in two related ways: first, as the smallest (non-zero, given the anomalous vanishing at the focal point) value of $r$ in the focal plane $z = 3d/2$ for which the above integral vanishes; and second, as the value of $r$ for which the integral is maximum. Note that this second definition is only non-trivial when the field vanishes at $r = 0$, since normally $r = 0$ would be the maximum of the field.

2.4.1 Approximating the Transmission Coefficients

The essential part of the calculation is the behavior of the transmission functions $T_E(q)$ and $T_M(q)$, which we shall now discuss. A graphical accompaniment to the analysis below is found in Appendix 5. From Equation 15, we have $T_M(q) = N(q)/D(q)$, where the numerator $N$ and the denominator $D$ are smooth functions of $q$ on $(0, \infty)$ defined
by:

\[ N(q) = 4\epsilon\kappa_0\kappa e^{\kappa_0 d/2} \]  

\[ D(q) = e^{\kappa d}(\kappa + \epsilon\kappa_0)^2 - e^{-\kappa d}(\kappa - \epsilon\kappa_0)^2 \]  

The denominator \( D(q) \) vanishes for some value \( q_0 \), when the \( e^{\kappa d} \) and \( e^{-\kappa d} \) terms are equal. Empirically, this occurs at sufficiently large \( q_0 \) such that \( e^{\kappa d} \) and \( e^{-\kappa d} \) are, respectively, growing and shrinking very quickly. Thus, they are comparable in size with each other only for a small range of \( q \) near \( q_0 \). It is therefore clear that for most \( q < q_0 \), the \( e^{-\kappa d} \) term dominates, and then

\[ T_M(q) \approx \frac{4\epsilon\kappa_0\kappa e^{\kappa_0 d/2}}{e^{-\kappa d}(\kappa - \epsilon\kappa_0)^2} \approx e^{3\kappa_0 d/2} \]  

(25)

since \( \epsilon \approx -1 \) and \( \kappa = \sqrt{q^2 - k_0^2} \approx \sqrt{q^2 - \epsilon\mu k_0^2} = \kappa \). Likewise, for most \( q > q_0 \), the \( e^{\kappa d} \) term dominates, and

\[ T_M(q) \approx \frac{4\epsilon\kappa_0\kappa e^{\kappa_0 d/2}}{e^{\kappa d}(\kappa + \epsilon\kappa_0)^2} = O(e^{-\kappa_0 d/2}) \approx 0 \]  

(26)

As for when \( q \) is near \( q_0 \), the above arguments no longer apply since both terms \( e^{\kappa d} \) and \( e^{-\kappa d} \) are significant. However, as discussed in the appendices, \( T_M(q) \) behaves more and more closely to a function that is odd about \( q = q_0 \), and thus the Cauchy principal value of an integral involving the product of \( T_M(q) \) and a non-singular function over a small range centered about \( q_0 \) ought to be small. This is justified to first order in the appendices. It is therefore apparent that, to good approximation, we can ignore the part of the integral in which \( q \) is near \( q_0 \). Furthermore, we have also seen that for \( q \) sufficiently larger than \( q_0 \), \( T_M(q) \approx 0 \). Thus the integral for the transmitted field becomes:

\[ H_y = -2\pi\alpha \int_0^{q_0} \frac{q^2}{\kappa_0} e^{-\kappa_0 z} J_1(qr) T_M(q) dq \approx -2\pi\alpha \int_0^{q_0} \frac{q^2}{\kappa_0} e^{-\kappa_0 z} J_1(qr) e^{3\kappa_0 d/2} dq \]  

(27)
In the focal plane \( z = 3d/2 \) this simplifies to:

\[
H_y = -2\pi\alpha \int_0^{q_0} \frac{q^2}{\kappa_0} J_1(qr)dq
\]

(28)

Note that the dependence on \( r \), the radial distance in the focal plane from the focal point, is accomplished through the \( r \) inside the Bessel function \( J_1(qr) \). Note also that the above approximations for \( T_M \) apply equally well to \( T_E \), once we know where the singularity \( q_0 \) occurs. Finally, we restrict ourselves to the near field and changes variables to the dimensionless parameter \( s = q/k_0 \) to find that:

\[
H_{y,\text{near}} = -2k_0^2\pi\alpha \int_1^{q_0/k_0} \frac{s^2}{\sqrt{s^2 - 1}} J_1((k_0r)s)ds
\]

(29)

2.4.2 An Estimate for the Resolution

In addition to the above analysis of the transmission functions, we only need to know that the Bessel function \( J_1 \) (or \( J_0 \) and \( J_2 \) if we wish to deal with orientations of the source dipole in which \( n_x \neq 0 \)) is oscillatory and decaying and that \( s^2/\sqrt{s^2 - 1} \) is increasing on all but a small domain over which it is absolutely integrable in order to show the logarithmic dependence of the resolution width on \( \Delta\epsilon \) and \( \Delta\mu \). The logarithmic dependence of the general problem, in which we consider all components of the magnetic field, will be heuristically obvious from the argument for \( n_x = 0 \) because replacing \( s^2 \) by \( s \) or taking away a factor of \( 1/\sqrt{s^2 - 1} \) does not alter the well-behavedness of the non-Bessel function portion of the integrand in Equation 29.

To determine the resolution, which is tantamount to the dependence on \( r \) of the intensity in the focal plane, we start with a simpler question. It is physically necessary that \( \lim_{r\to\infty} |H_{y,\text{near}}(r)| = 0 \), so it is sensible to ask what feature of Equation 29 forces this to occur. The only effect of an increase in \( r \) is to shift the argument of the Bessel function
$J_1(k_0 rs)$ in the integrand of Equation 29 to higher values, so that it oscillates more rapidly and decreases as $r^{-1/2}$ due to the asymptotic properties of $J_0$. Either of these effects, oscillation and decay, suffice to cause the integral to tend toward zero as $r$ increases. The decay part is obvious, since $\lim_{r \to \infty} r^{-1/2} = 0$, whereas the vanishing as a consequence of oscillation is essentially equivalent to the simply-proven fact from real analysis that for a function $f(x)$ that is Lebesgue-integrable over a set $E$, $\lim_{n \to \infty} \int_E f(x) \sin(nx) dx = 0$.

Since in the above approximation we only integrate from 1 to $q_0/k_0$, the conditions under which the oscillations of the Bessel function cause the integral to vanish are met. We note that for $s$ not too small, $s^2/\sqrt{s^2-1}$ grows approximately as $s$, while the amplitude of $J_1(k_0 rs)$ shrinks as $s^{-1/2}$. Thus the integrand in Equation 29 has an amplitude that grows as $s^{1/2}$, modulated by the nearly periodic oscillations of the Bessel function $J_1(k_0 rs)$. Since the amplitude is increasing, the size of integral over each “hump” (see Appendix 7) increases with $s$. This means that if $r$ is such that $J_1(k_0 rs)$ has two humps, i.e., goes up, then down, then back to zero, in the interval $[0, q_0/k_0]$ the second, negative hump should contribute more to the integral, and thus the integral in Equation 29 is negative for such an $r$. This means that the resolution length, defined as the first value of $r$ besides $r = 0$ where the transmitted field vanishes, must occur somewhere in the first-occurring range wherein $J_1(k_0(q_0/k_0)r) = J_1(q_0r)$ is negative. Hence we have a resolution length $R$ that must satisfy:

$$J_{1,1} \leq q_0 R \leq J_{1,2} \quad (30)$$

where $J_{n,m}$ denotes the $m$th non-zero zero of $J_n$.

The resolution length defined as the value of $r$ for which the intensity in the focal plane is maximal is easier to estimate precisely. From Appendix 7, In the appendices,
we see that the maximum value seems to occur when the first maximum of the Bessel function $J_1(k_0rs)$ occurs at the cutoff $s_0 = q_0/k_0$, which is also the maximum of the $r$-independent part of the integrand in Equation 29. Intuitively, this appears to be a consequence of the rearrangement inequality from mathematics, since $s^2/\sqrt{s^2-1}$ and $J_1(k_0rs)$ are essentially monotonic on the range where $k_0rs$ does not exceed the first maximum of the Bessel function $J_1$. Call the value of this maximum $\beta$. Then, by the definition in terms of maxima, the resolution width $R$ is given by:

$$Rk_0s_0 = \beta \Rightarrow R = \beta/q_0$$  \hspace{1cm} (31)

Thus, the problem of the resolution width is solved if we can estimate the location $q_0$ of the singularity in the transmission function $T_M(q)$. In Appendix 3, we do this and find that, as long as $\epsilon \to -1$ not much slower than $\mu \to -1$, $q_0 \approx (1/d) \ln |2/\Delta \mu|$. Hence the resolution of the lens is inversely proportional to $\ln \Delta \mu$, as we sought to establish.

Furthermore, we observe that for fixed $\epsilon$ and $\mu$, the amplitude of the transmitted field as a function of $r$ when $r \to \infty$ is of the order $O(r^{-3/2})$, since the width of the non-cancelling part of the integrand (see Appendix 7), i.e., the part left over after the positive and negative oscillations of the Bessel function cancel each other out, is $O(r^{-1})$ and the Bessel functions $J_{0,1,2}$ are $O(r^{-1/2})$. Thus the intensity in the focal plane is roughly proportional to $r^{-3}$. In addition to the appendices that demonstrate the above arguments, Appendix 9 contains a plot of numerically calculated resolution widths in excellent agreement with the inverse logarithmic prediction.

### 2.4.3 Conclusion

It is important to understand that the lens does attain sub-wavelength resolution under relatively lenient conditions on $\Delta \epsilon$ and $\Delta \mu$. The limitation is that the improvement in
the resolution is extremely slow. A lens with $\epsilon = \mu = 1.0001$ is only three times as powerful as one with $\epsilon = \mu = 1.1$. Significant improvement upon the diffraction limit does not, therefore, seem possible, even for a lossless lens as in the model considered here.
3 Appendices

3.1 Appendix 1: Derivation of the Transmission Coefficients

The transverse electric and transverse magnetic transmission coefficients \(T_E(q)\) and \(T_M(q)\) were defined as the functions such that an incident plane wave \(\mathbf{E}_i = \hat{y} e^{iqx} e^{-\kappa_0(z+d/2)}\) in the region \(z < 0\) yields a transmitted plane wave \(\mathbf{E}_t = T_E(q)\hat{y} e^{iqx} e^{-\kappa_0(z+d/2)}\) in the region \(z > d\), and likewise for waves whose magnetic fields are parallel to the slab’s surface. To find the coefficient \(T_E(q)\), we must solve Maxwell’s equations inside and outside the slab. If we define regions I, II, and III as \(z < 0\), \(0 < z < d\), \(d < z\), respectively, and define \(\kappa\) and \(\kappa_0\) as before, then we have the general solution:

\[
E_y = E_0 e^{iqx-\kappa_0(z+d/2)} + E_R e^{iqx+\kappa_0(z+d/2)} \quad \text{in region I} \tag{32}
\]

\[
E_y = E_- e^{iqx-\kappa z} + E_+ e^{iqx+\kappa z} \quad \text{in region II} \tag{33}
\]

\[
E_y = E_T e^{iqx-\kappa_0 z} \quad \text{in region III} \tag{34}
\]

Because \(\nabla \times \mathbf{E} = -i\mu \omega /c \mathbf{H}\), the boundary conditions are continuity of \(\frac{1}{\mu} \frac{\partial E_y}{\partial z}\) and \(E_y\) at \(z = 0\) and \(z = d\). Hence we require that:

\[
E_+ e^{\kappa d} + E_- e^{-\kappa d} = ET + e^{-\kappa_0 d} \tag{35}
\]

\[
\frac{\kappa}{\mu} (E_+ e^{\kappa d} - E_- e^{-\kappa d}) = -\kappa_0 E_T e^{-\kappa_0 d} \tag{36}
\]

\[
E_0 e^{-\kappa_0 d/2} + E_R e^{\kappa_0 d/2} = E_+ + E_- \tag{37}
\]

\[
\kappa_0 (E_R e^{\kappa_0 d/2} - E_0 e^{-\kappa_0 d/2}) = \frac{\kappa}{\mu} (E_+ - E_-) \tag{38}
\]

We can eliminate \(E_-\) and \(E_+\) from the boundary condition equations at \(z = d\) to obtain the two expressions:
\[ E_+ = (1 - \kappa_0 \mu / \kappa)(E_T/2)e^{-(\kappa_0 + \kappa)d} \]  
(39)

\[ E_+ = (1 + \kappa_0 \mu / \kappa)(E_T/2)e^{-(\kappa_0 - \kappa)d} \]  
(40)

And we can eliminate \( E_R \) from the boundary equations at \( z = 0 \) to obtain:

\[ 2E_0 e^{-\kappa_0 d/2} = (1 - \kappa/(\mu \kappa_0))E_+(1 + \kappa/(\mu \kappa_0))E_- \]  
(41)

\[ \Rightarrow 2E_0 e^{-\kappa_0 d/2} = \left[ (1 - \frac{\kappa}{\mu \kappa_0})^2 e^{-\kappa d} + (1 + \frac{\kappa}{\mu \kappa_0})^2 e^{\kappa d} \right] \frac{E_T}{2} e^{-\kappa_0 d} \]  
(42)

This simplifies readily to the desired expression:

\[ E_T = \frac{4\mu \kappa \kappa_0 E_0 e^{-\kappa_0 d/2}}{e^{\kappa d}(\kappa + \mu \kappa_0)^2 - e^{-\kappa d}(\kappa - \mu \kappa_0)^2} \]  
(43)

The derivation of the transmission coefficient for waves with magnetic fields parallel to the boundary is virtually identical due to the symmetry in Maxwell’s equations and is omitted.
3.2 Appendix 2: Integral Identities

The following definite integral identities involving Bessel functions were used.

\[ \int_{0}^{2\pi} e^{iqr \cos \phi} d\phi = 2\pi J_0(qr) \]  \hspace{1cm} (44)

\[ \int_{0}^{2\pi} \cos(\phi)e^{iqr \cos \phi} d\phi = 2\pi i J_1(qr) \]  \hspace{1cm} (45)

\[ \int_{0}^{2\pi} \cos(2\phi)e^{iqr \cos \phi} d\phi = -2\pi J_2(qr) \]  \hspace{1cm} (46)

\[ \int_{0}^{2\pi} \sin(\phi)e^{iqr \cos \phi} d\phi = 0 \]  \hspace{1cm} (47)

\[ \int_{0}^{2\pi} \sin(2\phi)e^{iqr \cos \phi} d\phi = 0 \]  \hspace{1cm} (48)
3.3 Appendix 3: Location of the Transmission Singularity

The transmission function has a pole [Figure 3] when the denominator

\[ e^{\kappa d}(\kappa + \mu \kappa_0)^2 - e^{-\kappa d}(\kappa - \mu \kappa_0)^2 \]  

(49)

vanishes; that is, recalling the definition of \( \kappa \) and \( \kappa_0 \), when

\[ e^{\kappa d}(\sqrt{q^2 - \epsilon \mu k_0^2} + \mu \sqrt{q^2 - k_0^2})^2 = e^{-\kappa d}(\sqrt{q^2 - \epsilon \mu k_0^2} - \sqrt{q^2 - k_0^2})^2 \]  

(50)

We can divide throughout by \( q^2 \) and use the approximation \( \kappa_0 \approx |q| \) when \( q \) is not small, which is the case at the singularity for the values of \( \epsilon \) and \( \mu \) that interest us. Defining the dimensionless variable \( s = q/k_0 \), we obtain:

\[ e^{2\kappa d}(\sqrt{1 - \epsilon \mu / s^2} + \mu \sqrt{1 - 1/s^2})^2 = (\sqrt{1 - \epsilon \mu / s^2} - \mu \sqrt{1 - 1/s^2})^2 \]  

(51)

Using the Taylor series approximation \( \sqrt{1 + x} \approx 1 + x/2 \), we find that

\[ e^{2|q|d} \left( 1 - \frac{\epsilon \mu}{2s^2} + \mu - \frac{\mu}{2s^2} \right)^2 \approx \left( 1 - \frac{\epsilon \mu}{2s^2} - \mu + \frac{\mu}{2s^2} \right)^2 \]  

(52)

\[ e^{2|q|d} \left( \Delta \mu - \frac{\Delta \epsilon}{2s^2} \right)^2 \approx e^{2|q|d} \left( \Delta \mu + \frac{\Delta \epsilon}{2s^2} \right)^2 \approx 4 \]  

(53)

\[ |q|d = \ln \left| \frac{2}{\Delta \mu + \frac{\Delta \epsilon}{2s^2}} \right| \]  

(54)

There are two limits of interest to us. As long as \( \Delta \epsilon / s^2 \) is small (empirically, we end up with \( s \) in the range of 5 to 20 for most reasonable physical values of \( \Delta \epsilon \) and \( \Delta \mu \)), we obtain:

\[ q_0 d \approx \ln \left| \frac{2}{\Delta \mu} \right| \]  

(55)

whereas if \( \Delta \mu \ll \Delta \epsilon \), we find that:

\[ k_0 s_0 d \approx 2 \ln \left| \frac{s_0}{\Delta \epsilon} \right| \]  

(56)
in which case the location of the singularity location $s_0$ does grow faster than logarithmically as a function of $\Delta \epsilon$ as $\Delta \epsilon$ decreases, but only briefly, in the domain where $\Delta \mu \ll \Delta \epsilon$.

Finally, we note that it is easy to find the zeroes of a function numerically using a binary search, and therefore it is almost trivial to verify the above prediction, as the following graph does. The horizontal axis is $|\ln \Delta \mu|$, and the relation between it and the singularity location is linear.
3.4 Appendix 4: A Note on the Numerical Computations

Numerical calculations were necessary to generate the images in this paper as well as to provide some insight before the analytic work could begin. In particular, I encountered the integral:

\[ \int_{\omega/c}^{\infty} \frac{q^2}{\kappa_0} J_1(qr) T_M(q) dq \]  

(57)

Substituting to the dimensionless parameter \( s = q/k_0 \), where \( k_0 = \omega/c \) leads, up to a constant multiple, to the integral:

\[ \int_{1}^{\infty} \frac{s^2}{\sqrt{s^2 - 1}} J_1(k_0 sr) T_M(q) dq \]  

(58)

I implemented a rather pedestrian algorithm based on Simpson’s rule to handle most of the domain of integration. Such a method works fails at the two singularities of the integral above, which are the singularity \( s = 1 \) of the square root term and the singularity \( q = k_0 s = q_0 \), where \( q_0 \) is the aforementioned singularity of the transmission coefficient. Fortunately, when \( 1/\sqrt{s^2 - 1} \) has large derivatives, \( T_M(k_0 sr) \) does not, and vice versa. Thus when we integrate over the singularity of one function, we can approximate the other either to zeroth or to first order.

The singularity of \( 1/\sqrt{s^2 - 1} \) is trivial, since it is absolutely integrable:

\[ \int_{1}^{1+\delta} \frac{1}{\sqrt{s^2 - 1}} ds = \int_{0}^{\sqrt{\delta^2 + 2\delta}} \frac{du}{\sqrt{1 + u^2}} \]  

(59)

where we obtain the right side of the equation from the left side using the substitution \( u = \sqrt{s^2 - 1} \). Written in this form, it is clear that, to first-order in \( \delta \),

\[ \int_{1}^{\infty} \frac{s^2}{\sqrt{s^2 - 1}} J_1(k_0 sr) T_M(k_0 s) dq = \sqrt{2\delta} J_1(k_0 r) T_M(k_0) \]  

(60)

Thus we can extend Simpson’s rule down to \( 1 + \delta \), correcting the size of the partition
near $s = 1$ to account for the error term in Simpson’s rule proportional to the fourth derivative, and then use the above approximation to handle the domain from 1 to $1 + \delta$.

The integral over the transmission function singularity is subtler because, as mentioned earlier, it is not absolutely convergent. This mathematical aberration represents the physical fact that there exists no steady-state equilibrium of energy flux into and out of the lens at resonant values of $q$. Physically, some damping effect would always render the expression integrable. Fortunately, the asymptotic behavior of the transmission function is odd near the singularity. That is, $T_M(q_0 + \delta) \sim -T_M(q_0 - \delta)$ as $\delta \to 0$. This can be easily verified by linearizing the singularity denominator

$$e^{\kappa d}(\kappa + \mu \kappa_0)^2 - e^{-\kappa d}(\kappa - \mu \kappa_0)^2$$

about its zero at $q_0$. Thus the Cauchy principal value, defined by:

$$PV \int_{x_0 - \Delta x}^{x_0 + \Delta x} f(x)dx = \lim_{\alpha \to 0} \left[ \int_{x_0 - \Delta x}^{x_0 - \alpha} f(x)dx + \int_{x_0 + \alpha}^{x_0 + \Delta x} f(x)dx \right]$$

It is easy to see that for a function with odd asymptotic symmetry about $x_0$, the two expressions above are equivalent to:

$$\int_0^{\Delta x} (f(x_0 + t) + f(x_0 - t))dt$$

By the odd asymptotic symmetry of the function $f(x)$, it is apparent that the integrand above is not only integrable; it is also bounded. Hence we expect this form to be particularly conducive to integrating $\frac{s^2}{\sqrt{s^2 - 1}} J_1(k_0 sr) T_M(k_0 s)$ near the singularity. However, this is not the case. Even when I wrote extremely accurate numerical codes to find the location $q_0$ of the singularity (these can be made very fast and accurate since finding the zero is essentially a binary search algorithm), the term $f(x_0 + t) + f(x_0 - t)$ was too sensitive to the precise value of $x_0$ to be of use. Instead, I came up with an
analytic estimate of the Cauchy principal value that did not rely on a numerical estimate of the singularity.

The idea behind it was simple. I wrote the integrand as a quotient of a non-singular numerator \( n(s) \) by the denominator \( d(s) := e^{\kappa d}(\kappa + \mu \kappa_0)^2 - e^{-\kappa d}(\kappa - \mu \kappa_0)^2 \). I thus sought to estimate:

\[
PV \int_{s_0 - \Delta s}^{s_0 + \Delta s} \frac{n(s)}{d(s)} \quad \text{where} \quad d(s_0) = 0
\]  

(64)

Taking Taylor series, we find that this equals, to first order,:

\[
PV \int_{s_0 - \Delta s}^{s_0 + \Delta s} \frac{n(s_0) + n'(s_0)(s - s_0)}{d'(s)(s - s_0)}
\]

(65)

This is equal to:

\[
PV \int_{-\Delta s}^{\Delta s} \left[ \frac{n(s_0)}{td'(s_0)} + \frac{st(s_0)}{d'(s_0)} \right] dt = \frac{2\Delta sn'(s_0)}{d'(s_0)} \quad \text{PV} \int_{-\Delta s}^{\Delta s} \frac{dt}{t} = \frac{2\Delta sn'(s_0)}{d'(s_0)}
\]

(66)

Integrating over the singularity of the transmission coefficient is thus reduced to numerical computation of two derivatives, which is not dangerously sensitive to small errors.
3.5 Appendix 5: Graphs of the Transmission Coefficient for Different $\epsilon$

The following are a series of graphs of $T_M(\epsilon, \mu = -1, s = q/k_0)e^{-3\kappa_0 d/2}$, where the added exponential term reflects the decay of evanescent waves from the lens boundary to the focal plane. As derived above, note that before the singularity these functions are nearly identical to 1, and that after the singularity they are nearly identical to 0. Also note that the function near the singularity $s_0 = q_0/k_0$ exhibits odd symmetry and that the region for which neither of the $T_Me^{-3\kappa_0 d/2} \equiv 1$ or $T_Me^{-3\kappa_0 d/2} \equiv 0$ approximations holds becomes proportionally less significant as $\Delta \epsilon \to 0$. Finally, note that the location of the singularity goes to increasing values of $s$ at a logarithmically slow rate.
Figure 1: Clockwise from upper left, $\epsilon = -1.1, \epsilon = -1.01, \epsilon = -1.0001, \epsilon = -1.000001, \\
\epsilon = -1.00001, \epsilon = -1.001$
3.6 Appendix 6: Plot of the Transmitted Field over $r$ and $z$

The following is a plot of the $n_x = 0, \Delta \mu = \Delta \epsilon = 0.001$ transmitted field magnitude for $z$ from the lens boundary to 2 wavelengths past the boundary and for $r$ from the origin to one-half wavelength. The ellipse denotes the focal region. The sub-wavelength resolution and interference pattern are clearly visible. Note the large magnitude of the field near the boundary and the vanishing of the field at $r = 0$.

![Color plot of the transmitted field for varying $r$ and $z$](image)

Figure 2: Color plot of the transmitted field for varying $r$ and $z$
3.7 Appendix 7: Graphs of the Transmitted Field Integrand at Varying $r$

Recall the approximation (Equation 29) for the transmitted near field intensity:

$$H^\text{near}_y = -2k_0^2\pi\alpha \int_1^{q_0/k_0} \frac{s^2}{\sqrt{s^2-1}} J_1((k_0r)s)ds$$  \hspace{1cm} (67)

The aim of this appendix is to demonstrate the conclusions about the resolution that we earlier drew from this equation. The integrand is built up from the function $f(s) := \frac{s^2}{\sqrt{s^2-1}}$, which is independent of $r$, and the Bessel function $J_1((k_0r)s)$, which causes the integrand to oscillate and decay more rapidly. The function $f(s)$ looks like this, when cut off at the singularity $s_0 = q_0/k_0$:

![Graph of the Transmitted Field Integrand at Varying $r$](image)

Notice the validity of the $f(s) \approx s$ approximation, since it is only incorrect for the narrow region near an integrable singularity at $s = 1$. We now present a series of paired graphs, with the Bessel function $J_1((k_0r)s)$ on the right and the effect of its modulation upon $f(s)$ (ie, the integrand above) on the right. The value of $s_0$ is derived from $\Delta\mu = 0.00001, \Delta\epsilon = 0$. 

31
Figure 3: $r = 0.01\lambda$

Figure 4: $r = 0.02\lambda$

Figure 5: $r = 0.03\lambda$
Figure 6: $r = 0.04\lambda$

Figure 7: $r = 0.05\lambda$

Figure 8: $r = 0.06\lambda$
Figure 9: $r = 0.07\lambda$

Figure 10: $r = 0.09\lambda$

Figure 11: $r = \lambda$
3.8 Appendix 8: A Dipole Source Resolved into Rings

Below is a plot of the transmitted field intensity in the focal plane for the \( n_x = 0 \) geometry. The width of the image is a single wavelength. We reiterate that the resolution surpasses the diffraction limit. Note that the intensities of the rings are not monotonically decreasing, but rather alternate.
3.9 Appendix 9: Calculated Values of the Resolution Width

Below we present numerical results for resolution widths of the lens, defined as the first positive-$r$ zero of the $n_x = 0$ transmitted intensity. The resolution is given in terms of the wavelength for the geometry $d = \lambda/(2\pi)$, while the horizontal axis is $1/|\ln(\Delta \mu)|$ (for $\Delta \epsilon = 0$). Note that the relationship becomes linearly asymptotically, as expected. The horizontal axis represents a range of fifteen orders of magnitude in $\Delta \mu$.

![Graph showing the relationship between resolution length and $1/|\ln(\Delta \mu)|$]
References


