Inventory Cost Rate Functions with Nonlinear Shortage Costs

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Abstract

This article considers five cost-rate models for inventory control, each summarizing the expected holding and shortage costs per period as a function of the inventory position. All models have linear holding costs and shortage cost coefficients of dimension \([$/unit/period]\), \([$/unit]\) and \([$/period]\). The latter two coefficients may be the shadow costs of a fill-rate and a ready-rate service constraint, respectively. One of the cost-rate models is a new suggestion, intended to facilitate modeling of periodic-review inventory systems.

If-and-only-if conditions on the demand process are presented for which the cost rate is quasiconvex in the inventory position. The typical sufficient condition requires that the cumulative demand distribution be logconcave, a condition that is met by most demand distributions commonly used in the inventory literature.

The results simplify optimization and extend the known optimality of \((s,S)\) and \((nQ,r)\) policies to cost structures common in applications and to the presence of typical service constraints. As a prerequisite for the study, a series of new monotonicity results are derived for compound renewal processes.
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1. Introduction

A key element of an inventory model is a cost-rate function that summarizes the expected holding and shortage costs incurred in a period as a function of the initial inventory position. When modeling inventory systems with backlogging, the costs are measured a leadtime ahead, and it is most commonly assumed that the shortage costs are linear. That is, they accumulate at a constant rate proportional to the number of backorders, so that the dimension of the cost coefficient is [$/unit/period], see e.g., Veinott (1966b). This assumption, along with the standard linear inventory holding cost assumption, results in an expected cost rate that is quasiconvex (in fact, convex) in the inventory position. [A function \( f(x) \), \( \mathbb{R} \rightarrow \mathbb{R} \), is quasiconvex if and only if \(-f(x)\) is unimodal.] This convex function was assumed by Scarf (1960) in his seminal work on \((s,S)\) policies. A quasiconvex cost rate is a key condition for many subsequent, important results in inventory theory, such as the optimality of \((s,S)\) and \((nQ,r)\) policies (Veinott 1965, 1966a, Zheng 1991, Chen 2000) and for efficient algorithms for such policies (Zheng and Federgruen 1991, Federgruen and Zheng 1992).

In this article, we consider two types of backorder costs that are nonlinear. They are quite common in the inventory literature and in applications. One is fixed for each unit backordered, regardless of how long the unit remains in backlog, i.e., the dimension of the cost coefficient is [$/unit]. This shortage cost is discussed in Hadley and Whitin (1963) and by Chen and Zheng (1993) in a spirit similar to the present study. The other is proportional to the time backlogs stay on the books, regardless of the volume of backlog that accumulates, i.e., the dimension of the cost coefficient is [$/period]. This shortage cost is discussed in Peterson and Silver (1979). It is easy to show that the cost rate of inventory models incorporating one or both of these two backorder costs is not convex (the latter are constant for negative inventory positions).

Then the important question arises: does the cost rate still remain quasiconvex (so that the above optimality results and efficient algorithms still apply)? This article derives conditions on the demand process for which the cost rate indeed remains quasiconvex, although it is not convex.
The usefulness of the two nonlinear shortage costs is further indicated by their logical relationship to two popular service constraints, the *fill-rate* and the *ready-rate*, which are often levied on inventory policies in applications as substitutes for difficult-to-estimate shortage costs. The *fill-rate* stipulates the fraction of demand that should be filled directly without being backlogged, and the *ready-rate* stipulates the fraction of time with no backlog on the books. These service constraints are discussed in *Peterson and Silver* (1979) and *Chen* (1996). If the constraints are handled by lagrange multipliers, then the dimension of the multiplier (shadow cost) would be \([$/unit]\) in case of the fill-rate, and in case of the ready-rate, \([$/period]\). (A shadow cost of dimension \([$/unit/period]\) would appear with a constraint on the expected time a demand is backlogged before it is satisfied.) In a companion paper (*Rosling* 1999) simple optimization techniques have been developed for models with these service constraints.

Here are the main results. For all nonnegative cost coefficients, the cost rate is quasiconvex if and only if the cumulative distribution (CDF) of leadtime demand is logconcave. We show that this holds for five different models of inventory systems that differ in their demand process and/or how they are reviewed. A sufficient condition for the leadtime demand CDF to be logconcave is that the single customer’s demand has a logconcave CDF. (For periodic demand processes, “customer demand” means total demand in a period.) With some restrictions on the cost coefficients, quasiconvexity appears for most of the models under the less restrictive assumption that the customer demand CDF be MCR, a concept coined in this article (see below) and interestingly related to logconcavity. The results extend those of *Chen and Zheng* (1993), who restricted their analysis to our *Model 1* (see below) and the compound Poisson version of our *Model 2* (see below), and did not consider the \([$/period]\) shortage cost component. The results also extend those of *Chen* (1996), who assumed strongly unimodal distributions in the context of ready-rate and fill-rate service constraints.

The remainder is organized as follows. *Section 2* is for prerequisites. It reviews and extends known results for distributions with logconcave (strongly unimodal) densities, \(f(x)\), and with
logconcave CDF, \( F(x) \) or \( 1-F(x) \). It further introduces the concept of Monotone Convolution Ratios (MCR), which is shown to be more general than logconcavity. It is shown that even the least general class – the one with logconcave densities – still contains most demand distributions commonly assumed in the inventory literature. The concepts are then applied to compound renewal processes and the prerequisite for the main study is obtained – a series of monotonicity results extending the work of Barlow et al. (1963). Following An (1998), the proofs avoid differentiability assumptions.

Using the results of section 2, section 3 provides the conditions for the cost rate to be quasiconvex. This is done for the following five models:

1. **Periodic demand and periodic review.**
2. **Demand generated by a compound renewal process under continuous review.**
3. **The latter model under periodic review** – this model is only approximately developed, except for compound Poisson demand (where it is exact).
4. **The idealized textbook model where review is continuous and the demand process has a continuous sample path with independent increments.**
5. **The latter model under periodic review** – this is a new suggestion related to the early models by Holt et al. (1960).

Section 4 concludes with a summary and extensions. There are two appendices. The rather long (but most readable) proofs of the major propositions are available on-line. Appendix 1 contains the URL. Appendix 2 contains a list of symbols.

### 2. The Demand Processes: Logconcavity and Related Concepts

#### 2.1 The Periodic Demand Process

We consider a single-item inventory system with periodic demand. The demands of consecutive periods are independent and identically distributed nonnegative random variables with CDF (cumulative distribution function) \( F(\cdot) \) and mean \( D \). The demand process may be understood as a periodically arriving customer, each time demanding a random quantity, \( x \), say. Assume \( F(x) \) be
continuous and differentiable, except possibly at \( x=0 \), where it is right differentiable and possibly discontinuous. The corresponding derivative is denoted \( f(x) \) and referred to as the “frequency function”. Thus, \( F(x) = F(0) + \int_0^x f(y)\,dy \) for \( x \geq 0 \), and if \( F(0)=0 \), then \( f(x) \) coincides with a regular density. The frequency function is continuous, except possibly at \( x=0 \), but not necessarily differentiable.

With some additional wording and care, the subsequent propositions may be extended to \( F(x) \) differentiable almost everywhere, but only discontinuous at \( x=0 \). The proofs of the crucial propositions \( P1-1,2,3 \) below are in fact carried out so as not to rely on differentiability. Most results similarly carry over to discrete probability distributions as noted.

**Logconcavity**

A frequency function \( f(x) \) is logconcave, or strongly unimodal, if \( \log f(x) \) is concave. When \( f(x) \) is logconcave in the present setting, it is assumed that \( F(0)=0 \). Logconcavity implies unimodality, and it can be demonstrated that \( f(x) \) logconcave is necessary and sufficient for the unimodality of the convolution of \( f(x) \) with an arbitrarily unimodal function (Ibragimov 1956).

Logconcavity implies that \( f(x) \) is differentiable almost everywhere. Logconcavity holds if \( f'(x)/f(x) \), the derivative of \( \log f(x) \), is nonincreasing. The notion of logconcavity can be extended to discrete distributions, which are logconcave if \( [f(x+1)-f(x)]/f(x) \), or equivalently \( f(x+1)/f(x) \), is nonincreasing in integer \( x \).

The CDF, \( F(x) \), or its complement, \( 1-F(x) \), is logconcave if \( f(x)/F(x) \), or \( -f(x)/[1-F(x)] \), respectively, are nonincreasing (for discrete distributions, \( f(x) \) should read \( f(x+1) \)). The distributions with logconcave complements are thus identical to those with increasing failure rates. Neither logconcavity of \( F(x) \) nor logconcavity of \( 1-F(x) \) implies unimodality of \( f(x) \). Logconcavity of \( 1-F(x) \) precludes a spike at \( x=0 \), and so \( F(0)=0 \), but logconcavity of \( F(x) \) allows \( F(0)>0 \).

The most important properties of the three classes of logconcave distributions are summarized in Proposition 1-1. The proof of (i), concerning \( F(x) \) and \( 1-F(x) \), is found in Barlow et al. (1963, Theorem 3.2 and Corollary 3.3) and concerning \( f(x) \), it is a direct consequence of the
result by Ibragimov mentioned above. (Let \( f \) and \( g \) be logconcave and \( h \) arbitrarily unimodal; unimodality of the convolution of \( f \), \( g \) and \( h \) then implies logconcavity of the convolution of \( f \) and \( g \).) The fact (ii) follows for \( F(x) \) by writing the inverse of \( f(x)/F(x) \) as \( \int_0^x [f(x-y)/f(x)]dy \) (since \( F(0)=0 \)). As \( f(x)=0 \) for \( x<0 \) and \( f(x-y)/f(x) \) is nondecreasing on \( x\geq0 \) by logconcavity, the expression within brackets is non-decreasing in \( x \); that is, \( f(x)/F(x) \) is nonincreasing and so, \( F(x) \) is logconcave. A similar argument is given by Barlow et al. (p.378) for \( 1-F(x) \).

**PROPOSITION 1-1**

(i) For each of the distribution functions, \( f(x) \), \( F(x) \), and \( 1-F(x) \), logconcavity is closed with regard to composition.

(ii) If \( f(x) \) is logconcave, so are \( F(x) \) and \( 1-F(x) \).

Most commonly used distributions have logconcave frequency functions. These include the Gamma family with shape parameter \( r \geq 1 \), the uniform distribution, the Beta distributions with parameters \( (r,q) \) such that \( r \geq 1 \) and \( q \geq 1 \), the Weibull distribution with shape parameter \( r \geq 1 \), the Normal distribution, the truncated Normal distribution defined as \( f(x) = \phi(x)/[1-\Phi(0)] \) on \( x \geq 0 \), and the similarly truncated logistic distribution. (For some helpful inventory related comments on the last five, see Fortuin 1980.) Moreover, the discrete Uniform distribution, the Poisson, the Binomial, the Hypergeometric, and the Negative Binomial (with shape parameter \( r \geq 1 \)) are all logconcave.

If only \( F(x) \) is required to be logconcave, the class of distributions expands to include, the Lognormal, Gamma, and Weibull distributions for all parameter values; the cut-off Normal distribution defined as \( F(x)=\Phi(x) \) for \( x \geq 0 \), and the similarly cut-off logistic distribution.

Concavity of a nonnegative function (on its support) implies logconcavity, so that in particular \( F(x) \) is logconcave if \( f(x) \) is nonincreasing, and \( 1-F(x) \) is logconcave if \( f(x) \) is nondecreasing. Moreover, \( 1-F(x) \) logconcave implies that all moments of the distribution exist. Recent summaries of logconcavity and related concepts are found in An (1998) and in Dharmadhikari and Joeg-dev (1988).
Monotone Convolutions

If the set \( \{f_n(x)\} \) of the frequency functions of all \( n \)-fold convolutions of \( F(x) \) is such that for all \( n \geq 1 \), \( f_{n+1}(x)/f_n(x) \) is nondecreasing on an interval of the real line and \( f_n(x) = 0 \) outside that interval, then we say that \( f(x) \) is MCR (has Monotone Convolution Ratios). We say similarly that \( F(x) \) or \( 1-F(x) \) is MCR, if \( F_{n+1}(x)/F_n(x) \) or \( [1-F_{n+1}(x)]/[1-F_n(x)] \), respectively, are nondecreasing in \( x \) in the similar way for all \( n \geq 1 \). When \( f(x) \) is MCR, it is assumed that \( F(0) = 0 \), but \( F(x) \) or \( 1-F(x) \) being MCR does not exclude a spike at \( x = 0 \).

Note that \( [1-F_n(x)]/[1-F_m(x)] \geq 1 \geq F_n(x)/F_m(x) \) for all \( x \) and \( n \geq m \). In addition, if \( F(x) \) (or \( f(x) \) or \( 1-F(x) \)) is MCR, then \( F_n(x)/F_m(x) \) (or \( f_n(x)/f_m(x) \) or \( [1-F_n(x)]/[1-F_m(x)] \)) is nondecreasing in \( x \) for all \( n \geq m \). When applied to frequency functions, MCR is also referred to as monotone likelihood ratios (Ross 1983). Chen and Zheng (1993) generalized the concept for cumulative distribution functions.

The notion of MCR is a generalization of logconcavity that turns out to be quite convenient, but we have no example of a common parametric distribution that is MCR, but not logconcave (there are many examples, though, for compound Poisson processes as noted).

**PROPOSITION 1-2**

(i) If \( f(x) \) is MCR, so are \( F(x) \) and \( 1-F(x) \).

(ii) If \( f(x) [F(x), 1-F(x)] \) is logconcave, then \( f(x) [F(x), 1-F(x)] \) is MCR.

The proof is found in Appendix 1 (available on-line). A proof of (i) for \( F(x) \) that assumes differentiability is found in Chen and Zheng (1993, Lemma 2).

(Fig. 1 about here)

The hierarchy implied by P1-1,2 is illustrated in Fig. 1. To the left are the distributions of major interest in the sequel, and to the right are those reminding of reliability theory. At the bottom and most specific are the distributions with logconcave frequency functions, which imply both \( F(x) \) logconcave and \( f(x) \) MCR. The distributions of these two sets are not clearly related, but membership in either implies that \( F(x) \) is MCR, the most general concept on the left-hand side.
2.2 The Compound Renewal Demand Process

The Arrival Process

We now turn to a continuous-time setting with customers arriving according to a renewal process with interarrival time, t ≥ 0, with CDF, Ω(t) with Ω(0)=0 and frequency function ω(t). The probability of n arrivals in a period of length t (following an arrival) is

\[
p_n(t) = Ω_n(t) - Ω_{n+1}(t),
\]

where Ω_n(t) is the n-th convolution of Ω(t) and Ω_0(t)=1 for all t≥0. The corresponding CDF is denoted P_n(t) = \sum_{k=0}^{n} p_k(t) = 1 - Ω_{n+1}(t).

The long run cost of an inventory system is closely related to the equilibrium renewal process, which is a delayed renewal process with the density of the first arrival time equal to

\[
\tilde{ω}_1(t) = \frac{1}{λ}[1-Ω(t)],
\]

where 1/λ is the expected interarrival time, i.e., \(1/λ = \int_0^∞ [1-Ω(t)]dt\). An observer who arrives randomly after the original renewal process has been run for "infinitely long" observes the equilibrium renewal process. The corresponding equilibrium distribution function is denoted

\[
\tilde{Ω}_1(t) = \int_0^t λ[1-Ω(x)] dx.
\]

The subsequent interarrival times follow the original distribution, Ω(t), and so \(\tilde{Ω}_n(t)\) is defined as the convolution of \(\tilde{Ω}_1(t)\) with \(Ω_{n-1}(t)\), i.e.,

\[
\tilde{Ω}_n(t) = \int_0^t Ω_{n-1}(t-x)λ[1-Ω(x)] dx.
\]

\(\tilde{ω}_n(t)\) is similarly defined, i.e., as Ω(0)=0,

\[
\tilde{ω}_n(t) = \int_0^t ω_{n-1}(t-x)λ[1-Ω(x)] dx = λ[Ω_{n-1}(t) - Ω_n(t)].
\]

Moreover,

\[
\bar{p}_n(t) = [\tilde{Ω}_n(t) - \tilde{Ω}_{n+1}(t)] \quad \text{and} \quad \bar{P}_n(t) = [1 - \tilde{Ω}_{n+1}(t)].
\]
When the interarrival distribution is exponential (customers arrive according to a Poisson process) then
\[ \tilde{\omega}(t) = \omega(t) = \lambda e^{-\lambda t}. \]
That is, the equilibrium process coincides with the regular renewal process and
\[ \tilde{p}_n(t) = p_n(t) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}. \]

Barlow et al. (1963, Theorem 5.1b) demonstrated that \( P_n(t) \) is logconcave in \( n \) if \( 1 - \Omega(t) \) is logconcave. For the subsequent purposes, the following more specialized results are required. The proof is found in Appendix 1 (available on-line).

**Lemma 1-3**

If \( \omega(t) \) is logconcave (strongly unimodal), then

1. \( p_n(t) \) and \( \tilde{p}_n(t) \) are logconcave in \( t \) for all \( n \geq 0 \).
2. \( p_n(t) \) and \( \tilde{p}_n(t) \) are logconcave in \( n \) for all \( t > 0 \).
3. \( \tilde{p}_n(t)/p_n(t) \) is nondecreasing in \( n \) for all \( t > 0 \).
4. \( \tilde{p}_n(t)/p_{n-1}(t) \) is nonincreasing in \( n \) for all \( t > 0 \).

**The Demand Process**

The demand of an arriving customer has a CDF, \( \Psi(x) \), and a frequency function, \( \psi(x) \). We make the same assumptions for \( \Psi(x) \) as for \( F(x) \) defined in subsection 2.1. Thus, the total demand, \( x \), in a period of length \( t \) following an arrival is distributed as

\[ F_{(0)}(x) = \sum_{n=0}^{\infty} p_n(t) \Psi_n(x), \]

where \( \Psi_0(x) = 1 \) for all \( x \geq 0 \). The mean demand per customer is denoted \( \mu \) and so, the expected demand over one period is \( D = \lambda \mu \).

We use the notation \( F_{(t)} \) to indicate that \( F_{(t)} \) will play a similar role for the continuous time demand process as \( F_n(x) \) does for the periodic demand process. It is interesting to note that for the case of Poisson arrivals, \( i.e., \) when \( \Omega(t) \) is exponential, \( F_{(nt)} \) is the \( n^{th} \) convolution of \( F_{(t)} \).

We also use the notation, \( f_{(t)}(x) = \sum_{n=0}^{\infty} p_n(t) \psi_n(x) \), from which then follows that

\[ F_{(t)}(x) = F_{(0)}(0) + \int_{0}^{x} f_{(0)}(y)dy \] for \( x \geq 0 \). The convolution of \( F_{(t)}(x) \) with \( \Psi(x) \) appears frequently in the
subsequent cost formulae. It is denoted \( F_{(t+1)}(x) \) and reads

\[
F_{(t+1)}(x) = \sum_{n=0}^{\infty} p_n(t)\Psi_{n+1}(x),
\]

and \( f_{(t+1)}(x) \) is defined analogously. One may interpret \( F_{(t+1)}(x) \) as the distribution of demand over the interval consisting of \( t \) plus the time till the next customer arrives after \( t \). For the equilibrium process, we have similarly that

\[
\tilde{F}_{(t)}(x) = \sum_{n=0}^{\infty} \tilde{p}_n(t)\Psi_n(x) \quad \text{and} \quad \tilde{F}_{(t+1)}(x) = \sum_{n=0}^{\infty} \tilde{p}_n(t)\Psi_{n+1}(x),
\]

and analogously for \( \tilde{f}_{(t)}(x) \) and \( \tilde{f}_{(t+1)}(x) \). Note that both \( \tilde{F}_{(t)}(x) \) and \( F_{(t)}(x) \) necessarily have a spike at \( x=0 \), but otherwise all assumptions concerning continuity and differentiability on \( \Psi(x) \) carry over directly.

**PROPOSITION 1-3**

Suppose \( \omega(t) \) is logconcave (strongly unimodal):

(i) If \( \Psi(x) \) is MCR, then both \( \tilde{F}_{(t)}(x)/F_{(t)}(x) \) and \( F_{(t+1)}(x)/\tilde{F}_{(t)}(x) \) are nondecreasing in \( x \) for all \( t>0 \).

(ii) If \( \Psi(x) \) is MCR, then both \( F_{(t)}(x)/F_{(t)}(s) \) and \( \tilde{F}_{(t)}(x)/\tilde{F}_{(t)}(s) \) are nondecreasing in \( x \) for all \( t \) and \( s, t \geq s > 0 \).

(iii) Suppose customers arrive according to a Poisson process (i.e., \( \omega(t) \) is exponential).

Then \( F_{(t)}(x) \) is logconcave in \( x \) for all \( t>0 \), if and only if \( \Psi(x) \) is concave on \( x \geq 0 \)

(i.e., if and only if \( \psi(x) \) is nonincreasing).

The proof is found in *Appendix 1* (available on-line). By virtually identical arguments, the conclusions of *P1-3(i)* and *P1-3(ii)* carry over to the complements of the distributions. For example, if \( [1-\Psi(x)] \) is MCR, then both \( [1-\tilde{F}_{(t)}(x)]/[1-F_{(t)}(x)] \) and \( [1-F_{(t+1)}(x)]/[1-\tilde{F}_{(t)}(x)] \) are nondecreasing in \( x \) for all \( t>0 \).

As all distribution functions approach 1.0 as \( x \to \infty \), the assertions of *P1-3(i)* also demonstrate the quite intuitive result that the equilibrium distribution \( \tilde{F}_{(t)}(x) \) is a compromise between \( F_{(t)}(x) \) and \( F_{(t+1)}(x) \) in the sense that \( F_{(t)}(x) \geq \tilde{F}_{(t)}(x) \geq F_{(t+1)}(x) \).
P1-3(i) is a prerequisite for the proof of P2-2 below. Note that the first point implies that $F_{(o+1)}(x)/F_{(o)}(x)$ is nondecreasing in x. The same result can be derived for $\tilde{F}_{(o+1)}(x)/\tilde{F}_{(o)}(x)$ and for the respective complement ratios.

The assertions of P1-3(ii) carry over the MCR property of $\Psi(x)$ to a similar property of $F_{(o)}(x)$ and $\tilde{F}_{(o)}(x)$, on which the proof of P2-3 below relies. In the case of a Poisson process, where $F_{(o)}(x)$ may be computed as the $n$-fold convolution of $F_{(o)}(x)$, the MCR property carries over exactly.

The condition for logconcave $F_{(o)}(x)$ specified in P1-3(iii) allows $\Psi(x)$ to be exponential or uniform; and $\Psi(0)$ positive (for the discrete version, the stuttering Poisson process would do). Note that logconcavity of $1-F_{(o)}(x)$ is ruled out because of the spike at zero. The only-if-part of (iii) could be generalized to general renewal processes, but we have not succeeded to do so with the if-part. It should be noted that for large $t$, the compound renewal distribution generally approaches a normal distribution, and so approximate logconcavity of $F_{(o)}(x)$, and of $1-F_{(o)}(x)$ and $f_{(o)}(x)$, could be expected for nearly any $\Psi$ and $\Omega$ if $t$ is large.

Examples of distributions that are MCR, but not logconcave, are easily generated by using P1-3(iii) when arrivals are Poisson. For example, if $\Psi(x)$ is logconcave, but not concave (e.g. gamma with $r \geq 2$), then $F_{(o)}(x)$ is MCR by P1-2(ii) and P1-3(ii), but not logconcave by P1-3(iii).

3. The Inventory Cost Rate Functions

The single-item inventory system under study is controlled by a replenishment policy that is a function of the inventory position, $x$ (the sum of stock on-hand and on-order minus backlogs), e.g., an $(s,S)$ policy or an $(nQ,r)$ policy. Replenishments arrive after a fixed leadtime, $L$. For the periodic demand case, $L$ is integer. All unsatisfied demand is backlogged. Let $G(x)$ be the expected rate at which costs are incurred $L$ time units later when the current inventory position is $x$. This cost rate means the expected cost per time period. The objective is to minimize the long-run average cost, which can be determined by $G(x)$ and the steady-state distribution of the inventory position that is controlled by the replenishment policy. Let $h$ be the inventory carrying cost per unit per time
period. Let $p$, $\pi$ and $b$ be the rates at which the various penalty costs for backorders accumulate. Specifically, let $p$ be the cost per time period per unit backlog on the book, $\pi$ the cost per unit of new backlogs, and $b$ the cost per time period when there are backlogs on the book.

The conditions for quasiconvexity of $G(x)$ are investigated for the five different models under three different scenarios:

(i) all cost coefficients are nonnegative,

(ii) $b=0$ and the others are nonnegative,

(iii) $\pi=0$ and the others are nonnegative.

The result for case (iii) is reported only for Model 1 in the propositions below. Its generalization to the other models is straightforward and found in Table 1 in section 4.

3.1 Model 1: Periodic Review - Periodic Demand

This model is the standard model of scientific papers (see, e.g., Arrow et al., 1958, Veinott, 1966b or Chen and Zheng, 1993) although typically not with all the types of shortage cost coefficients considered here. We assume that all events occur at the beginning of each period in the following sequence: first, a replenishment order, if any, is placed; second, the order placed $L$ periods ago is received; then backlogs, if any, are satisfied on a first-come first-serve basis; finally, the customer arrives and demands a random quantity with mean $D$. The demand is satisfied and any shortage is backlogged.

The order quantity decision in a period does not affect costs until $L$ periods later. All presently outstanding orders will arrive by then; and the inventory has been depleted by the demands of $L+1$ periods. So, if the inventory position is $x \geq 0$ at time $t$, then the expected physical stock on-hand in the period between $t+L$ and $t+L+1$ (the “target period”) is

$$
I(x) = xF_{L+1}(0) + \int_0^x [x-y] f_{L+1}(y) dy = [x-(L+1)D] + \int_x^\infty [y-x] f_{L+1}(y) dy
$$

$$
= [x-(L+1)D] + \int_x^\infty [1-F_{L+1}(y)] dy. \tag{7}
$$
Since the second term of the RHS of the last equation equals zero for $x<0$, (7) holds for all $x$.

Similarly, for $x \geq 0$ the expected backlog in $L$ periods is

$$B_p(x) = \int_x^\infty [y-x]f_{L+1}(y)dy = \int_x^\infty [1-F_{L+1}(y)]dy,$$

which is valid also for $x<0$ as it then equals $[-x+(L+1)D]$. The expected number of new backlogs, $B_n(x)$ incurred in $L$ periods is the difference between the expected backlogs at the end and the beginning of the target period. That is,

$$B_n(x) = \int_x^\infty [1-F_{L+1}(y)]dy - \int_x^\infty [1-F_L(y)]dy = \int_x^\infty [F_L(y)-F_{L+1}(y)]dy,$$

which equals $D$ for $x \leq 0$. Finally, the expected time with backlog in the target period is

$$B_b(x) = [1-F_{L+1}(x)].$$

Multiplying (8)-(10) with respective cost coefficients and summing up give the cost rate:

$$G(x) = h[x-(L+1)D] + (p+h)\int_x^\infty [1-F_{L+1}(y)]dy + \pi\int_x^\infty [F_L(y)-F_{L+1}(y)]dy + b[1-F_{L+1}(x)].$$

Note that if $F(0)>0$ and $b>0$, then $G(x)$ is non-differentiable (and discontinuous) at $x=0$, but only there. If either $F(0)=0$ or $b=0$, then $G(x)$ continuous and differentiable everywhere.

**PROPOSITION 2-1**

(i) $G(x)$ in (11) is quasiconvex for all nonnegative values of $h$, $p$, $\pi$ and $b$, if and only if both $F_L(x)/F_{L+1}(x)$ and $f_{L+1}(x)/F_{L+1}(x)$ are nonincreasing in $x$ (which conditions hold if $F(x)$ is logconcave).

(ii) $G(x)$ in (11) is quasiconvex for $b=0$ and all nonnegative values of $h$, $p$ and $\pi$, if and only if $F_L(x)/F_{L+1}(x)$ is nonincreasing in $x$ (which holds if $F(x)$ is MCR).

(iii) $G(x)$ in (11) is quasiconvex for $\pi=0$ and all nonnegative values of $h$, $p$ and $b$, if and only if $f_{L+1}(x)/F_{L+1}(x)$ is nonincreasing in $x$ (which holds if $F(x)$ is logconcave).

The proof is found in the Appendix 1 (available on-line). The results of P2-1 carry over to discrete demands in the analogous way, but $f_{L+1}(x)/F_{L+1}(x)$ should then be read $f_{L+1}(x+1)/F_{L+1}(x)$.

There is an alternative model where the period shortage cost $b$ is not incurred in partially satisfied periods. The term $b[1-F_{L+1}(x)]$ is then replaced by $b[1-F_L(x)]$. One can demonstrate that
then still holds, with the conditions on \( f_{L+1}(x)/F_{L+1}(x) \) and \( f_{L+1}(x)/[1–F_{L+1}(x)] \), respectively, replaced by the similar conditions on \( f_{L}(x)/F_{L+1}(x) \) and \( f_{L}(x)/[1–F_{L+1}(x)] \).

There is yet another variation, which alters the order of events, so that decisions are made after, rather than before, the demand of the present period is known. This variation is logically equivalent to the variant with Poisson arrivals of the model of the next subsection.

3.2 Model 2: Continuous Review – Renewal Arrivals

In this model customers arrive according to a renewal process with mean interarrival time \( 1/\lambda \), each demanding a random quantity with mean \( \mu \). The long-run expected demand per period is \( D = \lambda \mu \).

The ordering decisions are made at customer arrivals only. This restriction may lead to suboptimal decisions for non-Poisson arrival processes. Versions of the model appear in Beckmann (1961) and Federgruen and Schechner (1983). The simplified version with Poisson arrivals appears in Hadley and Whitin (1963) with each customer demanding a single unit, and in Archibald and Silver (1978) and Chen and Zheng (1993) with general demands.

Consider this inventory system at an arbitrary point in time. Since an equilibrium process is observed, the unconditioned distribution of the time till the next arrival is \( \Omega_0(1) \) as in (3). Moreover, since the inventory position is entirely determined at customer arrivals, the time till the next arrival is independent of the inventory position. Thus, the leadtime demand distribution is \( \tilde{F}_{L}(\cdot) \) as in (6), which is independent of the inventory position at the time the order is placed, as in Model 1. Therefore,

\[
B_b(x) = 1- \tilde{F}_{L}(x), \quad B_p(x) = \int_{x}^{\infty} [1- \tilde{F}_{L}(y)] \, dy \quad \text{and} \quad I(x) = [x-LD] + \int_{x}^{\infty} [1- \tilde{F}_{L}(y)] \, dy.
\]

The rate at which new shortages are incurred is the derivative of \( B_p(x) \) with respect to time, i.e.,

\[
B_\pi(x) = \frac{d}{dL} \int_{x}^{\infty} [1- \tilde{F}_{L}(y)] \, dy. \]

It is simpler, though, to consider an arrival and compute the rate directly as the difference between the expected backlog a leadtime ahead just after and before the arrival of the next customer. That is,

\[
I(x) = \int_{x}^{\infty} [1-F_{L+1}(y)] \, dy - \int_{x}^{\infty} [1-F_{L}(y)] \, dy,
\]

which difference is then divided by the average interarrival time:
\[ B_n(x) = \lambda \int_0^x [F_{(L)}(y) - F_{(L)+1}(y)] dy, \] which equals \( D \) for \( x \leq 0 \).

Multiplying with the respective cost coefficients and summing up render the cost rate:

\[ G(x) = h[x - LD] + (p+h) \int_0^x [1 - \tilde{F}_{(L)}(y)] dy + \lambda \pi \int_0^x [F_{(L)}(y) - F_{(L)+1}(y)] dy + b[1 - \tilde{F}_{(L)}(y)]. \] (12)

**PROPOSITION 2-2**

(i) \( G(x) \) in (12) is quasiconvex for all nonnegative values of \( h, p, \pi \) and \( b \), if and only if both \( [F_{(L)}(x) - F_{(L)+1}(x)]/F_{(L)+1}(x) \) and \( \tilde{f}_{(L)}(x)/\tilde{F}_{(L)}(x) \) are nonincreasing in \( x \) (which conditions hold if \( \omega(t) \) is exponential and \( \Psi(x) \) is concave).

(ii) \( G(x) \) in (12) is quasiconvex for \( b=0 \), and all nonnegative values of \( h, p \) and \( \pi \), if and only if \( [F_{(L)}(x) - F_{(L)+1}(x)]/\tilde{F}_{(L)}(x) \) is nonincreasing in \( x \) (which holds if \( \omega(t) \) is logconcave and \( \Psi(x) \) is MCR).

The main assertions follow as in the proof of P2-1 (available on-line: consistently replace the expressions \( [F_{(L)}(x) - F_{(L)+1}(x)]/F_{(L)+1}(x) \) and \( f_{(L)}(x)/F_{(L)+1}(x) \) there by \( [F_{(L)}(x) - F_{(L)+1}(x)]/\tilde{F}_{(L)}(x) \) and \( \tilde{f}_{(L)}(x)/\tilde{F}_{(L)}(x) \), respectively). The parenthesis of \((i)\) follows by P1-3(iii), P1-3(i) and P1-2(ii), as concavity of \( \Psi(x) \) implies logconcavity, and the parenthesis of \((ii)\) by P1-3(i).

Customers, none of whose demand is satisfied, arrive at an average rate of \( \lambda B_b(x) \), and so the cost term \( B_b(x) = b[1 - \tilde{F}_{(L)}(x)] \) may alternatively be interpreted as the cost per unsatisfied customer. The same formula is valid, but \( b/\lambda \) should then be understood as the fixed stock-out cost per customer. Contrary to Model 1, we cannot guarantee quasi-convexity if customers are unsatisfied when just a part of their demand is so, i.e., \( B_b(x) \) must not equal \( b[1 - \tilde{F}_{(L)+1}(x)] \).

With Poisson arrivals, \( \tilde{F}_{(0)}(y) = F_{(0)}(y) \) for all \( t \). So, the Poisson model is logically very close to Model 1. The major difference is that in Model 2 decisions are made immediately after a demand is observed, and not immediately before (and the narrower interpretation of \( b \)).
3.3 Model 3: Periodic Review - Renewal Arrivals

This model assumes a compound renewal demand process as in Model 2, but review is periodic as in Model 1. The expected demand per review period is \( D = \lambda \mu \). The discrete version of this model with each customer demanding a single unit appeared in Hadley and Whitin (1963). The model is more realistic than Model 1, with particular effects on the estimates of \( B_p(x) \) and \( B_b(x) \).

At review epochs, the inventory position is inspected, but the time elapsed since the last arrival is ignored. (Again, this may be suboptimal for non-Poisson arrivals.) Thus, for systems that have run for a long time, the arrival time of the first customer after a review obeys the equilibrium renewal distribution, and we shall assume this is true, independent of the inventory position. With Poisson arrivals this assumption is indeed true, but in general it is not, and should be understood as an approximation of a case that is analytically complicated (dependent on the replenishment policy). The approximation only concerns the first arrival in a period, so the effect should be small when the arrival intensity is not too low. With this approximation, the total demand during the next \( t \) time periods after the review is distributed \( \tilde{F}_0(x) \) as in (6). We will need the average of \( \tilde{F}_0(x) \) over \( L \leq t \leq L+1 \):

\[
F_L(x) = \int_L^{L+1} \tilde{F}_0(x) \, dt. \tag{13}
\]

The current decisions have effect in the period that begins in \( L \) (not necessarily integer) periods. The expected physical inventory level on-hand decreases throughout this period, and at \( t \), \( L \leq t \leq L+1 \), it is \( [x-D(L+t)] + \int_x^\infty [1-\tilde{F}_0(y)] \, dy \), as for Model 2, and hence, summed up,

\[
I(x) = \int_L^{L+1} [x-D(L+t)] + \int_x^\infty [1-\tilde{F}_0(y)] \, dy \, dt = [x - (L+1/2)D] + \int_x^\infty [1-F_L(y)] \, dy.
\]

The average expected backlog is similarly \( B_p(x) = \int_L^{L+1} \int_x^\infty [1-\tilde{F}_0(y)] \, dy \, dt = \int_x^\infty [1-F_L(y)] \, dy \). The average expected new backlog and the expected time with backlog are respectively,

\[
B_a(x) = \int_x^\infty [\tilde{F}_0(L) - \tilde{F}_0(L+1)] \, dy \quad \text{and} \quad B_b(x) = \int_L^{L+1} [1-\tilde{F}_0(x)] \, dt = 1-F_L(x).
\]

Thus, recalling (13), the entire cost-rate function may be summarized as
G(x) = h[x–(L+1/2)D] +(p+h) \int_{y=x}^{\infty} \left[ 1-F_L(y) \right]dy + \pi \int_{y=x}^{\infty} \left[ \tilde{F}_{(L+1)}(y) - \tilde{F}_L(y) \right]dy + b[1-F_L(x)]. \quad (14)

**PROPOSITION 2-3**

(i) For all nonnegative values of h, p, π and b, G(x) in (14) is quasiconvex, if and only if both \( \left[ \tilde{F}_{(L)}(x) - \tilde{F}_{(L+1)}(x) \right]/F_L(x) \) and \( f_L(x)/F_L(x) \) are nonincreasing in x.

(ii) For b=0, and all nonnegative values of h, p, and π, G(x) in (14) is quasiconvex, if and only if \( \left[ \tilde{F}_{(L)}(x) - \tilde{F}_{(L+1)}(x) \right]/F_L(x) \) is nonincreasing in x (which holds if \( \omega(t) \) is logconcave and \( \Psi(x) \) is MCR).

We skip the formal proof of this proposition since it would be parallel to that for P2-2. To see the assertion in the parenthesis of P2-3(ii), note that by (13),

\[
\left[ \tilde{F}_{(L)}(x) - \tilde{F}_{(L+1)}(x) \right]/F_L(x) = 1/\{ \int_{L}^{L+1} \frac{\tilde{F}_{(L)}(x)}{f_L(x)}dt \} - 1/\{ \int_{L}^{L+1} \frac{\tilde{F}_{(L+1)}(x)}{f_L(x)}dt \},
\]

which is nonincreasing in x if \( \Psi(x) \) is MCR, by P1-3(ii), as \( L \leq t \leq L+1 \).

Note that P2-3(i) provides no condition on \( \Psi(x) \) for quasiconvexity of G(x). However, for the special case of Poisson arrivals with unit demand, G(x) is quasiconvex. We also note that in this case the model degenerates into a discrete version of Model 5 (see below) with \( F(x) \) Poisson and therefore logconcave, see P2-5 and the relevant comments in subsection 3.5.

### 3.4 Model 4: Approximate Costs: Continuous Review - Continuous Demand

In this subsection we consider the continuous review model that assumes that the cumulative demand has a continuous sample path with nonnegative, independent increments. This model seems implicitly assumed in most (elementary) textbooks on inventory management. However, the assumptions of independence and continuity are conflicting, therefore, rigorously speaking, the model is approximate (Browne and Zipkin 1991).

\( D \) denotes the average demand rate. The replenishment leadtime, \( L \), is not necessarily an integer. The distribution of leadtime demand is \( F_L(x) \) (with mean \( LD \)). Given the current inventory position \( x \), let \( I(x) \) and \( B_p(x) \) be the expected inventory on hand and the expected backorders \( L \) time periods later. Then, as discussed in detail for Model 1, we have
\[ I(x) = (x - LD) + \int_{x}^{\infty} [1 - F_L(y)] \, dy \quad \text{and} \quad B_p(x) = \int_{x}^{\infty} [1 - F_L(y)] \, dy. \]

As the cumulative demand process has a continuous sample path, the incremental demand \( L \) periods later is either entirely satisfied or entirely unsatisfied. Thus, the expected rate at which new backlogs are incurred \( L \) periods later is \( B_n(x) = D[1 - F_L(x)] \). So, as the probability of having backlogs a leadtime ahead is \( B_b(x) = [1 - F_L(x)] \), there is in effect no distinction between shortage costs per item and per time period. Consequently,

\[ G(x) = h[x - LD] + (p+h) \int_{x}^{\infty} [1 - F_L(y)] \, dy + (b+D\pi)[1 - F_L(x)] \tag{15} \]

**PROPOSITION 2-4**

\( G(x) \) in (15) is quasiconvex for all nonnegative values of \( h, p, \pi \) and \( b \),

if and only if \( F_L(x) \) is logconcave.

The assertion is a direct corollary to \( P2-1(iii) \), as the present model is a special case of Model 1.

### 3.5 Model 5: Approximate Costs: Periodic Review - Continuous Demand

In this subsection we consider a periodic review model with the approximate continuous demand model used in the previous subsection. The development is motivated by the complications of Model 3. We use the same notation as for Model 4, and in the similar way, the distinction between shortage costs per unit and per period vanishes:

\[ B_n(x) = \int_{x}^{\infty} [1 - F_{L+1}(y)] \, dy - \int_{x}^{\infty} [1 - F_L(y)] \, dy = D \int_{x}^{\infty} [F_L(y) - F_{L+1}(y)] \, dy = DB_b(x) \]

We approximate the average backorders in the target period as follows. Let the current inventory position be \( x > 0 \) and let the total demand over \( L+1 \) periods be \( y \). Suppose \( y > x \), so there are backlogs. Ignoring the fact that \( y \) is greater than \( x \), we assume (approximately) that the demand rate remains \( D \) per period during the stockout time. Thus, the expected length of the stockout time is \( (y-x)/D \), and the total unit-periods of backlogs, incurred in the \( L+1 \) periods, are \( (y-x)^2/2D \) (for the discrete case, the analogy would be \( (y-x+1)(y-x)/2D \)). Subtract the unit-periods of backlogs incurred in the first \( L \) periods, and take expectations,

\[ B_p(x) = \int_{x}^{\infty} (y-x)^2/2D \, f_{L+1}(y) \, dy - \int_{x}^{\infty} (y-x)^2/2D \, f_L(y) \, dy \tag{16} \]
\[
\int_{y}^{x} (y-x)[F_L(y) - F_{L+1}(y)]/Dy = \int_{y}^{x} (1 - \int_{0}^{y} [F_L(z) - F_{L+1}(z)]/Dz)dy
\]

Assuming this formula be valid also for \(x<0\), the expected inventory level on hand in the target period will be

\[
I(x) = \int_{1}^{x+i} (x-Dt)dt + B_p(x) = [x-(L+1/2)D] + B_p(x).
\]

Multiplying with the cost coefficients then gives the approximate cost function:

\[
G(x) = h[x - (L+1/2)D] + (p+h)\int_{L}^{y} [1-H_L(y)]dy + (b+D\pi)[1-H_L(x)], \quad \text{where} \quad (17)
\]

\[
H_L(x) = \int_{0}^{x} h_L(y)dy \quad \text{and} \quad h_L(x) = [F_L(x) - F_{L+1}(x)]/D.
\]

Note that \(G(x)\) is continuous even if \(F(0)>0\). With an analogous argument for the discrete case, the similar formula appears, but "\(x\)" should be replaced by "\(x-I\)" in the last two terms of (17).

**PROPOSITION 2-5**

\(G(x)\) in (17) is quasiconvex for all nonnegative values of \(h, p, \pi\) and \(b\),

if and only if \(H_L(x)\) is logconcave (which holds if \(F_L(x)\) is logconcave).

The proposition is apparently a direct corollary to \(P2-4\), so only the assertion in the parenthesis has to be demonstrated. To this end, note that

\[
H_L(x) = \int_{0}^{x} \{F_L(0)[1-F(y)]/D + \int_{0}^{x} f_L(y-t)[1-F(t)]/Dt\}dy
\]

\[
= \int_{0}^{x} \{F_L(0)[1-F(t)]/D + \int_{t}^{x} f_L(y-t)[1-F(t)]/Dy\}dt = \int_{0}^{x} F_L(x-t)[1-F(t)]/Dydt,
\]

which may be understood as the distribution function of the convolution of two stochastic variables, one with density \([1-F(x)]/D\), and the other with CDF, \(F_L(x)\). As \([1-F(x)]/D\) is non-increasing and thus the associated CDF logconcave, it follows by \(P1-1(i)\) that \(H_L(x)\) is logconcave if \(F_L(x)\) is so.

Note that the parenthetical requirement only concerns \(F_L(x)\). Thus, if \(F_L(x)\) should be logconcave, but not \(F(x)\), quasiconvexity of \(G(x)\) is still guaranteed by \(P2-5\).

Before closing this subsection, we remark that the approximated \(B_p(x)\) contains a systematic error. To see this, let \(\sigma^2\) denote the variance of \(F(x)\), so that the variance of \(F_L(x)\) and \(F_{L+1}(x)\) are \(L\sigma^2\) and \((L+1)\sigma^2\), respectively (due to the independent increment assumption). The suggested
formula \( \int_{\infty}^{\infty} [1-H_L(y)]dy \) then implies that \( B_p(x) = -x + (L+1/2)D + \sigma^2/(2D) \) for \( x \leq 0 \), which apparently is an overestimate by \( \sigma^2/(2D) \) [for the discrete case, the error term would read \( \sigma^2/(2D)+1/2 \)].

A possible improvement could be to subtract \( \sigma^2/(2D)[1-H_L(x)] \) in the continuous case, \( i.e. \)
\[
B_p(x) = \int_{\infty}^{\infty} [1-H_L(y)]dy - [\sigma^2/(2D)][1-H_L(x)].
\]

[The analogous suggestion for the discrete case – subtracting \( \sigma^2/(2D)+1/2[1-H_L(x-1)] \) – would be exact for Poisson arrivals with unit demand, for which \( B_p(x) = \sum_{\infty} \{1-H_L(y)\} \); cf. the comments following (17)] This would eliminate the bias when \( x \leq 0 \), and it would not interfere with the above formula for \( G(x) \), as it essentially suggests that the cost coefficient of \( B_b(x) \) be determined as \( \{b+\pi D-(p+h)\sigma^2/(2D)\} \).(Unfortunately though, the improved formula typically turns a little negative for large values of \( x \), which is unrealistic, of course.)

There is an alternative approximation of \( B_p(x) \), inspired by Holt et al. (1960) in a somewhat different context, namely, for \( x \geq 0 \),
\[
B_p(x) = (L+1) \int_{\infty}^{\infty} (y-x)^2/2y f_{L+1}(y)dy - L \int_{\infty}^{\infty} (y-x)^2/2y f_L(y)dy.
\]

Here, the fraction of the stockout time over \( L \) time periods is estimated as the ratio of the shortage, \( y-x \), and the actual demand, \( y \); not the average demand, \( LD \). This suggestion is quite reasonable, as there is no obvious bias or misbehavior. However, the resulting formula becomes computationally intractable except for the gamma distribution, and even for this demand distribution, there is no guarantee that \( G(x) \) would be quasiconvex in general.

Assume that the one-period demand is gamma with density \( f(y) \) and shape parameter, \( r \). Then both \( f_L(y) \) and \( f_{L+1}(y) \) are gamma with shape parameters \( Lr \) and \( (L+1)r \), respectively \( (Lr>1) \). Quite interestingly, one can then show that (19) is equivalent to
\[
B_p(x) = \int_{\infty}^{\infty} (y-x)^2/2Df_{M+1}(y)dy - \int_{\infty}^{\infty} (y-x)^2/2Df_M(y)dy,
\]
which coincides with (16), except that \( L \) is now replaced by a fictitious leadtime, \( M = L-1/r>0 \).
Thus, if \( r \) is large (the one-period density \( f(x) \), with shape parameter \( r \), would then be approximately normal) one would guess that the difference between (20) and (16) would be small. Moreover, as \( L>M \), the observation again suggests that (16) may typically overestimate the average backlog.

4. Conclusion

4.1 Summary and Illustrations

Table I summarizes the sufficient conditions for \( G(x) \) to be quasiconvex for all 15 possible combinations of the five models and the three scenarios. The conditions are with regard to the “basic” demand distributions. Note that for Model 3, no sufficient conditions on the basic distribution were found for (i) and (iii). For all models, we have included case (iii) with \( \pi=0 \) and \( h, p, b \geq 0 \), which was investigated in \( P2-1(iii) \). The sufficient requirements on the basic distributions coincide with those for (i). The necessary and sufficient requirements are that \( F_{L+1}(\tilde{F}, \tilde{F}_L, F_L, H_L) \) be logconcave.

**TABLE I. Summary of Sufficient Conditions for Quasiconvexity of \( G(x) \)**

<table>
<thead>
<tr>
<th>Cost Coefficient</th>
<th>Model 1</th>
<th>Model 2</th>
<th>Model 3</th>
<th>Model 4</th>
<th>Model 5</th>
</tr>
</thead>
<tbody>
<tr>
<td>(i) ( h,p,\pi,b \geq 0 )</td>
<td>F logconcave</td>
<td>( \Psi ) concave (^1)</td>
<td>–</td>
<td>( F_L ) logconcave</td>
<td>( F_L ) logconcave</td>
</tr>
<tr>
<td>(ii) ( b=0, h,p,\pi \geq 0 )</td>
<td>F MCR</td>
<td>( \Psi ) MCR (^2)</td>
<td>( \Psi ) MCR (^2)</td>
<td>( F_L ) logconcave</td>
<td>( F_L ) logconcave</td>
</tr>
<tr>
<td>(iii)( \pi=0, h,p,b \geq 0 )</td>
<td>F logconcave</td>
<td>( \Psi ) concave (^1)</td>
<td>–</td>
<td>( F_L ) logconcave</td>
<td>( F_L ) logconcave</td>
</tr>
</tbody>
</table>

The three shortage cost components, \( B_p(x) \), \( B_{\pi}(x) \) and \( B_b(x) \), are similar in shape across the five models: \( B_p(x) \) is convex and decreasing towards zero. For negative \( x \), it is linearly decreasing at unit rate. \( B_{\pi}(x) \) equals \( D \) for \( x \leq 0 \), and is then continuously nonincreasing towards zero. \( B_b(x) \) equals 1.0 for \( x \leq 0 \) and is possibly discontinuous at \( x=0 \). For \( x \geq 0 \), \( B_b(x) \) is nonincreasing, and approaches zero as \( x \) increases. (In the last two sentences, ”nonincreasing” may be replaced by ”decreasing” if \( F(x) \), or \( \Psi(x) \), is increasing.) Note that the conditions for \( G(x) \) to be quasiconvex also apply to each
of the three shortage cost components \( G(x) \) reduces to one of them when the other cost coefficients are set to zero).

The cost-rate function \( G(x) \) is illustrated in Fig. 2. It is asymptotically linear in both directions. It is convex if \( \pi = b = 0 \). There can be a downward discontinuity at \( x = 0 \) if \( b > 0 \) (or \( \pi \) in Model 4 - but not in Model 5, where \( B_\delta(x) \) is continuous); and necessarily so for Models 2 and 3, if \( b > 0 \), because of the spike at zero of the leadtime demand distributions.

(Fig. 2 about here)

The following proposition sharpens the quasiconvexity results of the last section in a way that may be very useful in optimization (Rosling 1999). It is stated without proof.

**PROPOSITION 3-1**

Consider any of the above models. Under the assumptions for which \( G(x) \) is quasiconvex, \( G(x) \) is also strictly quasiconvex with a unique minimum, if in addition \( h \) and \( p \) are positive and, for Model 1, 2 and 3, if \( F(x) \) and \( \Psi(x) \), respectively, are strictly increasing where they are positive (which they are if they are logconcave).

The strict quasiconvexity of \( G(x) \) does not generalize to discrete demand distributions. It is then generally possible to find values of the cost coefficients for which \( G(x) \) attains its minimum at two adjacent values of \( x \).

**4.2 Extensions**

The cost-rate models trivially generalize to stochastic leadtimes under the conditions clarified by Zipkin (1986). The distribution of demand over the deterministic leadtime is just replaced by the distribution of total demand over the stochastic leadtime. However, when the leadtime demand distribution must be derived from the leadtime distribution and the underlying demand process, extending the above propositions to conditions on the leadtime distribution is far from trivial.

The above framework can be used, though, to analyze stochastic leadtimes in Model 1, provided leadtime is a Poisson random variable, \( t = 1, 2, \ldots \) with probability \( e^{L} / t! \), where \( L \) is the expected leadtime. If the single period demand distribution is denoted \( \Psi(x) \), then the leadtime
demand distribution is $F_{A}(x)$ as in (4). Thus, Model 1 degenerates into the simplified version of Model 2 with Poisson arrivals, so that $\widetilde{F}_{A}(x)=F_{A}(x)$. (For an interesting result on continuously distributed leadtimes, see Dharmadhikiri and Joag-dev, 1988, Theorem 4.5.)

The results for Model 1, 4 and 5 generalize to distributions, $F(x)$ and $F_{L}(x)$, whose support extends into the negative half-line, provided the functions are continuous on their entire support, as is the Normal distribution. There is only a problem of realism, because a minimum of $G(x)$ now appears for $x<0$ if the holding cost coefficient is sufficiently large. With a discontinuity at $x=0$, there could then be two minima, excluding quasiconvexity. Thus, unless $b=0$, negative customer demands cannot be allowed in Model 2 and 3, because of the necessary spike at zero of their leadtime demand distributions.

The results for Model 1 and 4 trivially generalize to the discounted cost case – just multiply $G(x)$ by the discount factor for $L$ periods to get its present value – but for the other models, additional investigations are required. A generalization of the results for all the present models that allows negative values of $b$ and $\pi$ is found in Rosling (1997).

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Appendix 1. Omitted Proofs

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Appendix 2. List of Symbols

\( F_n(x) \) the n-period demand distribution in the model with periodic demand,
\( \Psi(x) \) the customer demand distribution in the renewal models; mean=\( \mu \),
\( \Omega_n(t) \) the arrival time distribution for n customers (renewals); mean=n/\( \lambda \),
\( \tilde{\Omega}_n(t) \) the corresponding distribution for the equilibrium renewal process,
\( p_n(t) \) the probability for n new customers arriving in time t after a renewal,
\( \tilde{p}_n(t) \) the corresponding equilibrium probability,
\( F_0(t) \) the distribution of demand over t, following a renewal,
\( \tilde{F}_0(t) \) the corresponding equilibrium distribution,
\( F_L(x) \) the average of \( \tilde{F}_0(t) \) over \( L \leq t \leq L+1 \).
\( x \) the present inventory position,
\( D \) expected demand per period = \( \lambda \mu \),
\( L \) the fixed leadtime
\( h \) inventory holding cost per unit per period,
\( p \) backlog cost per unit per period,
\( \pi \) backlog cost per unit new backlog,
\( b \) backlog cost per period (with backlogs on the books),
\( I(x) \) the expected physical stock on-hand a leadtime ahead, given \( x \),
\( B_p(x) \) expected backlog per period a leadtime ahead, given \( x \),
\( B_s(x) \) expected new backlogs incurred a leadtime ahead, given \( x \),
\( B_b(x) \) the expected time with backlogs on the books a leadtime ahead, given \( x \).
Fig. 1. Summary of P1-1 and P1-2
Fig.2 $G(x)$ - all cost coefficients are positive.
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Appendix 1. Omitted Proofs

This appendix contains the proofs of Proposition 1-2, Lemma 1-3, Proposition 1-3 and Proposition 2-1. The texts of the propositions have been copied from the main text to enhance readability. Equations that only appear in this appendix are numbered (A1) – A(4). In the proofs, references are made to equations and propositions of the main text.

PROPOSITION 1-2

(i) If \( f(x) \) is MCR, so are \( F(x) \) and \( 1-F(x) \).

(ii) If \( f(x) \) (\( F(x) \), \( 1-F(x) \)) is logconcave, then \( f(x) \) (\( F(x) \), \( 1-F(x) \)) is MCR.

Proof (i): To see the result for \( 1-F(x) \), pick \( 0 \leq x < y \) such that \( [1-F_{n+1}(y)] > 0 \), and consider

\[
\frac{1-F_{n+1}(y)}{1-F_n(y)} - \frac{1-F_{n+1}(x)}{1-F_n(x)} = \frac{F_n(y)-F_n(x)}{1-F_n(y)} - \frac{F_n(y)-F_n(x)}{1-F_n(x)} = \frac{[F_n(y)-F_n(x)](1-F_{n+1}(y)) - [F_n(y)-F_n(x)](1-F_{n+1}(x))}{[1-F_n(y)][1-F_n(x)]} = \int_y^\infty \int_x^y \{ f_n(s)/f_{n+1}(s)-f_n(t)/f_{n+1}(t) \} f_{n+1}(s)f_{n+1}(t) \, ds \, dt \geq 0 \quad \text{by assumption, as } x \leq s \leq y \leq t.
\]

For \( x \leq 0 \) such that \( F_{n+1}(x) = 0 \), also \( F_n(x) = 0 \), and then by convention, \( F_n(x)/F_{n+1}(x) = -\infty \), and the result follows. The proofs for \( F(x) \) and \( 1-F(x) \) are identical, except that \( F(0) = 0 \).

Q.E.D.

(ii): To see the result for \( F(x) \), recall that

\[
F_{n+1}(x) = F(0)F_n(x) + \int_0^\infty f(s)f_n(x-s) \, ds.
\]

Pick \( 0 \leq x < y \) such that \( F_n(x) > 0 \) and consider

\[
F_n(x)F_{n+1}(y)F_{n+1}(x) = F(0)[F_n(x)F_n(y)-F_n(y)F_n(x)] + \int_0^\infty f(s)[F_n(x)F_n(y-s)-F_n(y)F_n(x-s)] \, ds
\]

= \[0^\infty f(s)[F_n(x)/F_n(x-s) - F_n(y)/F_n(y-s)]F_n(x-s)F_n(y-s) \, ds \geq 0,
\]

as \( x < y \) and \( F_n(x) \) is logconcave by P1-1(i). For \( x \leq 0 \) such that \( F_{n+1}(x) = 0 \), also \( F_n(x) = 0 \), and then by convention, \( F_n(x)/F_{n+1}(x) = -\infty \), and the result follows. The proofs for \( f(x) \) and \( 1-F(x) \) are identical, except that \( F(0) = 0 \).
LEMMA 1-3

If \( \omega(t) \) is logconcave (strongly unimodal), then

(i) \( p_n(t) \) and \( \tilde{p}_n(t) \) are logconcave in \( t \) for all \( n \geq 0 \).

(ii) \( p_n(t) \) and \( \tilde{p}_n(t) \) are logconcave in \( n \) for all \( t > 0 \).

(iii) \( \tilde{p}_n(t)/p_n(t) \) is nondecreasing in \( n \) for all \( t > 0 \).

(iv) \( \tilde{p}_n(t)/p_{n-1}(t) \) is nonincreasing in \( n \) for all \( t > 0 \).

Proof (i): \( p_n(t) = \Omega_n(t) - \Omega_{n+1}(t) = (1/\lambda) \int_0^t \omega_n(x) \lambda [1 - \Omega(t-x)] dx \) is logconcave in \( t \) by P1-1(i) as the latter expression may be understood as the convolution of two logconcave densities, \( \omega_n(t) \) and \( \lambda [1 - \Omega(t)] \), which both are logconcave, by assumption and P1-1(i) and P1-1(ii), respectively.

\[ \tilde{p}_n(t) = \tilde{\Omega}_n(t) - \tilde{\Omega}_{n+1}(t) = \int_0^t \tilde{\omega}_1(x) \left[ \Omega_{n+1}(t-x) - \Omega_n(t-x) \right] dx = \int_0^t \lambda [1 - \Omega(x)] p_{n-1}(t-x) dx \]

is logconcave in \( t \) as the last expression is the convolution of the two densities, \( \lambda [1 - \Omega(t)] \) and \( \lambda [\Omega_{n+1}(t) - \Omega_n(t)] = \lambda p_{n-1}(t) \), which both are logconcave as just noted.

(ii): The result is a direct corollary to (iii) and (iv) below, dividing \( \tilde{p}_n(t)/p_{n-1}(t) \) by \( \tilde{p}_n(t)/p_n(t) \).

(iii): The conclusion follows if \( p_{n+1}(t)/\tilde{p}_{n+1}(t) \leq p_n(t)/\tilde{p}_n(t) \), i.e., if \( p_{n+1}(t)/\tilde{p}_n(t) - \tilde{p}_{n+1}(t)/p_n(t) \leq 0 \) for all \( n \geq 0 \). Note that \( \tilde{p}_{n+1}(t) = \int_0^t \alpha(x)p_n(t-x) dx \) and \( \tilde{p}_n(t) = \int_0^t \lambda [1 - \Omega(x)] p_n(t-x) dx \).

So for \( n \geq 1 \), \( p_{n+1}(t)/\tilde{p}_n(t) - \tilde{p}_{n+1}(t)/p_n(t) \)

\[ = \int_0^t \int_0^s \left\{ \alpha(x)p_n(t-x)\lambda [1 - \Omega(s)] p_{n-1}(t-s) - \lambda [1 - \Omega(x)] p_n(t-x)\alpha(s)p_{n-1}(t-s) \right\} ds \ dx \]

\[ = \int_0^t \int_0^s p_n(t-x) p_{n-1}(t-s) \alpha \left\{ [1 - \Omega(s)] \alpha(x) - [1 - \Omega(x)] \alpha(s) \right\} ds \ dx \]

\[ = \int_0^t \int_{x \geq s} \left\{ p_n(t-x)p_{n-1}(t-s) - p_n(t-s)p_{n-1}(t-x) \right\} \left\{ [1 - \Omega(s)] \alpha(x) - [1 - \Omega(x)] \alpha(s) \right\} dx \ ds \]

\[ = \int_0^t \int_{x \geq s} \int_0^x \alpha(y) \left\{ p_n(t-x-y)p_{n-1}(t-s) - p_n(t-s-y)p_{n-1}(t-x) \right\} \left\{ [1 - \Omega(s)] \alpha(x) - [1 - \Omega(x)] \alpha(s) \right\} dy \ dx \ ds \]

\[ \leq 0, \] because the first expression in curly parenthesis is nonpositive by logconcavity of \( p_n(t) \) in \( t \), which result appeared above at (i), and the fact that \( t-s \geq t-x \), as then \( p_{n+1}(t)/p_{n-1}(t-s-y) \leq \]
\[ p_{n-1}(t-x)/ p_{n-1}(t-x-y); \text{ and the second expression in curly parenthesis is nonnegative by logconcavity of } [1-\Omega(t)], \text{ which is implied by the assumed logconcavity of } \omega(t) \text{ according to } \]

P1-1(ii) as then \{\omega(x)/[1-\Omega(x)] - \omega(s)/[1-\Omega(s)]\} \geq 0 \text{ for } x \geq s. \text{ Thus, } p_{n+1}(t)/p_n(t) \leq \tilde{p}_{n+1}(t)/ \tilde{p}_n(t) \text{ for all } n \geq 1. \text{ For } n=0, \text{ } p_1(t)/p_0(t) = \{1-\Omega_1(t)\} \left[ \Omega_2(t) - [1-\Omega(t)][\tilde{\Omega}_1(t)-\tilde{\Omega}_2(t)] \right] \]

\[ \leq 0 \text{ by logconcavity of } [1-\Omega(t)] \text{ as } x \geq s. \]

(iv): The conclusion follows if \( p_0(t)/\tilde{p}_{n+1}(t) \geq p_{n-1}(t)/\tilde{p}_n(t) \), i.e., if \( p_0(t)/\tilde{p}_n(t) - p_{n-1}(t)/\tilde{p}_{n+1}(t) \geq 0 \).

Pick \( n \geq 1 \) and note that \( p_0(t)/\tilde{p}_n(t) - p_{n-1}(t)/\tilde{p}_{n+1}(t) = \int_0^t \omega(s) \left[ p_{n-1}(t-s)/p_{n+1}(t) - p_{n-1}(t)/\tilde{p}_{n+1}(t-s) \right] \text{ds}, \)

which is nonnegative if the term in brackets is so. To see that this is the case, note that \( \tilde{p}_n(t)/ p_{n-1}(t) = \int_0^t \tilde{\omega}_1(s) \{p_{n-1}(t-s)/p_{n-1}(t)\} \text{ds}, \) which is nondecreasing in \( t \) as the term in curly parenthesis is so by the logconcavity of \( p_0(t) \) in \( t \), which was proved above at (i). \quad \text{Q.E.D.} \]

**PROPOSITION 1-3**

Suppose \( \omega(t) \) is logconcave (strongly unimodal):

(i ) If \( \Psi(x) \) is MCR, then both \( \tilde{F}_0(x)/F_0(x) \) and \( F_{0+1}(x)/\tilde{F}_0(x) \) are nondecreasing in \( x \) for all \( t>0. \)

(ii ) If \( \Psi(x) \) is MCR, then both \( F_0(x)/F_0(x) \) and \( \tilde{F}_0(x)/\tilde{F}_0(x) \) are nondecreasing in \( x \) for all \( t \) and \( s, t \geq s>0. \)

(iii ) Suppose customers arrive according to a Poisson process (i.e., \( \omega(t) \) is exponential).

Then \( F_0(x) \) is logconcave in \( x \) for all \( t>0 \), if and only if \( \Psi(x) \) is concave on \( x \geq 0 \)

(i.e., if and only if \( \psi(x) \) is nonincreasing).
Proof (i): \( \tilde{F}_0(x)/F_0(x) \) is nondecreasing in \( x \) if \( F_0(x)\tilde{F}_0(y) - F_0(y)\tilde{F}_0(x) \geq 0 \) for \( y \geq x \). Now,

\[
F_0(x)\tilde{F}_0(y) - F_0(y)\tilde{F}_0(x) = \sum_{n=0}^{\infty} p_n(t)\Psi_n(x) - \sum_{n=0}^{\infty} p_n(t)\Psi_n(y)
\]

\[
= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} p_n(t)\tilde{p}_m(t)[\Psi_n(x)\Psi_m(y) - \Psi_n(y)\Psi_m(x)]
\]

\[
\geq 0, \text{ as the first term in brackets is nonpositive by L1-3(iii) and so is the second term by assumption, since } n>m \text{ and } y \geq x.
\]

\[
F_0(x)/\tilde{F}_0(x) \text{ is nondecreasing in } x \text{ if } F_0+1(x)\tilde{F}_0(y) - F_0(y)\tilde{F}_0(x) \leq 0 \text{ for } y \geq x. \text{ Now,}
\]

\[
F_0+1(x)\tilde{F}_0(y) - F_0+1(y)\tilde{F}_0(x) = \sum_{n=0}^{\infty} p_n(t)\Psi_{n+1}(x) - \sum_{n=0}^{\infty} p_n(t)\Psi_{n+1}(y)
\]

\[
= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} p_n(t)\tilde{p}_m(t)[\Psi_n(x)\Psi_m(y) - \Psi_n(y)\Psi_m(x)]
\]

\[
+ \sum_{n=0}^{\infty} p_n(t)\tilde{p}_n(t)[\Psi_{n+1}(x)\Psi_{n+1}(y) - \Psi_{n+1}(y)\Psi_{n+1}(x)] \quad [\text{as } \Psi_0(x)=0]
\]

\[
\leq \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} p_{n-1}(t)\tilde{p}_m(t)[\Psi_n(x)\Psi_m(y) - \Psi_{n+1}(y)\Psi_m(x)] \quad [\text{as } y \geq x]
\]

\[
= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} [p_{n-1}(t)\tilde{p}_m(t) - p_{n-1}(t)\tilde{p}_n(t)][\Psi_n(x)\Psi_m(y) - \Psi_n(y)\Psi_m(x)] \quad [\text{by symmetry}]
\]

\[
\leq 0, \text{ as the first term in brackets is nonnegative by L1-3(iv) and the second term is nonpositive by the MCR assumption, since } n>m \text{ and } y \geq x.
\]

(ii): Pick \( s \leq t, x \leq y \) and consider \( F_{(s)}(y)/F_{(s)}(x) - F_{(t)}(x)/F_{(s)}(y) \), i.e.,

\[
F_{(s)}(y)F_{(t)}(x) - F_{(s)}(x)F_{(t)}(y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} p_n(s)p_n(t)[\Psi_m(y)\Psi_n(x) - \Psi_m(x)\Psi_n(y)] \quad [\text{by (2) and (3)}]
\]

\[
= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} [p_n(s)p_n(t)-p_n(s)p_m(t)][\Psi_m(y)\Psi_n(x) - \Psi_m(x)\Psi_n(y)] \quad [\text{by symmetry}]
\]
\[
\sum_{m=0}^{\infty} \sum_{n=m+1}^{\infty} \int_0^t \omega_{m-n}(y) \left[ p_n(s-y)p_n(t-y)p_n(t-y) \right] \Psi_m(y) \Psi_n(x) - \Psi_m(y) \Psi_n(y) \] dy

(by definition of \( p_m(t) \), the notation \( p_n(x)=0 \) for \( x<0 \), and the fact \( t \geq s \))

\[ \leq 0, \quad \text{as the first term in brackets is nonpositive, as } t \geq s \text{ and } p_n(t) \text{ is logconcave in } t \text{ by L1-3(i)}; \]

and the second term is nonnegative by the MCR assumption, as \( m>n \) and \( y \geq x \).

(iii): To prove the if-part, approximate the Poisson process by dividing the period of length \( t \) into \( n>\lambda t \) subperiods of length \( t/n \), and let one customer arrive in each subperiod with probability \( q_1=\lambda t/n \), and no customer with probability \( q_0=1-q_1 \). The distribution function of one subperiod’s demand is then \( F(x) = (1-q_1) + q_1 \Psi(x) \), and the frequency function \( f(x) = q_1 \psi(x) \).

Apparently, \( f(x)/F(x) \) is nonincreasing if \( \psi(x) \) is so. Thus, \( F(x) \) is logconcave, and by P1-1(i), the total demand over the entire \( t \)-period is also logconcave. As \( n \) approaches infinity, this compound binomial distribution converges towards the compound Poisson process, and so the result follows. To see the only-if-part, recall that for a Poisson distribution, the probability of no arrival during a period \( t \) approaches \( p_0=1-\lambda t \), and the probability of one arrival approaches \( p_1=\lambda t \) as \( t \) approaches zero. Thus, as \( t \) approaches zero, \( f_0(x)/F_0(x) \) approaches \( p_1 \psi(x) / [(1-p_1) + p_1 \Psi(x)] \), which increases with \( x \) where \( \psi(x) \) does, provided \( p_1 \) is sufficiently small. Thus, \( F_0(x) \) is not logconcave for sufficiently small \( t \), unless \( \psi(x) \) is nonincreasing.

Q.E.D.
The following equation (11) copied from the main text is referred to in the next proposition:

\[ G(x) = h[x-(L+1)D] + (p+h) \int_{-\infty}^{x} [1-F_{L+1}(y)]dy + \pi \int_{-\infty}^{x} [F_{L}(y)-F_{L+1}(y)]dy + b[1-F_{L+1}(x)]. \]

**PROPOSITION 2-1**

(i) G(x) in (11) is quasiconvex for all nonnegative values of h, p, \( \pi \) and b, if and only if both \( F_L(x)/F_{L+1}(x) \) and \( f_{L+1}(x)/F_{L+1}(x) \) are nonincreasing in x

(which conditions hold if \( F(x) \) is logconcave).

(ii) G(x) in (11) is quasiconvex for b=0 and all nonnegative values of h, p and \( \pi \), if and only if \( F_L(x)/F_{L+1}(x) \) is nonincreasing in x (which holds if \( F(x) \) is MCR).

(iii) G(x) in (11) is quasiconvex for \( \pi=0 \) and all nonnegative values of h, p and b, if and only if \( f_{L+1}(x)/F_{L+1}(x) \) is nonincreasing in x (which holds if \( F(x) \) is logconcave).

**Proof (i):** The assertion in parenthesis and the necessary conditions follow as corollaries from (ii) and (iii) below. To see sufficiency, note that G(x) is linearly nonincreasing on x<0, and possibly jumps downwards at x=0. Recall that G(x) is differentiable on x>0 and write the derivative as

\[ G'(x) = h - (p+h)[1-F_{L+1}(x)] - \pi[F_{L}(x)-F_{L+1}(x)] - b f_{L+1}(x) \]

\[ = F_{L+1}(x) [-p/F_{L+1}(x) - \pi[F_{L}(x)-F_{L+1}(x)])/F_{L+1}(x) + (p+h) - b f_{L+1}(x)/F_{L+1}(x)]. \]  \hspace{1cm} (A1)

Note that the terms in brackets are nondecreasing: the first as \( F_{L+1}(x) \) is so, the third as it is constant, and the second and fourth by assumption. Thus, the derivative can change sign at most once, from - to +, and the conclusion follows.

(ii): The proof of necessity is omitted as it is very similar to (iii) below. In addition, an alternative proof under similar conditions is given in Chen and Zheng (1993, Lemma 1).

Sufficiency follows as in (i), putting b=0. The parenthesis follows by the definition of MCR.

(iii): The assertion in parentheses follows from the definition of logconcavity and P1-(i).

Sufficiency follows as in (i) above, putting \( \pi=0 \). To prove necessity of the requirement on
\( f_{L+1}(x)/F_{L+1}(x) \), assume the assertion is incorrect. That is, \( F(x) \) exists such that \( G(x) \) is quasiconvex for all nonnegative cost parameters although there are two points for which
\[
f_{L+1}(x_1)/F_{L+1}(x_1) < f_{L+1}(x_2)/F_{L+1}(x_2) \quad \text{and} \quad 0 \leq x_1 < x_2.
\] (A2)

Set \( \pi = 0 \) and \( h = 1 \) in (A1). It follows that the derivative \( G'(x) \) approaches \( h = 1 \) as \( x \) approaches infinity. Thus, the assumption on quasiconvexity is contradicted if we can find values of \( p \) and \( b \) such that the continuous derivative \( G'(x_1) > 0 > G'(x_2) \). By (A1), the inequality \( G'(x_1) > 0 > G'(x_2) \) is equivalent to
\[
[h - bf_1]/[1 - F_1] > p + h > [h - bf_2]/[1 - F_2],
\] (A3)

where I have written \( f_1, F_1, f_2, \) and \( F_2 \) instead of \( f_{L+1}(x_1), F_{L+1}(x_1), f_{L+1}(x_2), \) and \( F_{L+1}(x_2), \) respectively. Thus, a nonnegative value of \( p \) is easy to find, provided a value of \( b \) can be found such that \( [h - bf_1]/[1 - F_1] > [h - bf_2]/[1 - F_2] > h \). These two inequalities are equivalent to
\[
F_2/f_2 > b/h > [F_2 - F_1]/\{f_2[1 - F_2] - f_1[1 - F_2]\}. \tag{A4}
\]

If this interval is not empty, a value of \( b \) can be found, and we are done (the right-hand side of (A4), which reappears below, is positive as (A2) implies that \( f_2 > f_1 \)). Now,
\[
F_2/f_2 - [F_2 - F_1]/\{f_2[1 - F_2] - f_1[1 - F_2]\} = [(f_2/F_2)/(f_1/F_1) - 1]f_1F_2/[f_2\{f_2[1 - F_1] - f_1[1 - F_2]\}] > 0
\]
as the first term in brackets is positive by (A2).

Thus, if we set \( h = 1, b \) in the middle of the interval (A4), and finally \( p + h \) in the middle of the interval (A3), then we have found a set of nonnegative cost coefficients such that \( G(x) \) is not quasiconvex, which contradicts the assumption, and proves necessity. \textbf{Q.E.D.}