Effective inseparability in a topological setting

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Abstract

Effective inseparability of pairs of sets is an important notion in logic and computer science. We study the effective inseparability of sets which appear as index sets of subsets of an effectively given topological $T_0$-space and discuss its consequences. It is shown that for two disjoint subsets $X$ and $Y$ of the space one can effectively find a witness that the index set of $X$ cannot be separated from the index set of $Y$ by a recursively enumerable set, if $X$ intersects the topological closure of an effectively enumerable subset of $Y$. As a consequence of a more general parametric inseparability result a theorem of Rice-Shapiro type is obtained. Moreover, under some additional requirements it follows that nonopen subsets have productive index sets. This implies a generalized Rice theorem: Connected spaces have only trivial completely recursive subsets. As application some decision problems in computable analysis and domain theory are studied. It follows that the complement of the halting problem can be reduced to the problem to decide of a number whether it is a computable irrational. The same is true for the problems to decide whether two numbers are equal, whether one is not greater than the other, and whether a number is equal to a given number. In the case of an effectively given continuous complete partial order the complexity of the last problem depends on whether the given element is the smallest element, in which case the complement of the halting problem is reducible to it, whether it is a base element and maximal, then the decision problem is recursively isomorphic to the halting problem, or whether it is none of these. In this case, both the halting problem and its complement are reducible to the problem. The same is true in nontrivial cases for the problems whether an element belongs to the basis, whether two elements of the partial order are equal, or whether one approximates the other. In general, for any nonempty proper subset of the partial order either the halting problem or its complement can be reduced to the membership problem of the subset.

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0. Introduction

Effective inseparability of pairs of sets is an important notion in logic and computer science. A careful discussion of various inseparability notions has recently been given by Case [6]. In this paper we consider the effective inseparability of one of two sets which appear as index sets of subsets of an effectively given topological space from the other by a recursively enumerable set and study some of its consequences.

Topological spaces appear quite naturally in computer science. Scott [36] and Smyth [37] pointed out that data types can be thought of as countably based topological $T_0$-spaces with basic open sets the finitely describable properties of the data objects. Most structures considered in programming are equipped with a canonical topology. Prominent examples are Scott domains and metric spaces. As is shown in Stoltenberg-Hansen and Tucker [46] many algebraic structures e.g. all term algebras over a finite signature can be canonically embedded in complete ultrametric spaces as well as Scott domains. Moreover, since set theory is only applicable in computer science when developed constructively, all sets are codable, i.e. (partially) numberable. On such sets there are natural topologies generated by some or all of their completely enumerable subsets. They have first been considered by Mal’cev [25]. Therefore, we call them Mal’cev topologies. If Scott domains or metric spaces are effectively given, their canonical topologies can be characterized in terms of Mal’cev topologies (cf. e.g. [33, 43, 50]).

Topological spaces that satisfy certain natural effectivity requirements have been studied by various authors (cf. [8, 14–24, 28–32, 42–45, 47, 48, 51]). We consider countable (i.e. (partially) numbered) topological $T_0$-spaces $(T, \tau)$ with a countable basis, on which a relation of strong inclusion is defined such that the property of being a topological basis holds effectively with respect to this relation instead of normal set inclusion. The further effectivity conditions that have to be satisfied are very natural. We require that elements of each basic open set can uniformly be enumerated as well as all basic open sets containing a given point. This implies that for each point we can enumerate a base of its neighbourhood filter. A second condition demands that we can also do the converse, i.e. from a (normed) enumeration of a filter base of basic open sets compute a point for which the corresponding filter is a neighbourhood filter. The numbering of the space can be thought of as a programming system for the effective generation of approximations. The indices are obtained by coding the programs.

For such spaces we then show our inseparability result: Let $X$ and $Y$ be two disjoint subsets of $T$. Then the index set of $X$ is effectively recursively enumerably (r.e.) inseparable from the index set of $Y$, i.e., there is a total recursive function which witnesses that no recursively enumerable set separates the first from the second, if $X$ intersects the topological closure of some effectively enumerable subset of $Y$. As we shall see, for singletons $X$ the condition is also sufficient. Moreover, under more complex, but less restricting assumptions it is shown that the function witnessing r.e. inseparability depends uniformly on the enumeration of the enumerable subset and the point in the intersection of a weakening of its closure with $X$. All this is done in
Section 2. Section 1 contains basic definitions and properties. In the following sections some consequences of the inseparability result are studied.

In Section [3] we derive a generalization of the classical Rice–Shapiro theorem (cf. [34]). If $T$ is effectively pointed, which means that with respect to the specialization order each basic open set of topology $\tau$ has a lower bound in a certain neighbourhood of it, then $\tau$ can be generated by all completely enumerable subsets of $T$, i.e., it is equivalent to the finest Mal'cev topology on $T$. This topology is also called Eršov topology. As is well-known, the characterization implies that effective operators between such spaces are effectively continuous, which generalizes the classical Myhill–Shepherdson theorem.

The more general result that $\tau$ is the recursively finest Mal'cev topology on $T$ compatible with it follows from the more general parametric inseparability result. Here, topology $\tau$ is recursively finer than topology $\eta$, if given a basic open set in $\eta$ and a point contained in it, we can effectively find a basic open set in $\tau$ which shows that the open set in $\eta$ is also open with respect to $\tau$. Moreover, saying that topology $\eta$ is compatible with topology $\tau$ roughly means that, if some basic open set of topology $\tau$ is not included in a given basic open set of topology $\eta$, then we must be able to find effectively a witness for this, i.e. an element of the basic open set of topology $\tau$ not contained in the basic open set of topology $\eta$. As is shown in Spreen [43], from this result one not only obtains the just mentioned characterization of effectively pointed topologies, but also a characterization of all effectively given semi-regular topologies and separable metric topologies; moreover, Spreen and Young’s general theorem on the effective continuity of effective operators [45].

As a further consequence of our inseparability result, in Section 4 we obtain a lower bound for the complexity of some decision problems. From the above-mentioned effectivity requirements it follows that every effectively enumerable union of basic open sets is completely enumerable. If, on the other hand, a subset of $T$ is not open and its complement contains a dense enumerable subset, then its index set is productive, which by Myhill’s theorem means that the complement of the halting set can be reduced to it. It follows that the index set of any singleton $\{y\}$ which is not closed is coproducive. If $T$ is recursively separable, i.e., if it contains a proper dense enumerable subset $D$, and $\{y\}$ is not open, then its index set is productive, if either $y \notin D$, or $y \in D$, $D \setminus \{y\}$ is dense in $T \setminus \{y\}$ and it is decidable for an element of $D$ whether it is equal to $y$. Moreover, the index set of $T \setminus D$ is productive. As a further consequence we obtain a generalization of Rice’s theorem: If $T$ is connected, then $\emptyset$ and $T$ are its only completely recursive subsets. As is shown in Eršov [11], Rice’s theorem holds for any precomplete numbered set. We present a necessary and sufficient condition for the numbering of $T$ being precomplete.

If $\tau$ is effectively pointed, the index set of any nonopen subset is productive. Thus, for any subset of $T$ the index set of at least the set or its complement is productive. Moreover, the index set of any singleton which is neither open nor closed is productive and coproducive. This improves Lemma 3.3 in [13]. Effectively pointed spaces are recursively separable. A singleton is neither open nor closed, if its element does either
not belong to the dense base or, if it does, is neither minimal nor maximal with respect to the specialization order. If the dense base is a proper subset of \( T \) or \( T \) is not discrete, it is not a \( T_1 \)-space. Hence, both equality and specialization order are not open in the product topology nor are their complements. This implies that their index sets are also productive and coproductive.

In the remaining sections the results obtained so far are applied to some important, standard examples of topological spaces. As is shown in [42, 43] the above effectivity requirements can be verified for effectively given separable metric spaces (cf. [7, 26, 49]). In Section 5 we consider the special case of the computable real numbers. If they are indexed in such a way that from a normed recursive sequence of rationals one can effectively pass to its limit, it follows that the problems to decide whether a number is a computable irrational or whether it equals some given number is at least as hard as to decide of a computation whether it does not terminate. The same is true for the problems whether two numbers are equal or whether one is not greater than the other.

Note that since the numbering of the computable reals is partial, the complement of the index set of a set is not the index set of the complement of this set. We therefore work with a refinement of the usual many-one reduction, which maps the complement of the halting set to the index set of a problem and the halting set to the index set of its complement.

Topological spaces consisting of the computable elements of an effectively given continuous complete partial order are an example of effectively pointed spaces. They are considered in Section 6. Since the space is not discrete in this case and the partial order coincides with the specialization order, it follows that both the halting problem as well as its complement can be reduced to the problems to decide the equality of two elements and whether one element approximates the other. The same is true for the problem to decide whether an element equals a given element \( y \), if \( y \) is neither the smallest element nor both maximal and a base element. If \( y \) is the smallest element, \( \{ y \} \) is closed. Hence, the complement of the halting problem is reducible to the decision problem for \( y \) in this case. This improves Theorem 3.4 in [35]. If \( y \) is both maximal and a base element, \( \{ y \} \) is open. Because one can choose the numbering of the space as total for this kind of spaces, it follows from the generalized Rice-Shapiro theorem that the decision problem for \( y \) is recursively isomorphic to the halting problem. If a programming language semantics is based on continuous complete partial orders, this decision problem is the problem to decide whether a program computes a specified function, i.e. whether it is correct.

The results in this paper partly generalize results in [5, 41].

1. Basic definitions and properties

In what follows, let \( \langle \cdot, \cdot \rangle : \omega^2 \to \omega \) be a recursive pairing function with corresponding projections \( \pi_1 \) and \( \pi_2 \) such that \( \pi_i(\langle a_1, a_2 \rangle) = a_i \), let \( P^{(n)} \) \((P^{(n)})\) denote the set of all \( n \)-ary partial (total) recursive functions, and let \( W_i \) be the domain of the \( i \)th partial recur-
sive function \( \varphi_i \) with respect to some Gödel numbering \( \varphi \). We let \( \varphi_i(a) \downarrow \) mean that the computation of \( \varphi_i(a) \) stops, \( \varphi_i(a) \in C \) that it stops with value in \( C \), and \( \varphi_i(a) \uparrow_n \) that it stops within \( n \) steps. In the opposite cases we write \( \varphi_i(a) \uparrow \) and \( \varphi_i(a) \downarrow_n \), respectively.

A pair \((A_1, A_2)\) of sets of natural numbers is \( m \)-reducible to another such pair \((B_1, B_2)\), if there is a function \( f \in \mathcal{R}(1) \) such that for \( i = 1, 2 \) and all \( a \in \omega \), if \( a \in A_i \) then \( f(a) \in B_i \). In the case that \( A_2 \) and \( B_2 \), respectively, are the complements \( \bar{A}_1 \) and \( \bar{B}_1 \) of \( A_1 \) and \( B_1 \), this definition reduces to the well-known many-one reducibility. We denote both kinds of reducibilities as usual by \( \leq_m \). Moreover, \( \leq_1 \) denotes one-to-one reducibility, \( \equiv_1 \) one-to-one equivalence, and \( A_1 \oplus A_2 \) is the join of \( A_1 \) and \( A_2 \).

A \( C \subseteq \omega \) is recursively enumerable (r.e.) inseparable from \( B \subseteq \omega \) if there is no r.e. set \( C \) with \( A \subseteq C \) and \( B \subseteq \bar{C} \). \( A \) is effectively r.e. inseparable from \( B \) if there is some function \( h \in \mathcal{R}(1) \) such that for all \( i \in \omega \), either \( h(i) \in A \setminus W_i \) or \( h(i) \in B \setminus W_i \). The function \( h \) witnesses the r.e. inseparability of \( A \) from \( B \).

Obviously, \( A \) is effectively r.e. inseparable from \( \bar{A} \), iff \( A \) is completely productive, hence, iff \( A \) is productive, and thus, by Myhill's theorem, iff \( \bar{K} \leq_1 A \), where \( K \) is the halting set. By using the function witnessing r.e. inseparability instead of that witnessing productivity in the proof of Myhill's theorem (cf. [40]), an analogous characterization of effective r.e. inseparability can be obtained also in the general case.

**Lemma 1.1** Let \( A \) and \( B \) be disjoint. Then \( A \) is effectively r.e. inseparable from \( B \) iff \((\bar{K}, K) \leq_1 (A, B)\).

Note that the above proof shows only that \((\bar{K}, K) \leq_m (A, B)\). One-to-one reducibility follows by an argument used in Eršov [9, Theorem 3']. Let \( v \) and \( v' \), respectively, map \( K \) and \( B \) to \( 1 \) and \( g \) and \( i? \) to \( 0 \). Then \( v \) and \( v' \) are total numberings of \( \{0, 1\} \). Now, let \( g \in \mathcal{P}(1) \). Then \( g^{-1}(K) \) is r.e. and hence \( g^{-1}(K) \leq_m K \). Let this be witnessed by \( k \in \mathcal{R}(1) \). Then for \( i \in \text{dom}(g) \) we have that \( v_{g(i)} = v_{k(i)} \), which means that \( v \) is precomplete. Let the function \( f \in \mathcal{R}(1) \) witness that \((\bar{K}, K) \leq_m (A, B)\). Then it also witnesses that there is a morphism from the numbered set \( (\{0, 1\}, v) \) into the numbered set \( (\{0, 1\}, v') \). Since \( v \) is precomplete, the morphism can be turned into a morphism with a one-to-one witness function \( h \in \mathcal{R}(1) \) (cf. [11, pp. 331/332]). To this end a function \( t \in \mathcal{R}(2) \) is constructed such that \( \lambda m.f(i((n, m))) \) is one-to-one and \( v_{t(n, m)} = v_n \), for all \( n, m \in \omega \). Then the function \( h \) is defined by \( h(0) = f(0) \) and \( h(n + 1) = f(t(n + 1, \mu m : f(t(n + 1, m)) > h(n))) \). Because \( f(\bar{K}) \subseteq A \), it also witnesses that \((\bar{K}, K) \leq_1 (A, B)\).

Now, let \( T = (T, \tau) \) be a countable topological \( T_0 \)-space with a countable basis \( \mathscr{B} \). The closure of a subset \( X \) of \( T \) is denoted by \( \text{cl}(X) \). In the special cases we have in mind, a relation between the basic open sets can be defined which is stronger than usual set inclusion, and one has to use this relation in order to derive the results we talked about in the introduction. We call a relation \( \prec \) on \( \mathscr{B} \) strong inclusion if for all \( X, Y \in \mathscr{B} \), from \( X \prec Y \) it follows that \( X \subseteq Y \). Furthermore, we say that \( \mathscr{B} \) is a strong basis if for all \( z \in T \) and \( X, Y \in \mathscr{B} \) with \( z \in X \cap Y \) there is a \( V \in \mathscr{B} \) such that \( z \in V \), \( V \prec X \) and \( V \prec Y \).
If one considers basic open sets as vague descriptions, then strong inclusion relations can be considered as "definite refinement" relations. Strong inclusion relations that satisfy much stronger requirements have also been used in Smyth's work on topological foundations of programming language semantics (cf. [38, 39]). Compared with these conditions, the above requirements seem to be rather weak, but as we go along, we shall meet further requirements, which in applications prevent us from choosing \( \prec \) to be ordinary set inclusion. For what follows we assume that \( \prec \) is a strong inclusion on \( \mathcal{B} \) and \( \mathbb{B} \) is a strong basis.

Let \( x: \omega \to T \) (onto) and \( B: \omega \to \mathcal{B} \) (onto), respectively, be (partial) indexings of \( T \) and \( \mathcal{B} \) with domains \( \text{dom}(x) \) and \( \text{dom}(B) \). The value of \( x \) at \( i \in \text{dom}(x) \) is denoted, interchangeably, by \( x_i \) or \( x(i) \). The same holds for the indexing \( B \). For a subset \( X \) of \( T \) and a relation \( R \subseteq T \times T \), respectively, \( \Omega(X) = \{ i \in \text{dom}(x) \mid x_i \in X \} \) and \( \Omega(R) = \{ (i,j) \mid i,j \in \text{dom}(x) \land (x_i,x_j) \in R \} \) are the sets of indices under \( x \). If \( X \) is a singleton \( \{ y \} \), we write \( \theta(y) \) instead of \( \Omega(\{ y \}) \).

\( X \) is completely enumerable if there is an r.e. set \( W_n \) such that \( x_i \in X \) iff \( i \in W_n \), for all \( i \in \text{dom}(x) \). Set \( M_n = X \), for any such \( n \) and \( X \), and let \( M_n \) be undefined, otherwise. Then \( M \) is an indexing of the class \( CE \) of all completely enumerable subsets of \( T \). If \( W_n \) is creative, \( X \) is called completely creative, and if \( W_n \) is recursive, it is said to be completely recursive. \( X \) is enumerable if there is an r.e. set \( A \subseteq \text{dom}(x) \) such that \( X = \{ x_i \mid i \in A \} \).

We say that \( x \) is computable if there is some r.e. set \( L \) such that for all \( i \in \text{dom}(x) \) and \( n \in \text{dom}(B) \), \( \langle i,n \rangle \in L \) iff \( x_i \subseteq B_n \). Furthermore, the space \( T \) is called effective if \( B \) is a total indexing and the property of being a strong basis holds effectively, which means that there exists a function \( p \in P^3 \) such that for \( i \in \text{dom}(x) \) and \( n, m \in \omega \) with \( x_i \subseteq B_m \cap B_n \), \( p(i,m,n) \subseteq \text{dom}(x) \) and \( B_p(i,m,n) \prec B_m \) and \( B_p(i,m,n) \prec B_n \). As it is shown in [42], \( T \) is effective if \( x \) is computable, \( B \) total, and \( \{ (m,n) \mid B_m \prec B_n \} \) r.e.

Note that very often the totality of \( B \) can easily be achieved if the space is recursively separable, which means that \( T \) has a dense enumerable subset, called its dense base.

As it is well-known, each point of a \( T_0 \)-space is uniquely determined by its neighbourhood filter and/or a base of it. If \( x \) is computable, a base of basic open sets can effectively be enumerated for each such filter. An enumeration \( (B_{f(a)})_{a \in \omega} \) with \( f: \omega \to \omega \) such that \( \text{range}(f) \subseteq \text{dom}(B) \) is said to be normed, if it is decreasing with respect to \( \prec \). If \( f \) is recursive, it is also called recursive and any Gödel number of \( f \) is said to be an index of it. In case \( (B_{f(a)}) \) enumerates a base of the neighbourhood filter of some point, we say it converges to that point. The following result is proved in [42]:

**Lemma 1.2** Let \( T \) be effective and \( x \) be computable. Then there are functions \( q \in P^{(1)} \) and \( p \in P^{(2)} \) such that for all \( i \in \text{dom}(x) \) and all \( n \in \omega \) with \( x_i \subseteq B_n \), \( q(i) \) and \( p(i,n) \) are indices of normed recursive enumerations of basic open sets which converge to \( x_i \). Moreover, \( B_{p(i,n)}(0) \prec B_n \).
The numbering $x$ is said to allow effective limit passing if there is a function $pt \in P^{(1)}$ such that, if $m$ is an index of a normed recursive enumeration of basic open sets which converges to some point $y \in T$, then $pt(m) \downarrow \in \text{dom}(x)$ and $x_{pt(m)} = y$. If $x$ allows effective limit passing and is computable, it is called acceptable. In the sequel we always assume that the space $T$ is effective and the numbering $x$ is acceptable.

By definition each open set is the union of certain basic open sets. In the context of effective topology one is only interested in such open sets where the union is taken over an effectively enumerable class of basic open sets. They are called Lacombe sets. This means $U \in \tau$ is a Lacombe set if there is some r.e. set $A \subseteq \text{dom}(B)$ such that $U = \bigcup \{B_a \mid a \in A\}$.

2. On inseparability

In this section we present sufficient conditions for the effective r.e. inseparability of the index set of one of two disjoint subsets $X$ and $Y$ of $T$ from the other.

Obviously, $X$ is inseparable from $Y$ by an open set if $X$ intersects the closure of $Y$, i.e., in this case for no open set $U$ one can have $X \subseteq U$ and $Y \subseteq \overline{U}$. Thus, $X$ is also inseparable from $Y$ by a Lacombe set. If $x$ is computable, every basic open set and hence each Lacombe set is completely enumerable. It follows that if $X$ intersects the closure of $Y$, then $\Omega(X)$ is inseparable from $\Omega(Y)$ by index sets of Lacombe sets. As is pointed out in [27, p. 45], this does not imply that $\Omega(X)$ is r.e. inseparable from $\Omega(Y)$. In order to obtain a sufficient condition for r.e. inseparability, the assumption has thus to be more constructive. The modifications we present already guarantee effective r.e. inseparability. The first result is a parametric inseparability theorem.

Let $q \in R^{(1)}$ be as in Lemma 1.2 and for $n \in \omega$ define $c_l(a)$ to be the set of all $x_j$ for which $B(q,(j)(a))$ intersects $Y$, for all $a \leq n$. Then $c(Y)$ is the intersection over all $c_l(a)$.

**Theorem 2.1** There is a function $t \in R^{(2)}$ with the following property for $j, m \in \omega$:

For all disjoint subsets $X$ and $Y$ of $T$ with $j \in \Omega(X)$ and such that, if $\varphi_{(j,m)}(i) \in W_i$, then $m$ is an index of an enumeration in $g(i, j, m) (= \mu : \varphi_i(\varphi_{(j,m)}(i)))$ steps of a subset $Y'$ of $Y$ with $x_j \in c_l(i,j,m)(Y')$, $\varphi_{(j,m)}$ witnesses the effective r.e. inseparability of $\Omega(X)$ from $\Omega(Y)$.

**Proof.** The function $t$ is constructed such that the following properties hold:

1. If $g(i, j, m) \uparrow$, then $\varphi_{(j,m)}(i) \downarrow \in \text{dom}(x)$ and $x(\varphi_{(j,m)}(i)) = x_j$.

2. If $g(i, j, m) \downarrow$ and $m$ is an index of an enumeration in $g(i, j, m)$ steps of a subset $Y'$ of $T$ which intersects $B(\varphi_{(j,m)}(g(i, j, m)))$, then $\varphi_{(j,m)}(i) \downarrow \in \text{dom}(x)$ and $x(\varphi_{(j,m)}(i)) \in B(\varphi_{(j,m)}(g(i, j, m))) \cap Y'$.

From this it follows that for all $X$ and $Y$ as in the theorem, $\varphi_{(j,m)}$ witnesses the effective r.e. inseparability of $\Omega(X)$ from $\Omega(Y)$.

To obtain a function $t$ with these properties we construct a self-referential normed
enumeration of basic open sets which follows the enumeration \( (B(\varphi_{q(j)}(n))) \) converging to \( x_j \), as long as an index of the point it converges to, if it does that at all, has not been found in \( W_i \). If such an index has been found, say after \( a \) steps, then it follows a normed enumeration converging to some point in the intersection of \( B(\varphi_{q(j)}(a)) \) and \( Y' \).

Let \( L \subseteq \omega \) and \( pt \in P(1) \) witness that \( x \) is acceptable, and be \( p \in P(2) \) as in Lemma 1.2. Moreover, define \( s(j, n, m) \) to be the first element enumerated in \( \{ b \mid (b, \varphi_{q(j)}(n)) \in L \} \cap \text{range}(\varphi_m \upharpoonright n) \). Then \( s \in P(3) \), and for \( j \in \text{dom}(x) \) and \( m, n \in \omega \) such that \( m \) is an index of an enumeration in \( n \) steps of a subset \( Y' \) of \( T \) which intersects \( B(\varphi_{q(j)}(n)) \), we have that \( s(j, n, m) \downarrow \in \text{dom}(x) \) and \( x_{s(j, n, m)} \in B(\varphi_{q(j)}(n)) \cap Y' \). Now, set \( \hat{g}(i, a) = \mu c : \varphi_i(p(a)) \downarrow \) and let \( k \in R(4) \) be such that

\[
\varphi_{k(j, m, i, a)}(n) = \begin{cases} 
\varphi_{q(j)}(n) & \text{if } n < \hat{g}(i, a), \\
\varphi_{p(s(j, \hat{g}(i, a), m), \varphi_{q(j)}(\hat{g}(i, a)))(n - \hat{g}(i, a))} & \text{otherwise}.
\end{cases}
\]

By the recursion theorem there is then a function \( f \in R(3) \) with \( \varphi_{f(j, m, i)} = \varphi_{k(j, m, i, f(j, m, i))} \). Let \( j \in \text{dom}(x) \). Then \( f(j, m, i) \) is an index of a normed recursive enumeration of basic open sets. If \( \hat{g}(i, f(j, m, i)) \uparrow \), it converges to \( x_j \). On the other hand, if \( \hat{g}(i, f(j, m, i)) \downarrow \) and \( m \) is an index of an enumeration in \( f(j, m, i) \) steps of a subset \( Y' \) of \( T \) which intersects \( B(\varphi_{q(j)}(\hat{g}(i, f(j, m, i)))) \), it converges to \( x_{s(j, \hat{g}(i, f(j, m, i)), m)} \), an element of this intersection. Thus, in both cases, \( pt(f(j, m, i)) \downarrow \). Let \( t \in R(2) \) be such that \( \varphi_{t(j, m)}(t) = pt(f(j, m, i)) \). Then \( g(j, i, m) = \hat{g}(i, f(j, m, i)) \), and (1) and (2) hold. \( \Box \)

The above result has a complex assumption. Next we derive a consequence which uses a simpler but very natural condition.

**Theorem 2.2** Let \( X \) and \( Y \) be disjoint subsets of \( T \). If \( Y \) has an enumerable subset \( Y' \) the closure of which intersects \( X \), then \( \Omega(X) \) is effectively r.e. inseparable from \( \Omega(Y) \).

**Proof.** Let \( L \subseteq \omega \) witness that \( x \) is computable and \( s \in R(1) \) such that \( Y' = \{ x_a \mid a \in \text{range}(s) \} \}. Moreover, let \( z \in X \cap \text{cl}(Y') \) and \( j \) be an index of \( z \). For \( n \in \omega \) define \( f(n) \) to be the first element enumerated in \( \{ b \mid (b, \varphi_{q(j)}(n)) \in L \} \cap \text{range}(s) \). Then \( f \in P(1) \) with \( \text{range}(f) \subseteq \text{dom}(x) \). Since \( z \in \text{cl}(Y') \), each basic open set containing \( z \) intersects \( Y' \). Thus, \( f \) is even total. For \( n \in \omega \) let \( Y'_n \) be the subset of \( Y' \) enumerated by \( f \) in \( n \) steps. Then \( z \in \text{cl}_{\omega}(Y'_n) \). By the preceding theorem it now follows that \( \varphi_{x(j, a)} \) where \( a \) is a Gödel number of \( f \) witnesses the r.e. inseparability of \( \Omega(X) \) from \( \Omega(Y) \). \( \Box \)

As is well-known, on \( T_0 \)-spaces there is a canonical partial order: the specialization order, which we denote by \( \leq \). For \( u, z \in T \), it is defined by \( u \leq z \) iff \( u \in \text{cl} \{ z \} \). If \( X \) and \( Y \), respectively, contain elements \( u \) and \( z \) with \( u \leq z \), the assumptions of the theorem can easily be verified. Set \( Y' = \{ z \} \). Then \( Y' \) is an enumerable subset of \( Y \) and by the definition of the specialization order \( u \in \text{cl}(Y') \). This shows:
Corollary 2.3 Let \( X \) and \( Y \) be disjoint subsets of \( T \). If there are \( u \in X \) and \( z \in Y \) with \( u \leq z \), then \( \Omega(X) \) is effectively r.e. inseparable from \( \Omega(Y) \).

An obvious question is whether one of the conditions in the above theorems characterizes effective r.e. inseparability. This is not known in general, but is true if \( X \) is a singleton.

Theorem 2.4 Let \( Y \) be a subset of \( T \) and \( z \in T \setminus Y \). Then \( Y \) has an enumerable subset \( Y' \) the closure of which contains \( z \), iff \( \theta(z) \) is effectively r.e. inseparable from \( \Omega(Y) \).

Proof. Because of Theorem 2.2 it remains to show the “only if” part. Let to this end \( L \subseteq \omega \) witness that \( x \) is computable, be \( j \) an index of \( z \), and \( k = \varphi_{g(j)} \). Then \( k \in R^{(1)} \).

Let \( g \in R^{(1)} \) with \( W_{g(n)} = \{ i \mid (i, k(n)) \in L \} \). By the assumption there exists a function \( h \in R^{(1)} \) such that for \( n \in \omega \) either \( h(g(n)) \in \theta(z) \setminus W_{g(n)} \) or \( h(g(n)) \in \Omega(Y) \cap W_{g(n)} \). In the first case we have \( x_{h(g(n))} = z \), but \( x_{h(g(n))} \notin B_{k(n)} \), which contradicts the definition of \( k \) and Lemma 1.2. Hence the second case holds. Set \( Y' = \{ x_a \mid a \in \text{range}(h \circ g) \} \). Then \( Y' \) is an enumerable subset of \( Y \). Furthermore, for each \( n \in \omega \) we have that \( x_{h(g(n))} \in B_{k(n)} \).

Since the set of all \( B_{k(n)} \) is a base of the neighbourhood filter of \( z \), this means that \( z \in \text{cl}(Y') \). \( \square \)

In the following sections we study some consequences of the above inseparability results.

3. Generalized Rice–Shapiro theorems

As is well-known, the classical Rice–Shapiro theorem has been lifted to effectively given Scott domains, saying that on the computable domain elements the Scott topology can be generated by the completely enumerable subsets of the domain. In [43] a theorem of this type is proved for effective topological spaces. From this result one obtains the just mentioned characterization of the Scott topology as a special case. Moreover, it shows that also other spaces can be characterized with the help of completely enumerable subsets. If the topology is semi-regular, it can be generated by the regular elements in the lattice of all completely enumerable subsets of the space. Any effectively given separable metric space is an example of such a space. A further consequence of this result is the theorem on the effective continuity of effective operators in [45]. In this section we show that this general result can be derived from the parametric inseparability theorem. Moreover, we shall see that for domain-like spaces the generalized Rice–Shapiro theorem follows already from Theorem 2.2.

Any topology \( \eta \) on \( T \) with a basis of completely enumerable subsets of \( T \) is said to be a Mal’cev topology and the basis a Mal’cev basis. The topology \( \delta \) generated by \( CE \) is called Eršov topology. With the help of the numbering \( M \) of \( CE \) every Mal’cev basis can be indexed in such a way that \( x \) is computable with respect to
this numbering. Moreover, each numbering of a Mal'cev basis with this property is reducible to $M$ (cf. [42]). Thus, each Mal'cev topology is recursively coarser than the Eršov topology.

A topology $\eta$ on $T$ with countable basis $C: \omega \to C$ (onto) of $C$, is said to be recursively coarser than $\tau$ and $\tau$ recursively finer than $\eta$, if there is some function $g \in P^{(2)}$ such that $g(i, m) \downarrow \in \text{dom}(B)$ and $x_i \in B_{g(i, m)} \subseteq C_m$ for all $i \in \text{dom}(x)$ and $m \in \text{dom}(C)$ with $x_i \in C_m$. If $\eta$ is both recursively finer than $\tau$ and recursively coarser than $\tau$, then $\eta$ and $\tau$ are called recursively equivalent.

In order to characterize the given topology $\tau$ by a class of Mal'cev topologies, we have to study which Mal'cev topologies $\eta$ are recursively coarser than $\tau$. The verification of this proceeds indirectly, i.e., for some $x_j, B_n$ and $C_m$ with $x_j \in C_m$ we assume that $x_j \in B_n$ but $B_n \not\subseteq C_m$ and derive a contradiction. To this end we need to compute a witness for $B_n$ not being included in $C_m$. The applications in [43, 44] show that one has to work with a witness that a certain neighbourhood of $B_n$ is not included in a completely enumerable subset of $C_m$. Note that by work of Beeson [1–4] a direct proof seems impossible.

For $X \subseteq T$, let

$$hl(X) = \cap \{O \subseteq \mathcal{B} | \exists O' X \subseteq O' \prec O\}.$$  

Then topology $\eta$ is compatible with $\tau$, if there are functions $s \in P^{(2)}$ and $r \in P^{(3)}$ such that for all $i \in \text{dom}(x)$, $n \in \text{dom}(B)$ and $m \in \text{dom}(C)$ the following hold:

1. If $x_i \in C_m$, then $s(i, m) \downarrow \in \text{dom}(M)$ and $x_i \in M_{s(i, m)} \subseteq C_m$.
2. Moreover if $B_n \not\subseteq C_m$, then also $r(i, n, m) \downarrow \in \text{dom}(x)$ and $x_{r(i, n, m)} \in hl(B_n) \setminus M_{s(i, m)}$.

**Theorem 3.1** Let the topology $\tau$ be compatible with itself. Then $\tau$ is the recursively finest Mal'cev topology on $T$ that is compatible with $\tau$.

**Proof.** Since $x$ is computable, each basic open set in $\tau$ is completely enumerable, i.e., $\tau$ is a Mal'cev topology. Let $\eta$ be a Mal'cev topology with Mal'cev basis $C$ which is compatible with $\tau$. We have to show that $\eta$ is recursively coarser than $\tau$. Let to this end $s \in P^{(2)}$, $r \in P^{(3)}$, $q \in R^{(1)}$, $g \in P^{(3)}$ and $t \in R^{(2)}$, respectively, be as in the definition of compatibility, Lemma 1.2 and Theorem 2.1. Moreover, let $k \in R^{(2)}$ with $\varphi_{k(j, m)}(a) = r(j, \varphi_{q(j)}(a + 1), m)$ and set $g(j, m) = g(s(j, m), j, k(j, m))$. We want to show for $j \in \text{dom}(x)$ and $m \in \text{dom}(C)$ with $x_j \in C_m$ that $\hat{g}(j, m) \downarrow$ and $B(\varphi_{q(j)}(\hat{g}(j, m) + 1)) \subseteq C_m$.

Assume to the contrary that if $\hat{g}(j, m) \downarrow$ then $B(\varphi_{q(j)}(\hat{g}(j, m) + 1)) \not\subseteq C_m$. Since the sequence $(B(\varphi_{q(j)}(n)))$ is decreasing with respect to strong inclusion, it follows from the compatibility of $\eta$ with $\tau$ that for all $a \leqslant \hat{g}(j, m)$

$$x_{r(j, \varphi_{q(j)}(a + 1), m)} \in hl(B_{\varphi_{q(j)}(a + 1)}) \setminus M_{s(j, m)} \subseteq B_{\varphi_{q(j)}(a)} \setminus M_{s(j, m)}.$$  

Thus, $k(j, m)$ is an index of an enumeration in $\hat{g}(j, m)$ steps of a subset $Y'$ of $\bar{M}_{s(j, m)}$ with $x_j \in \text{cl}_{\hat{g}(j, m)}(Y')$. With Theorem 2.1 we obtain that $\Omega(M_{s(j, m)})$ is effectively r.e.
inseparable from $\Omega(M_{\alpha}(j,m))$, which contradicts the complete enumerability of $M_{\alpha}(j,m)$. It follows that $\lambda j,m.\varphi_{q,j}(j,m+1)$ witnesses that $\eta$ is recursively coarser than $\tau$.

In the case of domain-like spaces a witness which shows that a basic open set is not included in a completely enumerable set can easily be given. As we shall show now, in this case the generalized Rice–Shapiro theorem can be obtained as a consequence of Theorem 2.2.

An essential property of Scott domains just as of Eršov’s f-spaces [10, 12] is that their canonical topology has a basis with each basic open set being an upper set generated by a single element. We say that a subset $X$ of $T$ is pointed if there is some $y \in \text{hl}(X)$ such that $y \leq z$, for all $z \in X$. $\mathcal{T}$ is said to be effectively pointed if there is some function $pd \in P(\omega)$ such that for all $n \in \text{dom}(B)$ with $B_n \neq \emptyset$, $pd(n) \downarrow \in \text{dom}(x)$, $x_{pd(n)} \in \text{hl}(B_n)$ and $x_{pd(n)} \leq z$, for all $z \in B_n$.

In [43] it is shown that completely enumerable subsets of $T$ are upwards closed under the specialization order. Thus, if $X$ is completely enumerable and $B_n \not\subset X$, this is witnessed by $x_{pd(n)}$. Now, assume that the Eršov topology $\mathcal{S}$ is not recursively coarser than $\tau$. Then there are $j \in \text{dom}(x)$ and $m \in \text{dom}(M)$ such that for all $n \in \omega$ $B(\varphi_{q,j}(n)) \not\subset M_m$. It follows for $Y' = \{x_0 | a \in \text{range}(pd \circ \varphi_{q,j}(j))\}$ that $Y'$ is disjoint from $M_m$. Because the sequence $(B(\varphi_{q,j}(n)))$ is decreasing with respect to strong inclusion, we have for all $a, n \in \omega$ with $a > n$ that $x(pd(\varphi_{q,j}(a))) \in B(\varphi_{q,j}(n))$. This shows that $x_j \in M_m \cap \text{cl}(Y')$. By Theorem 2.2 it follows that $\Omega(M_m)$ is effectively r.e. inseparable from $\Omega(M_m)$, which contradicts the complete enumerability of $M_m$. Hence, there is some $n \in \omega$ with $B(\varphi_{q,j}(n)) \subseteq M_m$. It follows that $x(pd(\varphi_{q,j}(n+1))) \in M_m$. Since $x$ is computable with respect to $M$, a number $n$ with this property can effectively be found, uniformly in $j$ and $m$. This shows that $\mathcal{S}$ is recursively coarser than $\tau$. As we have already seen if $x$ is computable, $\tau$ is a Mal’cev topology and hence recursively coarser than $\mathcal{S}$. This shows:

**Theorem 3.2** Let $\mathcal{S}$ be effectively pointed. Then $\tau$ is recursively equivalent to the Eršov topology on $T$.

4. Some decision problems

We have already seen that the index set of a Lacombe set is r.e. if $x$ is computable. As a consequence of Theorem 2.2 we now obtain a classification of the index sets of nonopen sets.

**Theorem 4.1** Let $X$ be a nonopen subset of $T$ such that its complement contains an enumerable dense subset. Then $(\mathcal{K}, \mathcal{K}) \leq_1 (\Omega(X), \Omega(\bar{X}))$, in particular $\Omega(X)$ is productive.

**Proof.** Let $Z$ be a dense enumerable subset of $\bar{X}$ and assume that $X$ is not open. Then there is some point $y \in X$ such that each open set containing $y$ intersects the
complement of $X$. Since $Z$ is dense in $\tilde{X}$, each such intersection contains some element of $Z$. Thus, $X$ intersects the closure of $Z$. With Theorem 2.2 and Lemma 1.1 we therefore obtain that $(\tilde{K}, K) \leq_1 (\Omega(X), \Omega(\tilde{X}))$. 

Obviously, the above requirement on $\tilde{X}$ is satisfied if this set is enumerable.

**Corollary 4.2.** For any nonopen subset $X$ of $T$ with enumerable complement, $(\tilde{K}, K) \leq_1 (\Omega(X), \Omega(\tilde{X})).$ In particular, $\Omega(X)$ is productive. If $\tilde{X}$ is completely enumerable, then $(\Omega(X), \Omega(\tilde{X})) \equiv_1 (\tilde{K}, K)$.

**Proof.** By the above theorem, $\Omega(X)$ is productive and $(\tilde{K}, K) \leq_1 (\Omega(X), \Omega(\tilde{X})).$ If $\tilde{X}$ is completely enumerable, there is some r.e. set $A$ with $\Omega(\tilde{X}) = A \cap \text{dom}(x)$. Since $A$ is r.e., we have that $A \leq_1 K$, which implies that $(\Omega(X), \Omega(\tilde{X})) \leq_1 (\tilde{K}, K)$. Thus, $(\Omega(X), \Omega(\tilde{X})) \equiv_1 (\tilde{K}, K)$. 

In the case that $\mathcal{T}$ is also connected, it follows for each Lacombe set $X$ that $(\Omega(X), \Omega(\tilde{X})) \equiv_1 (K, \tilde{K})$, which in particular means that $X$ is completely creative. Moreover, if $\mathcal{T}$ is recursively separable and $D$ is its proper enumerable dense base, then $(\tilde{K}, K) \leq_1 (\Omega(T \setminus D), \Omega(D))$. As a further consequence we obtain a lower bound for the complexity of the problem to decide for a given element $y \in T$ whether $x_j = y$. If one thinks of the index $j$ as coding a program for the effective generation of approximations, then this is the problem to decide of a program whether it computes the approximation of a specified datum, i.e., whether it is correct.

**Corollary 4.3** (1) Let $y \in T$. Then $(K, \tilde{K}) \leq_1 (\Theta(y), \Omega(T \setminus \{y\}))$, if $\{y\}$ is not closed.

(2) If $\mathcal{T}$ is recursively separable and for some $g \in \mathcal{R}(1)$, $D = \{x_a | a \in \text{range}(g)\}$ is its dense base, then $(\tilde{K}, K) \leq_1 (\Theta(y), \Omega(T \setminus \{y\}))$, if $\{y\}$ is not open and either $y \in T \setminus D$ or $y \in D$, $D \setminus \{y\}$ is dense in $T \setminus \{y\}$ and $a | x_{g(a)} = y$ is recursive.

**Proof.** If $\{y\}$ is not closed, $T \setminus \{y\}$ is not open. Moreover, $\{y\}$ is enumerable. Thus, Corollary 4.2 is applicable. In the case that $D$ is dense in $T$ and enumerable, $y \in T \setminus D$ and $\{y\}$ is not open, the conditions in Theorem 4.1 are satisfied, for then $D$ is dense in $T \setminus \{y\}$ as well. If $y \in D$ and $a | x_{g(a)} = y$ is recursive, $D \setminus \{y\}$ is enumerable. Thus, Theorem 4.1 applies also in the remaining case. 

If $\mathcal{T}$ is a $T_1$-space, the requirement that $D \setminus \{y\}$ is dense in $T \setminus \{y\}$ is redundant.

**Lemma 4.4** Let $\mathcal{T}$ be a separable $T_1$-space with dense base $D$. Moreover, let $y \in D$ such that $\{y\}$ is not open. Then $D \setminus \{y\}$ is dense in $T \setminus \{y\}$.

**Proof.** Assume that $D \setminus \{y\}$ is not dense in $T \setminus \{y\}$. Then there is an open set $U$ which contains $y$, but no element from $D$. Since $\{y\}$ is not open, $U$ cannot contain only $y$. Let $z \in U \setminus \{y\}$. Since $\mathcal{T}$ satisfies the $T_1$-axiom, there is some open subset $V$ of $U$ with $z \in V$ but $y \notin V$. Because of the density of $D$, $V$ must contain an element of $D$. This
contradicts the fact that \( U \) contains no element from \( D \) other than \( y \). Thus, \( D \setminus \{ y \} \) is dense in \( T \setminus \{ y \} \). □

Let \( i_0 \in \theta(y) \) and define \( f(i) = \langle i_0, i \rangle \). Then \( f \) witnesses that \( (\theta(y), \Omega(T \setminus \{ y \})) \leq_1 (\Omega(=), \Omega(\neq)) \). If \( \mathcal{F} \) is recursively separable with dense base \( D \) and \( \{ y \} \) is open, then \( y \in D \). Hence, if \( D \neq T \), there is some \( y \in T \setminus D \) such that \( \{ y \} \) is not open. With Corollary 4.3(2) we thus obtain:

**Corollary 4.5** Let \( \mathcal{F} \) be recursively separable such that the dense base is properly included in \( T \). Then \( (\mathcal{F}, \mathcal{K}) \leq_1 (\Omega(=), \Omega(\neq)) \).

An additional outcome of Theorem 4.1 is a generalization of Rice’s theorem: On connected effective spaces there are no nontrivial decidable properties.

**Theorem 4.6** Let \( \mathcal{F} \) be connected and \( X \) be a subset of \( T \). Then \( X \) is completely recursive, iff \( X \) is empty or the whole space.

**Proof.** The “if” part is obvious. For the “only if” part assume that \( X \) is neither empty nor the whole space, and completely recursive. Then there is some recursive set \( A \) such that for \( i \in \text{dom}(x) \), \( x_i \in X \) iff \( i \in A \). Thus, both \( X \) and its complement are completely enumerable. If \( X \) were not open, Corollary 4.2 would imply that \( \mathcal{F} \leq_1 A \). This means \( A \) would be productive, a contradiction. Hence \( X \) is open. In the same way it follows that \( \tilde{X} \) is open, which is impossible by the connectedness of \( T \). □

As is shown in [11], Rice’s theorem is true for arbitrary precomplete numbered sets. (Ersov considers only total numberings, but the proof remains valid also for partial numberings.) Here, for a numbered set \( (Y, v) \) the numbering \( v : \omega \to Y(\text{onto}) \) is called precomplete if for any function \( g \in P(\omega) \) there is a function \( f \in R(\omega) \) such that \( v_f(i) = v_{g(i)} \), for \( i \in \text{dom}(g) \) with \( g(i) \in \text{dom}(v) \). It is not known, whether acceptable numberings of effective topological spaces are also precomplete. The next lemma presents a necessary and sufficient condition.

**Lemma 4.7** The numbering \( x \) is precomplete, iff there is a total function \( pt \in R(\omega) \) which witnesses that \( x \) allows effective limit passing.

**Proof.** For the “if” part let \( g \in P(\omega) \). Moreover, let \( q \in P(\omega) \) be as in Lemma 1.2 and \( k \in R(\omega) \) with \( \varphi_{k(i)} = \varphi_{q(g(i))} \). Then, for \( i \in \text{dom}(g) \) such that \( g(i) \in \text{dom}(x) \), \( k(i) \) is an index of a normed recursive enumeration of basic open sets which converges to \( x_{g(i)} \). Thus, \( pt(k(i)) \) is \( \text{dom}(x) \) and \( x_{pt(k(i))} = x_{g(i)} \). Set \( f = pt \circ k \). Since \( pt \in R(\omega) \), also \( f \in R(\omega) \).

For the “only if” part assume that \( x \) is precomplete, and let \( pt \in P(\omega) \) witness that \( x \) allows effective limit passing. Then there is a function \( \overline{pt} \in R(\omega) \) with \( x_{\overline{pt}(m)} = x_{pt(m)} \), for \( m \in \text{dom}(pt) \) with \( pt(m) \in \text{dom}(x) \). Then also \( \overline{pt} \) witnesses that \( x \) allows effective limit passing. □
If $\mathcal{T}$ is effectively pointed, the above results can be improved.

**Theorem 4.8** Let $\mathcal{T}$ be effectively pointed. Then for any nonopen subset $X$ of $T$, $(\bar{K}, K) \leq_1 (\Omega(X), \Omega(\bar{X}))$.

**Proof.** If $X$ is not upwards closed with respect to the specialization order, the above reducibility statement follows from Corollary 2.3 and Lemma 1.1. Let us therefore assume that $X$ is upwards closed with respect to the specialization order, and let $q \in \mathcal{P}^{(1)}$ and $pd \in \mathcal{R}^{(1)}$, respectively, be as in Lemma 1.2 and the definition of being effectively pointed. If $X$ is not open, there is some $x_j \in X$ such that for no $n \in \omega$, $B(q_{(j)}(n))$ is contained in $X$. Hence for all $n$, $x(pd(q_{(j)}(n))) \not\in X$. Since the set of all $B(q_{(j)}(n))$ is a base of the neighbourhood filter of $x_j$, it follows that $x_j$ is an accumulation point of the set of all $x(pd(q_{(j)}(n)))$. The above reducibility statement is now a consequence of Theorem 2.2 and Lemma 1.1. $\square$

Obviously, an effectively pointed space is recursively separable. The set $D$ of all points $x_{pd(n)}$ is a dense base. If $D$ is not already the whole space, it is not open. With the remark in the beginning of this section we thus have that $(\bar{K} \oplus K, K \oplus \bar{K}) \leq_1 (\Omega(T \setminus D), \Omega(D))$. If $\mathcal{T}$ is also connected, it follows that for any nonempty proper subset $X$ of $T$ at least one of the index sets $\Omega(X)$ and $\Omega(\bar{X})$ is productive. For the decision problem whether $x_j$ is equal to a given element $y \in T$ we obtain the following improvement.

**Corollary 4.9** Let $\mathcal{T}$ be effectively pointed and $y \in T$. Then the following hold:

1. If $y \in T \setminus D$, then $(\bar{K} \oplus K, K \oplus \bar{K}) \leq_1 (\theta(y), \Omega(T \setminus \{y\}))$.
2. If $y \in D$ and not maximal with respect to the specialization order, then $(\bar{K}, K) \leq_1 (\theta(y), \Omega(T \setminus \{y\}))$. If in addition $y$ is not minimal, then also $(K, \bar{K}) \leq_1 (\theta(y), \Omega(T \setminus \{y\}))$.
3. If $y \in D$ with $y = x_{pd(n)}$, $B_n$ is not empty, and $y$ is maximal and not minimal with respect to the specialization order, then $(\theta(y), \Omega(T \setminus \{y\})) \equiv_1 (K, \bar{K})$.

**Proof.** Let $y \in T$ be such that $y$ is not maximal with respect to the specialization order, if $y \in D$, and assume that $\{y\}$ is open. Then there is some basic open set $B_n$ which equals $\{y\}$. It follows that $y$ is maximal. By Lemma 1.2 there is a further basic open set $B_m$ which contains $y$ and is strongly included in $B_n$. Hence, $y = x_{pd(n)}$. This means that $y \in D$, a contradiction. With Theorem 4.8 we obtain that $(\bar{K}, K) \leq_1 (\theta(y), \Omega(T \setminus \{y\}))$.

Since $y$ is contained in some basic open set $B_n$, we have that $y = x_{pd(n)}$ and hence that $y \in D$, if $y$ is minimal with respect to the specialization order. If $y$ is not minimal, it follows from Corollary 2.3 and Lemma 1.1 that $(\bar{K}, K) \leq_1 (\Omega(T \setminus \{y\}), \theta(y))$.

If $y = x_{pd(n)}$, $B_n$ is not empty and $y$ is maximal with respect to the specialization order, then $\{y\}$ is a basic open set. Moreover, $\{y\}$ is not open. Since $y$ is not minimal with respect to the specialization order, there is some $z \in T$ with $z \not\leq_T y$. Then $z \not\in \overline{\{y\}}$. If $\overline{\{y\}}$ were open, we would have that $y \in \overline{\{y\}}$, a contradiction. With Corollary 4.2 it thus follows that $(\theta(y), \Omega(T \setminus \{y\})) \equiv_1 (K, \bar{K})$. $\square$
Assume that \( \mathcal{T} \) is a \( T_1 \)-space, then the specialization order coincides with equality. Thus, for each basic open set \( B_n \) we have that \( B_n = \{ x_0d(n) \} \). It follows that \( T \) contains only the points of the dense base \( D \) and the topology is discrete. Thus, if either \( D \) is a proper subset of \( T \) or the topology is not discrete, \( \tau \) does not satisfy the \( T_1 \)-axiom. This implies that there are two different points \( y \) and \( z \) in \( T \) with \( y \leq z \). It is now easy to see that \( \{(u,v) \in T \times T | u = v\} \) and \( \{(u,v) \in T \times T | u \leq v\} \) as well as their complements are not open in the product topology. With Theorem 4.8 we therefore obtain:

**Corollary 4.10** Let \( \mathcal{T} \) be effectively pointed such that either \( D \) is a proper subset of \( T \) or the topology is not discrete. Then \( (\mathcal{K} \oplus K, K \oplus \overline{K}) \leq_1 (\Omega(=), \Omega(\neq)) \) and \( (\overline{K} \oplus K, K \oplus \overline{K}) \leq_1 (\Omega(\leq), \Omega(\not\leq)) \).

In the next two sections we apply the results of this and the preceding section to two important special cases: the computable real numbers and constructive domains.

### 5. The computable real numbers

Let \( \mathbb{R} \) denote the real number set, and \( v \) be a canonical indexing of the set \( \mathbb{Q} \) of rational numbers. Then \( \{(a, b) | v_a \leq v_b\} \) is recursive. A real number \( z \) is said to be computable if there is a function \( f \in R^{(1)} \) such that for all \( m, n \in \omega \) with \( m \leq n \), the inequality \(|v_{f(m)} - v_{f(n)}| < 2^{-m}\) holds and \( z = \lim v_{f(m)} \). Any Gödel number of the function \( f \) is called an index of \( z \). This defines a partial indexing \( \gamma \) of the set \( \mathbb{R}_c \) of all computable real numbers.

For \( i, m \in \omega \) set \( B_{(i,m)} = \{ z \in \mathbb{R}_c | |v_i - z| < 2^{-m}\} \). Then the collection of all sets \( B_{(i,m)} \) is a basis of the canonical topology on \( \mathbb{R}_c \). Define

\[
B_{(i,m)} < B_{(j,n)} \iff |v_i - v_j| + 2^{-m} < 2^{-n}.
\]

Using the triangular inequality it is readily verified that \( \prec \) is a strong inclusion, and \( \{B_a | a \in \omega\} \) is a strong basis. In [43] it is shown that \( \mathbb{R}_c \) is effective if \( x \) allows effective limit passing.

Let \( (y_a)_{a \in \omega} \) be a sequence of rational numbers. Then \( (y_a) \) is said to be normed if \(|y_m - y_n| < 2^{-m}\), for all \( m, n \in \omega \) with \( m \leq n \). Moreover, \( (y_a) \) is recursive, if there is some function \( f \in R^{(1)} \) such that \( y_a = v_{f(a)} \), for all \( a \in \omega \). Any Gödel number of \( f \) is called an index of \( (y_a) \).

**Lemma 5.1** Let \( x \) be a numbering of \( \mathbb{R}_c \). Then \( x \) allows effective limit passing, iff there is a function \( li \in P^{(1)} \) such that if \( m \) is an index of a normed recursive sequence \( (y_a) \) of rational numbers, then \( li(m) \downarrow \in \text{dom}(x) \) and \( x_{li(m)} = \lim y_a \).

**Proof.** The "if" part has been shown in [43]. For the "only if" part let \( (v(\varphi_m(a)))_{a \in \omega} \) be a normed recursive sequence of rational numbers, and let \( k \in R^{(1)} \) with \( \varphi_{k(m)}(a) = \langle \varphi_m(a + 1), a \rangle \). Then \( (B(\varphi_{k(m)}(a))) \) is a normed recursive enumeration of basic open sets
which converges to $\lim v(\varphi_m(a))$. Let $pt \in P^{(1)}$ be the function witnessing that $x$ allows effective limit passing, and set $li = pt \circ k$. Then $x_{li(m)} = \lim v(\varphi_m(a))$. \qed

It follows that the numbering $\gamma$ allows effective limit passing. Since $\gamma_i \in B_{(i,m)}$ iff there is some $a \in \omega$ with $|v(i) - v(\varphi_j(a))| + 2^{-a} < 2^{-m}$, $\gamma$ is also computable. For the remainder of this section, $x$ is assumed to be an acceptable numbering of $\mathbb{R}_c$.

Let $f \in R^{(1)}$ be strictly increasing, and $k \in R^{(1)}$ with $\varphi_k(i)(m) = (i, f(m))$. Then, for any $i \in \omega$, $(B(\varphi_k(i)(m)))$ is a normed recursive enumeration of basic open sets that converges to $v_i$. Hence $v_i = x_{pt(k(i))}$. It follows that $\mathbb{R}_c$ is recursively separable.

We now apply the results of the last section to obtain lower bounds for the complexity of some decision problems. Since $\mathbb{R}_c$ is connected, we first obtain from Theorem 4.6 that there are no nontrivial decidable properties of computable real numbers, i.e. Rice's theorem for $\mathbb{R}_c$. Moreover, we have:

**Theorem 5.2**

1. $(K', K) \leq_1 (\Omega(\mathbb{R}_c \setminus \mathbb{Q}), \Omega(\mathbb{Q}))$.
2. $(K', K) \leq_1 (\Omega(y), \Omega(T \setminus \{y\}))$, for all $y \in \mathbb{R}_c$.
3. $(K', K) \leq_1 (\Omega(=), \Omega(\neq))$, $(K', K) \leq_1 (\Omega(\leq), \Omega(\neq))$.

**Proof.** (1) follows from Corollary 4.2, and (2) from Corollary 4.3(2) and Lemma 4.4, since no singleton is open in the topology on $\mathbb{R}_c$. The first part of (3) follows from Corollary 4.5, the second is a consequence of Theorem 4.1, because the set $\{(y, z) \in \mathbb{R}_c \times \mathbb{R}_c | y \leq z\}$ is closed in the product topology and $\{(y, z) \in \mathbb{Q} \times \mathbb{Q} | y > z\}$ is dense in its complement and enumerable. \qed

6. Constructive domains

Let $\mathcal{D} = (Q, \sqsubseteq)$ be a partial order with smallest element $\bot$. A nonempty subset $S$ of $Q$ is directed if for all $y_1, y_2 \in S$ there is some $u \in S$ with $y_1, y_2 \sqsubseteq u$. $\mathcal{D}$ is a directed-complete partial order (cpo) if every directed subset $S$ of $Q$ has a least upper bound $\sup S$ in $Q$. Let $\ll$ denote the way-below relation on $Q$, i.e., let $y_1 \ll y_2$ iff for directed subsets $S$ of $Q$ the relation $y_2 \sqsubseteq \sup S$ always implies the existence of a $u \in S$ with $y_1 \sqsubseteq u$.

A subset $Z$ of $Q$ is a basis of $\mathcal{D}$ if for any $y \in Q$ the set $Z_y = \{z \in Z | z \ll y\}$ is directed and $y = \sup Z_y$. A cpo that has a basis is called continuous. Every basis contains all elements $y \in Q$ with $y \ll y$. They are called compact. As is well-known, on each cpo there is a canonical topology: the Scott topology. A subset $X$ of $Q$ is open if $X$ is upwards closed with respect to $\sqsubseteq$ and with each $u \in X$ there is some $y \in X$ with $y \ll u$. In case that $\mathcal{D}$ is continuous, this topology is generated by the sets $O_z = \{y \in Q | z \ll y\}$ with $z \in Z$. Note that the specialization order on $\mathcal{D}$ coincides with the partial order $\sqsubseteq$.

Let $\mathcal{D}$ be continuous, then it is called effective if there exists an indexing $e : \omega \rightarrow Z$ (onto) of $Z$ such that $\{(i, j) | e_i \ll e_j\}$ is r.e. An element $y \in Q$ is said to be computable,
if \( \{ i | e_i \ll y \} \) is r.e. Let \( Q_c \) denote the set of all computable elements of \( Q \), then \((Q_c, \sqsubseteq, Z, e)\) is called constructive domain. Let \( \sigma \) be the induced Scott topology on \( Q_c \). Then \( \mathcal{Q}_c = (Q_c, \sigma) \) is a countable connected \( T_0 \)-space with basic open sets \( B_n = O_{e(n)} \cap Q_c \). Moreover, \( Z \) is dense in \( \mathcal{Q}_c \).

An indexing \( x : \omega \to Q_c \) (onto) of the computable cpo elements is called admissible if \( \{ (i, j) | e_i \ll x_j \} \) is r.e. and there is a function \( d \in R^{(1)} \) with \( x_d(i) = \sup e(W_i) \), for all indices \( i \in \omega \) such that \( e(W_i) \) is directed. As it is shown in [50] such total numberings exist. In what follows, we always assume that constructive domains are admissibly indexed.

Define

\[
B_m \prec B_n \leftrightarrow e_n \ll e_m.
\]

Then \( \prec \) is a strong inclusion and \( \{ B_n | n \in \omega \} \) is a strong basis. In [43] it is shown that \( \mathcal{Q}_c \) is effective and effectively pointed, and any admissible numbering \( x \) is acceptable. From the results of the preceding sections we thus obtain that there are no nontrivial decidable properties on \( Q_c \) and that the induced Scott topology on \( Q_c \) is recursively equivalent to the Eršov topology, i.e. the Rice and the Rice–Shapiro theorem for constructive domains. Moreover, we have:

**Theorem 6.1** Let \((Q_c, \sqsubseteq, Z, e)\) be a constructive domain that contains at least two elements. Then the following hold:

1. The index set of any subset of \( Q_c \) which is not open in the induced Scott topology is productive, and for each nonempty proper subset \( S \) of \( Q_c \) at least one of the index sets \( \Omega(S) \) and \( \Omega(\overline{S}) \) is productive.

2. If \( Z \) is properly contained in \( Q_c \), then \( K \oplus \overline{K} \leq_1 \Omega(Z) \).

3. \( K \oplus \overline{K} \leq_1 \Omega(\sqsubseteq) \) and \( K \oplus \overline{K} \leq_1 \Omega(\sqsupseteq) \).

4. If \( y \in Q_c \setminus Z \), then \( K \oplus \overline{K} \leq_1 \theta(y) \).

5. If \( y \in Z \setminus \{ \bot \} \) and not maximal with respect to \( \sqsubseteq \), then \( K \oplus \overline{K} \leq_1 \theta(y) \).

6. \( \overline{K} \leq_1 \theta(\bot) \).

7. If \( y \in Z \) is compact and maximal with respect to \( \sqsubseteq \), then \( \theta(y) \) is recursively isomorphic to \( K \).

**Proof.** The first part of (1) follows from Theorem 4.8. The second as well as (2) are a consequence of a remark preceding Corollary 4.9. (3) is a special case of Corollary 4.10, and (4)–(7) is a consequence of Corollary 4.9. If \( y \) is compact, \( y \in O_y \). Hence \( O_y \cap Q_c \) is not empty. □

**References**


[33] G. Plotkin, Domains, Course Notes, Dept. of Computer Science, Univ. of Edinburgh, 1983.


