A comparison of well-known ordinal notation systems for $\varepsilon_0$

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Abstract

We consider five ordinal notation systems of $\varepsilon_0$ which are all well-known and of interest in proof-theoretic analysis of Peano arithmetic: Cantor’s system, systems based on binary trees and on countable tree-ordinals, and the systems due to Schütte and Simpson, and to Beklemishev.

The main point of this paper is to demonstrate that the systems except the system based on binary trees are equivalent as structured systems, in spite of the fact that they have their origins in different views and trials in proof theory. This is true while Weiermann’s results based on Friedman-style miniaturization indicate that the system based on binary trees is of different character than the others.

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1. Introduction

Starting with Gentzen’s consistency proof for Peano arithmetic (PA) in [13], a number of primitive recursive ordinal notation systems have been defined to give consistency proofs for PA, each of them provided with a primitive recursive well-ordering.

Most famous one is probably Cantor’s system of the ordinals in normal form less than $\varepsilon_0$ with a natural well-ordering. Using a transfinite induction along the ordering, the consistency of PA can be shown while, for any proper initial segment of the ordering, it is PA-provable that the transfinite induction on the segment can be applied to arbitrary formulas. This kind of characterization of PA is true for all of its well-known ordinal number systems.

Another popular characterization of a theory is formulated by the question whether there is a natural ordinal classification of all provably total functions of the theory. In a significant number of well-known theories in proof theory there do exist plausible solutions, such as the classifications of provably recursive functions through Hardy–Wainer or Schwichtenberg–Wainer hierarchies.

However, there still remain many questions about ordinal notation systems of a theory: How is it to be explained that proof-theoretic ordinals are sensitive to the choice of particular proof systems? What are the intrinsic properties

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that distinguish some ordering from the others? In proof theory, in fact, it is one of the so-called conceptual problems to give criteria for natural or canonical ordinal notations. See [18,14,12] for intensive discussions.

One of the latest approaches to this problem is done by Beklemishev [2]. He was concerned with the question how to recover an ordinal notation system from a given theory. He posed the question what would make a formal theory possible to rigorously specify its canonical ordinal notation system? He pointed out that an algebraic viewpoint of proof theory, e.g., a well-behaved notion of graded provability algebra, could give an answer. He showed this for PA in a canonical way.

Beside these two systems, Cantor’s and Beklemishev’s systems, three further ordinal notation systems for ε₀ are taken into account in this paper:

- **System based on rooted binary trees**: A rooted binary tree is a set of nodes such that, if it is not empty, there is one distinguished node called the root and the remaining nodes are partitioned into two rooted binary trees. The homeomorphic embeddability relation ≤ on the set B of all rooted binary trees is well-founded and has the maximal order type ε₀. The system (B, <) can also be obtained from the Feferman–Schütte notation system for Γ₀ by omitting the addition operator, where < is a canonical extension of ≤.
- **System based on countable tree-ordinals**: A certain set of countable tree-ordinals resembles the Cantor system. The subtree ordering on the set builds no well-ordering, although it is well-founded and has the height ε₀.
- **Schütte and Simpson’s system**: The system is obtained from Buchholz’s ordinal notation system in [5] by omitting the addition and the construction of ω^n as basic operators. This system contains a subset with a well-ordering of order type ε₀.

We will show that the four systems except the system based on binary trees build equivalent structured systems though they have all different historical backgrounds in proof theory. The equivalence between any two structured system will be established by a relatively simple construction of order-preserving isomorphism. See Section 3.

A structured system of countable ordinals is a system where an arbitrary, but fixed fundamental sequence has been assigned to each limit. In case of Cantor’s system, e.g., the fundamental sequence for any limit λ < ε₀ can be defined as follows: Let λ = ω^β₁ + · · · + ω^βₖ be in Cantor normal form.

\[ \lambda[n] := \begin{cases} \omega^β₁ + \cdots + \omega^β_{k-1} + \omega^{β_k[n]} & \text{if } \lambda_k \text{ is a limit,} \\ \omega^β₁ + \cdots + \omega^β_{k-1} + \omega^{β_k-1} \cdot (n+1) & \text{otherwise} \end{cases} \]

Then \( \lambda[n] < \lambda[n+1] \) and \( \lim_{n \to \infty} \lambda[n] = \lambda \).

Given \( f : \mathbb{N} \to \mathbb{N} \), we define the iterations: \( f^{(0)}(i) := i \) and \( f^{(l+1)}(i) := f(f^{(l)}(i)) \). Then the Hardy–Wainer hierarchy \( (H_α)_{α < ε₀} \) and the Schwichtenberg–Wainer hierarchy \( (F_α)_{α < ε₀} \) are defined as follows:

\[
\begin{align*}
H_0(i) &= i & F_0(i) &= i + 1 \\
H_{α+1}(i) &= H_α(i + 1) & F_{α+1}(i) &= F_α(i + 1) \\
H_α(i) &= H_α[i](i) & F_α(i) &= F_α[i](i)
\end{align*}
\]

Further let \( H_{ε₀}(i) := H_{ε₀}(i) \) and \( F_{ε₀}(i) := F_{ε₀}(i) \). Then \( F_{α}(i) = H_{ω·α}(i) \). And it is a folklore in proof theory that \( H_α \) (resp. \( F_α \)) is provably recursive in PA iff \( α < ε₀ \). See [9] or [6] for details.

**Friedman-style miniaturization and phase transition.** Phase transition in physics means the transformation of a thermodynamic system from one phase to another. The distinguishing characteristic of phase transition is an abrupt change in physical properties with a small change in a thermodynamic variable such as the temperature.

It was the paper [21] by Loebl and Matoušek on Friedman-style miniaturization of Kruskal’s theorem which indicated that similar phenomenon could happen in proof theory; abrupt change of provability of a sentence with a small change of a parameter. And it is one of the starting points of Weiermann’s pioneer works on phase transition.

A finite rooted tree is a finite partial ordering \((T, ≤)\) such that, if \( T \) is not empty,

- \( T \) has a smallest element called the root of \( T \);
- for each \( b \in T \), the set \( \{a \in T : a ≤ b\} \) is totally ordered.

Let \( a ∧ b \) denote the infimum of \( a \) and \( b \) for \( a, b ∈ T \). Given finite rooted trees \( T_1 \) and \( T_2 \), a homeomorphic embedding of \( T_1 \) into \( T_2 \) is an one-to-one mapping \( f : T_1 \to T_2 \) such that \( f(a ∧ b) = f(a) ∧ f(b) \) for all \( a, b ∈ T_1 \).
We write $T_1 \preceq T_2$ if there exists a homeomorphic embedding $f : T_1 \to T_2$. The following true $II^1_1$-sentence says that $(\mathbb{T}, \preceq)$ is a well-partial ordering, where $\mathbb{T}$ is the set of all finite rooted trees, cf. Kruskal [19].

**Theorem 1 (Kruskal’s Theorem).** For any infinite sequence of finite rooted trees $(T_k)_{k<\omega}$, there are indices $\ell < m$ satisfying $T_\ell \preceq T_m$.

Let $\|T\|$ denote the number of nodes of the finite tree $T$. Assume further that the set of finite rooted trees is coded primitive recursively into a set of natural numbers as usual. Given $f : \mathbb{N} \to \mathbb{N}$, the slowly well-partially orderedness is defined as follows:

For any $k$ there exists a constant $n$ so large that, for any finite sequence $T_0, \ldots, T_n$ of finite rooted trees with $\|T_i\| \leq k + f(i)$ for all $i \leq n$, there are indices $\ell < m \leq n$ satisfying $T_\ell \preceq T_m$.

Let $\text{SWP}(\mathbb{T}, \preceq, f)$ be the above $II^0_2$-sentence.

**Theorem 2 (Friedman [28], Smith [29]).** $\text{ATR}_0 \not\vdash \text{SWP}(\mathbb{T}, \preceq, \text{id})$.

**Theorem 3 (Loeb and Matoušek [21]).** Let $f_r(i) := r \cdot |i|$. Then

$$\text{PA} \not\vdash \text{SWP}(\mathbb{T}, \preceq, f_k) \quad \text{and} \quad \text{PA} \vdash \text{SWP}(\mathbb{T}, \preceq, f^r_1)$$.

These very interesting results made one speculate that there would be a threshold in view of the PA-provability of $\text{SWP}(\mathbb{T}, \preceq, f_r)$ depending on the numbers between 1/2 and 4. Indeed, Weiermann found such a point which is closely related to the so-called Otter’s tree constant $\alpha = 2.955765 \ldots$:

$$t(n) \sim \beta \cdot \alpha^n \cdot n^{-\frac{3}{2}}$$

for some real number $\beta$, where $t(n) = \text{card}(\{T : \|T\| = n\})$, cf. Otter [22].

**Theorem 4 (Weiermann [30]).** Let $c = \frac{1}{\log(\alpha)}$ and $r$ be a primitively recursive real number. Set $f_r(i) := r \cdot |i|$. Then

$$\text{PA} \vdash \text{SWP}(\mathbb{T}, \preceq, f_r) \iff r > c.$$ 

In fact, one can even show that the same phase transition holds with respect to $\text{ACA}_0 + II^1_2$-BI, cf. Lee [20]:

$$\text{ACA}_0 + II^1_2$-BI $\not\vdash \text{SWP}(\mathbb{T}, \preceq, f_r) \iff r > c$.

Through investigations on ordinal notation systems, e.g. with regard to phase transition, Weiermann has tried to give a possible approach to the question like what the intrinsic properties that distinguish some orderings from the others are or if the systems are independent of their possible use in proof-theoretic work. Cf. [30,32,31] for his pioneer works on phase transition in logic. In this sense it is a natural consequence that the structurally equivalent systems show the same behavior in view of slowly well-orderedness, see Section 4.

**Notational conventions.** Small Latin letters $i, m, n, \ldots$ range over natural numbers while Greek letters $\alpha, \beta, \ldots$ range over ordinals or finite trees. Given a non-negative real number $x$, $\lceil x \rceil$ is the largest natural number not bigger than $x$ and $\lfloor x \rfloor$ is the least natural number not less than $x$. Define

$$|x| := \lceil \log_2(x + 1) \rceil,$$

i.e., the length of the binary representation of $x$. We iterate the $|\cdot|$-function: $|x|_0 := x$ and $|x|_{m+1} := ||x|m \rceil$. And inv is the inverse function of the superexponential function:

$$\text{inv}(i) = \min \{ m : |i|m \leq 1 \}.$$

A function $f : \mathbb{N} \to \mathbb{N}$ is said to be unbounded if it is weakly monotone-increasing and the values are not bounded. For any unbounded function $f$ its inverse function is defined as follows:

$$f^{-1}(i) := \min \{ \ell : i < f(\ell) \}.$$ 

Note that $f^{-1}(i) \leq \ell$ if and only if $i < f(\ell)$. 


2. An difference

We start with two combinatorial results of Weiermann. They indicate that the system based on binary trees is of different character than Cantor’s system. His works are based on Friedman-style miniaturization. He found out that they behave differently in view of the slowly well-orderedness.

2.1. Cantor system

For any nonzero ordinal $\alpha < \varepsilon_0$ there exist an unique natural number $n$ and uniquely determined ordinals $\alpha_1, \ldots, \alpha_n < \varepsilon_0$ such that $\alpha = \omega^\alpha + \cdots + \omega^{\alpha_n}$ and $\alpha > \alpha_1 \geq \cdots \geq \alpha_n$. It is denoted by $\alpha =_{NF} \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$ and said to be in Cantor normal form.

Given $\alpha, \beta \in \varepsilon_0$ put

$$
\alpha_0(\beta) := \beta, \quad \alpha_{n+1}(\beta) := \alpha^{\alpha_n(\beta)}, \quad \text{and} \quad \alpha_n := \alpha_n(1).
$$

The fundamental sequence for a limit $\lambda$ is defined as above. For convenience, we set also $(\alpha + 1)[n] := \alpha$ for any $n$ and $\varepsilon_0[n] = \omega_n + 1$. The norm $N\alpha$ is the number of occurrences of $\omega$ in $\alpha$: $N\alpha := 0$ and if $\alpha =_{NF} \omega^{\alpha_1} + \cdots + \omega^{\alpha_n}$, then

$$
N\alpha := n + N\alpha_1 + \cdots + N\alpha_n.
$$

Definition 5 (Slowly Well-Orderedness, SWO). Let $(X, \preceq)$ be a linear ordering.

1. A function $g : X \rightarrow \mathbb{N}$ is called a norm function if for every $n \in \mathbb{N}$ the set $\{ \alpha \in X : g(\alpha) \leq n \}$ is finite.

2. Given a norm function $g : X \rightarrow \mathbb{N}$ and a function $f : \mathbb{N} \rightarrow \mathbb{N}$ the slowly well-orderedness of $(X, \preceq)$ is given by

$$
\text{SWO}(X, \preceq, f, g) := \left( \forall k \exists n \text{ such that for any finite sequence of ordinals } \alpha_0, \ldots, \alpha_n \in X \right.
$$

with $g(\alpha_i) \leq k + f(i)$ for all $i \leq n$, there exist indices $\ell$ and $m$ such that $\ell < m \leq n$ and $\alpha_\ell \preceq \alpha_m$.

Set $\text{SWO}(\varepsilon_0, <, f) := \text{SWO}(\varepsilon_0, <, f, N)$. Note that $\text{SWO}(\varepsilon_0, <, f)$ is a II$^0_2$ if $f$ is e.g. a primitive recursive function.

Theorem 6 (Friedman [28, 29]). $\text{SWO}(\varepsilon_0, <, id)$ is not PA-provable.

Using this, Weiermann and Arai gave an interesting characterization of the class of functions $f$ such that $\text{SWO}(\varepsilon_0, <, f)$ is not PA-provable.

Theorem 7 (Weiermann [30]). Let $m \in \mathbb{N}$.

1. $\text{SWO}(\varepsilon_0, <, f_0)$ is PRA-provable for $f_0(i) := |i| \cdot \text{inv}(i)$.

2. $\text{SWO}(\varepsilon_0, <, f_1)$ is not PA-provable for $f_1(i) := |i| \cdot |i|_m$.

Let $L(\cdot; F^{-1}_\alpha)$ be defined by

the least $n$ such that for any finite sequence of ordinals $\alpha_0, \ldots, \alpha_n < \varepsilon_0$ with $N\alpha_i \leq k + |i| \cdot |i|_{F^{-1}_\alpha(i)}$ for all $i \leq n$, there exist indices $\ell$ and $m$ such that $\ell < m \leq n$ and $\alpha_\ell \preceq \alpha_m$.

Theorem 8 (Arai [1]). Let $\alpha \leq \varepsilon_0$.

1. $L(\cdot; F^{-1}_\alpha)$ is primitive recursive in $F_\alpha$ and vice versa. Therefore, $L(\cdot; F^{-1}_\alpha)$ is provably total in PA iff $\alpha < \varepsilon_0$.

2. Let $f_\alpha(i) := |i| \cdot |i|_{F^{-1}_\alpha(i)}$. Then

$\text{SWO}(\varepsilon_0, <, f_\alpha)$ is PA-provable iff $\alpha < \varepsilon_0$. 

2.2. Binary trees

The system \((B, \prec)\) obtained from the Feferman–Schütte notation system\(^1\) for \(\Gamma_0\) by omitting the operator \(\text{‘}+\text{’}\) constitutes a well-ordering of order type \(\varepsilon_0\). Moreover, \(\leq\) is a canonical extension of the homeomorphic embeddability relation \(\trianglelefteq\) on the set of rooted binary trees: A rooted binary tree \(T\) is a set of nodes such that, if it is not empty, there is one distinguished node called the root of \(T\) and the remaining nodes are partitioned into two rooted binary trees. The following definition of the set \(B\) of all rooted binary trees is very useful for our study.

Assume that a constant symbol \(o\) and a binary function symbol \(\varphi\) are given. In the following, \(\varphi\alpha\beta\) stands for \(\varphi(\alpha, \beta)\).

**Definition 9.** The set \(B\) and the homeomorphic embeddability relation \(\trianglelefteq\) are defined inductively as follows:

1. \(o\in B\) and \(o\trianglelefteq \beta\) for all \(\beta\in B\);
2. if \(\alpha_1, \beta_1\in B\), then \(\varphi(\alpha_1, \beta_1)\in B\), and \(\varphi\alpha_1\beta_1\trianglelefteq \varphi\alpha_2\beta_2\) if
   - (a) \(\varphi\alpha_1\beta_1\trianglelefteq \alpha_2\) or \(\varphi\alpha_1\beta_1\trianglelefteq \beta_2\); or
   - (b) \(\alpha_1\trianglelefteq \beta_1\) and \(\alpha_2\trianglelefteq \beta_2\).

**Theorem 10** (Higman [16]). \((B, \trianglelefteq)\) is a well-partial ordering.

Given a well-partial ordering \((X, \preceq)\) define its **maximal order type** by

\[o(X, \preceq) := \sup\{\text{otype}(\prec^+) : \prec^+\text{ is a well-ordering on } X \text{ extending } \prec}\],

where \(\text{otype}(\prec^+)\) denotes the order type of the well-ordering \(\prec^+\).

**Theorem 11** (de Jongh and Parikh [8]). Assume \((X, \preceq)\) is a well-partial ordering. Then there is a well-ordering \(\prec^+\) extending \(\preceq\) such that \(o(X, \preceq) = \text{otype}(\prec^+)\).

In case of \((B, \trianglelefteq)\) we easily find such a well-ordering \(<\) on \(B\): < is the least binary relation on \(B\) defined as follows:

1. if \(\alpha = o\) and \(\beta \neq o\), then \(\alpha < \beta\);
2. if \(\alpha = \varphi\alpha_1\alpha_2\) and \(\beta = \varphi\beta_1\beta_2\), then \(\alpha < \beta\) if one of the following hold:
   - (a) \(\alpha_1 < \beta_1\) and \(\alpha_2 < \beta_2\);
   - (b) \(\alpha_1 = \beta_1\) and \(\alpha_2 < \beta_2\); or
   - (c) \(\alpha_1 > \beta_1\) and \(\alpha_2 \leq \beta_2\).

The following is a folklore in proof theory.

**Lemma 12.** \(<\) is a well-ordering on \(B\) extending \(\trianglelefteq\), and \(\text{otype}(<) = \varepsilon_0\).

The following unpublished result by de Jongh is well-known. For further details see, for example, Schmidt [24].

**Theorem 13** (de Jongh). \(o(B, \trianglelefteq) = \varepsilon_0\).

We want to here present an alternative, relatively simple proof of the theorem. A cumulative hierarchy is used.

Given a natural number \(d\) we define \(B^d\) recursively as follows:

- \(o \in B^d\);
- if \(d > 0\), \(\alpha \in B^{d-1}\), and \(\beta \in B^d\), then \(\varphi\alpha\beta \in B^d\).

And define \(\rho^d(\alpha)\) for \(\alpha \in B\) as follows:

- \(\rho^0(\alpha) = \alpha\);
- \(\rho^{d+1}(\alpha) = \varphi\rho^d(\alpha)\).

**Lemma 14.** Let \(d \in \mathbb{N}\).

1. \(B = \bigcup\{B^d : d \in \omega\}\).
(2) If \( a \in B^d \), then \( a < \rho^{d+1}(\alpha) \) and \( \rho^k(\alpha) \in B^{d+k} \).

(3) \( \rho^{d+1}(\alpha) \in B^{d+1} \setminus B^d \).

(4) If \( \alpha < \beta \), then \( \rho^d(\alpha) < \rho^d(\beta) \).

(5) If \( \alpha \leq \beta \), then \( \rho^d(\alpha) \subseteq \rho^d(\beta) \).

(6) If \( \alpha \in B^{d+1} \setminus B^d \) and \( \beta \in B^d \), then \( \alpha \not\preceq \beta \) and \( \beta < \alpha \).

**Proof.** The first five claims are obvious. The last one will be shown by induction on \( \beta \). If \( \beta = 0 \) there is nothing to show. Let \( \alpha = \varphi \alpha_1 \varphi_2 \cdots \varphi \alpha_m \circ \) and \( \beta = \varphi \beta_1 \beta_2 \cdots \beta_n \circ \). If \( \alpha_1 \in B^{d-1} \), then \( \beta_2 \in B^{d+1} \setminus B^d \). Hence \( \beta < \alpha_2 < \alpha \) by I.H. Now assume \( \alpha_1 \in B^d \setminus B^{d-1} \). Then \( \beta_1 < \alpha_1 \) and \( \beta_2 < \alpha_2 \) by I.H., so \( \beta < \alpha \) and \( \alpha \not\preceq \beta \). \( \Box \)

We are now going to compute \( o(B^d) \) by comparing \( (B^d, \leq) \) with a well-known well-partial ordering. Higman’s Lemma plays an important role.

**Definition 15.** Let \( (A, \preceq) \) be a partial ordering and \( A^* \) be the set of finite lists of members of \( A \). The **Higman embedding** \( \succeq_h \) is a partial ordering on \( A^* \) defined by:

\[
a_1 \cdots a_m \succeq_h b_1 \cdots b_n
\]

if there is a strictly increasing function \( g : [1, m] \to [1, n] \) such that \( a_i \preceq b_{g(i)} \) for any \( i \in \{1, \ldots, m\} \).

**Theorem 16 (Higman’s Lemma).**

(1) If \( A \) is a well-partial ordering, then \( A^* \) is a well-partial ordering with respect to the Higman embedding.

(2) If \( (A, \preceq) \) is a well-partial ordering with \( o(A, \preceq) = \alpha \), then

\[
o(A^*, \succeq_h) = \begin{cases} 
\omega^{\alpha-1} & \text{if } \alpha \in \omega \setminus \{0\}, \\
\omega^{\alpha} & \text{if } \alpha = \beta + m, \text{ where } \beta \geq \omega, \beta \not= \omega^0, \text{ and } m \in \omega, \\
\omega^{\alpha+1} & \text{otherwise}.
\end{cases}
\]

**Proof.** See e.g. [8, 24, 15]. \( \Box \)

Note that there is a similarity between \( B^{d+1} \) and \( (B^d)^* \). In fact, every \( \alpha \in B^{d+1} \) is of the form \( \alpha = \varphi \alpha_1 \varphi \alpha_2 \cdots \varphi \alpha_m \circ \), where \( \alpha_i \in B^d \). If \( \beta = \varphi \beta_1 \varphi \beta_2 \cdots \varphi \beta_n \circ \in B^{d+1} \) and \( \alpha_1 \cdots \alpha_m \succeq_h \beta_1 \cdots \beta_n \), then \( \alpha \succeq \beta \). Unfortunately the relation above is not isomorphic. Nevertheless, we will show that \( o(B^{d+1}, \preceq) = o((B^d)^*, \succeq_h) \) with the aid of the following obvious lemma.

**Lemma 17.** Let \( (A, \preceq_1) \) and \( (B, \preceq_2) \) be well-partial orderings and \( f : A \to B \) an injective function such that

\[
a \preceq_1 b \iff f(a) \preceq_2 f(b)
\]

for all \( a, b \in A \). Then \( o(A, \preceq_1) \leq o(B, \preceq_2) \).

**Theorem 18.** For any \( d > 0 \), \( o(B^{d+1}, \preceq) = o(B^{d+1} \setminus B^d, \preceq) = o((B^d)^*, \succeq_h) \).

**Proof.** Consider \( f : B^{d+1} \to (B^d)^* \) and \( g : (B^d)^* \to B^{d+1} \setminus B^d \) defined by:

\[
f(\alpha) := \begin{cases} 
\emptyset & \text{if } \alpha = \circ, \\
\alpha & \text{if } \alpha = \varphi \alpha_1 \alpha_2 \in B^d, \\
\alpha, \alpha_1, f(\alpha_2) & \text{if } \alpha = \varphi \alpha_1 \alpha_2 \not\in B^d
\end{cases}
\]

and

\[
g(\alpha_1, \ldots, \alpha_m) := \varphi \alpha_1 \varphi \alpha_2 \cdots \varphi \alpha_m \rho^{d+1}(\circ),
\]

where \( \emptyset \) denotes the empty sequence. It suffices to show that \( f \) and \( g \) satisfy the conditions in Lemma 17.
(1) Let \( \alpha, \beta \in B^d \).

(a) Assume \( \alpha \leq \beta \). We show \( f(\alpha) \leq_h f(\beta) \) by induction on the number of occurrences of \( \varphi \) in \( \alpha \) and \( \beta \). If \( \alpha \in B^d \), then it is obvious. Assume now \( \alpha = \varphi \alpha_1 \alpha_2 \notin B^d \). Then \( \beta = \varphi \beta_1 \beta_2 \notin B^d \) for some \( \beta_1, \beta_2 \) by Lemma 14. Furthermore, we have \( \alpha \not\leq \beta_1 \) since \( \beta_1 \in B^d \). Hence there are two possible cases, i.e. either \( \alpha \leq \beta_2 \) or \( \alpha_i \leq \beta_i \), \( i = 1, 2 \). In any case the claim follows from the I.H. since \( f(\alpha) = \alpha, f(\alpha_2) \) and \( f(\beta) = \beta, f(\beta_2) \).

(b) Assume \( f(\alpha) \leq_h f(\beta) \). We show \( \alpha \leq \beta \) by induction on the number of occurrences of \( \varphi \) in \( \alpha \) and \( \beta \). If \( \alpha \in B^d \), then it is again obvious since each component in \( f(\beta) \) is a subtree of \( \beta \). Assume \( \alpha \notin B^d \), then \( \beta \notin B^d \) and \( \alpha \not\leq \beta_1 \). Since then \( f(\beta) = \beta, f(\beta_2) \) we have either \( \alpha \leq \beta \) or \( f(\alpha) \leq_h f(\beta_2) \). In the latter case the I.H. implies \( \alpha \leq \beta_2 \leq \beta \).

(2) Let \( \alpha_1, \ldots, \alpha_m, \beta_1, \ldots, \beta_n \in B^d \).

(a) Assume \( \alpha_1, \ldots, \alpha_m \leq_h \beta_1, \ldots, \beta_n \). Then there is an increasing one-to-one function \( \mu : m \rightarrow n \) such that \( \alpha_i \leq \beta_{\mu(i)} \) for any \( i = 1, \ldots, m \). Then we can easily show that \( g(\alpha_1, \ldots, \alpha_m) \leq g(\beta_1, \ldots, \beta_n) \).

(b) Assume \( \varphi \alpha_1 \cdots \varphi \alpha_m \rho^d(\beta) \leq \varphi \beta_1 \cdots \varphi \beta_n \rho^d(\beta) \). By induction on \( n \) we show that \( \alpha_1, \ldots, \alpha_m \leq_h \beta_1, \ldots, \beta_n \). Assume \( n = 0 \). Then \( m = 0 \) since \( \varphi \gamma \rho^d(\beta) \leq \rho^d(\beta) \) for any \( \gamma \in B \). Assume \( n > 0 \). If \( m = 0 \), it is obvious. If \( m > 0 \), then there must be some \( j \in \{1, \ldots, n\} \) such that \( \alpha_1 \leq \beta_j \) and \( \varphi \alpha_2 \cdots \varphi \alpha_m \rho^d(\beta) \leq \varphi \beta_{j+1} \cdots \varphi \beta_n \rho^d(\beta) \). This is because \( \rho^d(\beta) \notin B^d \). The claim follows now by I.H. \( \square \)

**Corollary 19.** For any \( d > 0 \), \( (B^d, \leq) \) is a well-partial ordering and \( o(B^d, \leq) = \omega_{2d-1}(1) \).

**Proof.** If \( d = 1 \), then \( B^1 = \{\varnothing, \varnothing \circ \varnothing, \varnothing \circ (\varnothing \circ \varnothing), \ldots\} \) is linearly ordered by \( \leq \), so \( o(B^1, \leq) = \omega \). If \( d > 1 \), use I.H., Theorem 18, and Higman’s Lemma. \( \square \)

**Theorem 20.** \( o(B, \leq) = \varepsilon_0 \).

**Proof.** By Corollary 19 we have \( o(B, \leq) \geq \varepsilon_0 \). Assume \( o(B, \leq) > \varepsilon_0 \). Then there would be a well-ordering \( \prec \) on \( B \) extending \( < \) such that \( otype(<) > \varepsilon_0 \). This is, however, impossible since each restriction of \( \prec \) to \( B^d \) has the order type less than \( \varepsilon_0 \) again by Corollary 19. \( \square \)

Let the norm of \( \alpha \in B \), \( \|\alpha\| \), be the number of occurrences of \( \varphi \) in \( \alpha \), i.e.

\[
\|\alpha\| = 0 \quad \text{and} \quad \|\varphi \alpha \beta\| = 1 + \|\alpha\| + \|\beta\|.
\]

It is evident that \( B, \leq, < \), and \( \| \cdot \| \) are all primitive recursively definable in PA.

**Definition 21** (Slowly well-partial-orderedness, SWP). Let \( (X, \leq) \) be a partial ordering. Then given a norm function \( g : X \rightarrow \mathbb{N} \) and a function \( f: \mathbb{N} \rightarrow \mathbb{N} \) the slowly well-partial-orderedness is given by SWP(X, \( \leq, f \)),

\[
\text{SWP}(X, \leq, f) :\equiv
\]

For any \( k \) there exists a constant \( n \) which is so large that, for any finite sequence \( \alpha_0, \ldots, \alpha_n \) of finite trees with \( |\alpha_i| \leq k + f(i) \) for all \( i \leq n \), there exist indices \( \ell < m \leq n \) satisfying \( \alpha_\ell \leq \alpha_m \).

**Theorem 22** (Weiermann [33]). Given a primitively recursive real number \( r \) set \( f_r(i) := r \cdot |i| \). And let \( \leq \in \{\leq, <\} \).

Then it holds that

\[
\text{SWP}(B, \leq, f_r) \text{ is PA-provable if } r \leq \frac{1}{2}.
\]

The proof is not published yet. But Section 4 in [30] indicates the existence of such a constant like \( \frac{1}{2} \). The proof in [33] is very similar. It uses just another counting method, hence a different constant concerning the binary trees than Otter’s tree constant which is specific for general finite trees.

**Remark 23.** The theorem above indicates that Cantor’s system for \( \varepsilon_0 \) is of different character than the system \( (B, <) \).

In fact, we know by Theorem 7 that SWO(\( \varepsilon_0, <, \lambda \cdot r \cdot |x| \)) is PA-provable for any \( r \), while SWO(\( (B, <, \lambda \cdot r \cdot |x|) \)) is not PA-provable if \( r > \frac{1}{2} \).
3. Structural equivalences

We now show that the systems except \( (B, <) \) are structurally equivalent. The functions themselves which establish the structural equivalences look very simple and somehow canonical while the proofs of the equivalences are not so quite obvious. We begin with one of the most recent systems.

3.1. Graded provability algebra and Beklemishev’s system

Let \( T \) be an elementarily represented, sound fragment of \( \text{PA} \) containing \( \text{I} \Sigma_1 \). The Lindembaum boolean algebra \( L_T \) is the set of all sentences modulo provable equivalence in \( T \).

Let \( n \cdot \text{Con}(T) \) denote a natural formula expressing that the theory \( T + \text{Th}_{\Pi_n}(\mathbb{N}) \) is consistent, where \( \text{Th}_{\Pi_n}(\mathbb{N}) \) is the set of all true arithmetical \( \Pi_n \) sentences. The graded provability algebra of \( T, M_T \), is the structure of Lindembaum boolean algebra \( L_T \) with the \( n \)-consistency operator \( (n)_T, n \in \mathbb{N} \), defined by \( (n)_T : = n \cdot \text{Con}(T + \varphi) \).

The subscript \( T \) will be suppressed if the underlying theory is known from the context. The graded provability algebra correspond to propositional polymodal formulas.

GLP based on the identities of \( M_T \) is an extension of the Gödel–Löb system GL\(^2\): Let \( m, n \in \mathbb{N} \).

(1) Axioms

- Boolean tautologies
- \( (n)(\varphi \vee \psi) \rightarrow (n)\varphi \vee (n)\psi \)
- \( \neg (n)\neg \top \)
- \( (n)\varphi \rightarrow (n)(\varphi \land \neg (n)\varphi) \)
- \( (n)\varphi \rightarrow (m)\varphi \) for \( m \leq n \)
- \( (m)\varphi \rightarrow [n](m)\varphi \) for \( m < n \).

(2) Rules

- modus ponens
- \( \varphi \vdash \psi \Leftrightarrow (n)\varphi \rightarrow (n)\psi \).

Then it holds that GLP \( \vdash \varphi(\bar{x}) \) iff \( M_T \models \forall \bar{x} (\varphi(\bar{x}) = \top) \).

Let \( S \) be the set of all finite words in the alphabet \( \mathbb{N} \), including the empty word \( \lambda \). \( S_n \) is the restriction of \( S \) to the alphabet \( \{n, n + 1, \ldots \} \). We identify each element \( \alpha = n_1 \cdot \ldots \cdot n_k \) of \( S \) with its modal interpretation \( (n_1) \cdot \ldots \cdot (n_k) \cdot \top \).

We write \( \alpha \sim \beta \) if GLP \( \vdash \alpha \leftrightarrow \beta \). And \( \alpha = \beta \) means the graphical identity. The orderings \( <_n \) are defined on \( S \) by:

\[ \alpha <_n \beta \text{ if GLP } \vdash \beta \rightarrow (n)\alpha. \]

Note that \( <_n \) are transitive and irreflexive because of Gödel’s incompleteness theorem. Below we summarize some results from Beklemishev [2].

Given \( \alpha \in S \) let \( \alpha^k \) denote the \( k \) times iterated concatenation of \( \alpha \). The function \( o : S \rightarrow \varepsilon_0 \) is given as follows:

- \( o(0^k) = k \);
- if \( \alpha = a_0 \cdot \ldots \cdot a_n \), where all \( a_i \in S_1 \) and not all of them empty, then
  \[ o(\alpha) = o(a_n) + \ldots + o(a_0). \]

Here \( \gamma^- \) is obtained from \( \gamma \in S_1 \) by replacing every letter \( m + 1 \) with \( m \).

Note that some of the elements of \( S \) are pairwise equivalent. However, there is a set of elements which represent each equivalence class, namely the set \( \text{NF} \) of words in normal form. We define \( \alpha \in \text{NF} \) by recursive induction on the width \( w(\alpha) \), i.e. the number of different letters occurring in \( \alpha \).

- if \( w(\alpha) \leq 1 \), then \( \alpha \in \text{NF} \);
- assume \( w(\alpha) > 2 \) and let \( n \) be the smallest letter in \( \alpha \) such that graphically \( \alpha = a_0 n \cdot \ldots \cdot n a_k \), where all \( a_i \in S_{n+1} \).

Then \( \alpha \in \text{NF} \) if all \( a_i \in \text{NF} \) and \( a_{i+1} \neq_{n+1} a_i \) for any \( i < k \).

\(^2\text{Cf. Boolos [4] for more about provability logic.}\)
**Theorem 24** (Beklemishev). Let $\alpha, \beta \in S$.

1. $(S, \langle_0, \rangle)$ is a well-partial ordering of height $\varepsilon_0$.
2. Every word $\alpha \in S$ has an uniquely defined equivalent normal form.
3. If $\alpha \sim \beta$ then $o(\alpha) = o(\beta)$.
4. If $\alpha <_0 \beta$ then $o(\alpha) < o(\beta)$.
5. $\alpha \upharpoonright NF: \text{NF} \rightarrow \varepsilon_0$ is an order-preserving isomorphism.

It is also possible to assign fundamental sequences to each element of $S$. For $\alpha \in S$ and $k \in \mathbb{N}$ we define $\alpha[k] \in S$ as follows:

- if $\alpha = \langle 0 \rangle \beta$ then $\alpha[k] = \beta$;
- if $\alpha = \langle n + 1 \rangle m \beta$, where $\gamma \in S_{n+1}$ and $m \leq n$, then $\alpha[k] = (n \gamma)^{k+1} m \beta$.

There will be no confusion with the notation for fundamental sequences with respect to the ordinals up to $\varepsilon_0$.

**Theorem 25** (Beklemishev). Let $\alpha = \langle n + 1 \rangle \beta$ and $k \in \mathbb{N}$.

1. If $\alpha \in \text{NF}$, then $\alpha[k] \in \text{NF}$.
2. $\alpha[k] <_0 \alpha[k+1] <_0 \alpha$.
3. For every $\beta \in S$ with $\beta <_0 \alpha$ there is an $\ell \in \mathbb{N}$ such that $\beta <_0 \alpha[\ell]$.

### 3.2. Worms, Hydras, and tree-ordinals

The **Hydra battle** introduced by Kirby and Paris in [17] has an isomorphic formulation in terms of ordinals, namely fundamental sequences for ordinals below $\varepsilon_0$. Let $\cdot\cdot\cdot$ denote the standard assignments of fundamental sequences and let $\alpha(0) = \alpha$ and $\alpha(i + 1) = \alpha(i)[i]$. Then the fact that chopping off the rightmost head is a winning strategy for Hercules is formalized by:

\[
\text{for any } n \text{ there exists an } i \text{ such that } \omega_n(i) = 0.
\]

This is a true $I_2^0$-sentence because of the well-foundedness, but PA-unprovable since $H_\alpha(0) \leq \min\{i : \alpha(i) = 0\}$.

Another similar combinatorial game is introduced by Beklemishev in [3] as an application of proof-theoretic analysis to his ordinal notation system $S$. It deals with objects called **worms** and is hence called **Worm principle**. See also Carlucci [7] where the isomorphism of the Worm and the Hydra battle was established (independently of this paper).

A **worm** is just a finite function with natural numbers as values. We identify the worm $f : [0, n] \rightarrow \mathbb{N}$ with the list $f(0) \cdot \cdot \cdot f(n)$ or $\langle f(0), \ldots, f(n) \rangle$. We call $f(n)$ the **head** of the worm. $\emptyset$ denotes the empty function. Let $W$ be the set of all worms and $W_n$ the subset of $W$ whose elements have values $\geq n$.

A Worm game begins with a worm and at each step we chop off its head. In response the worm grows in length according to some rules. Formally, we specify a function $\text{next} : W \times \mathbb{N} \rightarrow W$. Let $\alpha$ range over worms.

1. $\text{next}(\emptyset, k) := \emptyset$.
2. Let $\alpha = a_0 \cdot \cdot \cdot a_n$.
   - If $a_n = 0$, then $\text{next}(\alpha, k) := a_0 \cdot \cdot \cdot a_{n-1}$.
   - If $a_n > 0$, let $m := \max\{i < n : a_i < a_n\}$. We define
     \[
     \text{next}(\alpha, k) := r * s * s * \ldots * s \quad \text{for } \underbrace{k+1 \text{ times}}_{m \text{ times}}
     \]
   
   where $r = \langle a_0, \ldots, a_m \rangle$, $s = \langle a_{m+1}, \ldots, a_{n-1}, a_n - 1 \rangle$.

Here $*$ means the concatenation function of worms. Now let $\alpha(0) := \alpha$ and $\alpha(n + 1) := \text{next}(\alpha(n), n + 1)$. Then the **Worm principle** says that **Every Worm Dies**:

\[
\text{EWD} := \text{for any worm } \alpha \text{ there exists an } n \text{ such that } \alpha(n) = \emptyset.
\]

---

3 Cf. Fairtlough and Wainer [9].
Note that EWD is a $\Pi^0_2$-sentence since $\alpha(n)$ is defined primitive recursively and that the size of maximal element of worms cannot increase. Hence $\alpha(n) = \beta$ can be written out as a $\Delta_0$ formula in three variables.

**Theorem 26** (Beklemishev [3]). (1) EWD is true, but not PA-provable.
(2) EWD is PA-equivalent to 1-Cons(PA).

What is responsible for the PA-unprovability of EWD? A proof-theoretical explanation is that the Skolem function of EWD should grow too fast to be provably total in PA. Below we give a characterization of the growth rate conditions responsible for the too fast growth. In order to emphasize the relevance to $S$ we prefer another notation to $\text{next}(\alpha, k)$.

**Definition 27.** Let $\alpha$, $\beta$, $\gamma \in W$.

$$\alpha[k] := \begin{cases} \emptyset & \text{if } \alpha = \emptyset, \\ \beta & \text{if } \alpha = \beta 0, \\ \beta m(\gamma n)^{k+1} & \text{if } \alpha = \beta m\gamma(n + 1), \gamma \in W_{n+1}, m \leq n. \end{cases}$$

Note that $\text{next}(\alpha, k) = \alpha[k]$. Given $f : \mathbb{N} \to \mathbb{N}$ and $\alpha \in W$ set

$$\alpha(f, 0) := \alpha, \quad \alpha(f, n + 1) := \alpha(f, n)[f(n + 1)],$$

and define

$$\text{EWD}(f) := \forall \alpha \exists n (\alpha(f, n) = \emptyset).$$

Then EWD = EWD($id$). And EWD($f$) remains true $\Pi^0_2$-sentence if $f$ is primitive recursive.

To analyze the growth rate of the Skolem function of EWD($f$) in terms of fast growing hierarchies one should notice that the function $\omega$ between $S$ and the Cantor system for $\varepsilon_0$ defined above cannot be used in its original form since the correspondence is not one-to-one. This is why we turn our attention to the tree-ordinals.

**Definition 28.** The set $\Omega$ of countable tree-ordinals is generated inductively as follows:

- 0 $\in \Omega$;
- if $\alpha \in \Omega$, then $\alpha + 1 := \alpha \cup \{\alpha\} \in \Omega$;
- if $\alpha_n \in \Omega$ for all $n \in \mathbb{N}$, then $\alpha := \langle \alpha_n \rangle_{n \in \mathbb{N}} \in \Omega$.

$\lambda$ will always denote a limit of the form $\lambda = \langle \lambda_n \rangle_n := \langle \lambda_n \rangle_{n \in \mathbb{N}}$. Addition, multiplication, and exponentiation are defined as usual:

- **Addition:** $\alpha + 0 := \alpha; \quad \alpha + (\beta + 1) := (\alpha + \beta) + 1; \quad \alpha + \lambda := \langle \alpha + \lambda_n \rangle_n$
- **Multiplication:** $\alpha \cdot 0 := 0; \quad \alpha \cdot (\beta + 1) := (\alpha \cdot \beta) + \alpha; \quad \alpha \cdot \lambda := \langle \alpha \cdot \lambda_n \rangle_n$
- **Exponentiation:** $\alpha^0 := 1; \quad \alpha^{(\beta + 1)} := \alpha^\beta \cdot \alpha; \quad \alpha^\lambda := \langle \alpha^{\lambda_n} \rangle_n$.

There is a set $\mathbb{T} \subseteq \Omega$ whose elements correspond to ordinals up to $\varepsilon_0$. Set

$$n := 0 + 1 + \cdots + 1 \quad \text{and} \quad \omega := \langle 1 + n \rangle_n.$$ 

Maybe it is an abuse of symbols to use the same names such as $n$ and $\omega$. However, it will always be clear from the context to which world they belong.

**Definition 29.** $\mathbb{T}$ is defined inductively as follows:

- 0 $\in \mathbb{T}$;
- if $\alpha_0, \ldots, \alpha_n \in \mathbb{T}$, then also $\omega^{\alpha_0} + \cdots + \omega^{\alpha_n} \in \mathbb{T}$.

Note that each tree-ordinal in $\mathbb{T}$ represents a unique determined (ordered) tree figure other than the ordinals in the Cantor system. And though there is a canonical fundamental sequence for each limit tree-ordinal, we shall make some modifications for technical reasons. This modifications will have no significant effect on the fast growing hierarchy we consider. We write $\alpha \cdot m$ for $\underbrace{\alpha + \cdots + \alpha}_{m \text{ times}}$. 


Definition 30 (Fundamental Sequences for Tree-Ordinals). Let $\alpha \in T$.

- If $\alpha = 0$, then $\alpha[k] = 0$.
- If $\alpha = \beta + 1$, then $\alpha[k] = \beta$.
- If $\alpha = n + \omega$ for some $n \in \mathbb{N}$, then $\alpha[k] = n + k + 1$.
- If $\alpha = \beta + \omega$ and $\gamma \neq n$ for any $n \in \mathbb{N}$, then $\alpha[k] = \beta + k + 2$.
- If $\alpha = \beta + \omega^{\gamma+1}$ and $\gamma \neq 0$, then $\alpha[k] = \beta + \omega^{ \gamma} \cdot (k + 1) + 1$.
- If $\alpha = \beta + \omega^\lambda$ and $\lambda$ a limit, then $\alpha[k] = \beta + \omega^\lambda[k]$.

Definition 31. The subtree ordering $\prec$ is the transitive closure of the rule:

\[ \alpha[m] \prec \alpha \lor \alpha \in T \setminus \{0\} \land m \in \mathbb{N}. \]

That the subtree ordering build a well-ordering can be proven easily as in Fairtlough and Wainer [9].

Theorem 32. The set $\{ \beta \mid \beta \prec \alpha \}$ is well-ordered by $\prec$ and of order type less than $\varepsilon_0$.

Once more we shall use the notation $o$ for an isomorphism between worms and tree-ordinals from $T$.

Definition 33. $o : W \to T$ is defined recursively as follows:

- $o(0^k) := k$;
- If $\alpha = a_00\cdots a_n$, where all $\alpha_i \in W_1$ and not all of them empty, then
  \[
  o(a_0a_10\cdots a_n) := \omega^{o(a_n)} + \cdots + \omega^{o(a_1)}.
  \]

There will be no confusion between $o : W \to T$ and $o : S \to \varepsilon_0$. One of the main differences is, however, that $o : W \to T$ is one-to-one and onto. The function $g : T \to W$ defined by

- $g(0^k) := 0^k$;
- If $\alpha = \omega^{\alpha_0} + \cdots + \omega^{\alpha_n}$ and $\alpha_i \neq 0$ for some $i \leq n$, then
  \[
  g(\alpha) = g(\alpha_0)^+0\cdots 0g(\alpha_n)^+,
  \]

is obviously the inverse function of $o : W \to T$, where $\beta^+$ is obtained from $\beta \in W$ by replacing every letter $m$ with $m + 1$.

Lemma 34. Let $\alpha, \beta \in W$. Then

\[
\begin{align*}
o(\alpha 0 \beta) &= \begin{cases} o(\alpha) + 1 + o(\beta) & \text{if } 0^m \in \{o(\alpha), o(\beta)\} \text{ for some } m \in \mathbb{N}, \\
o(\alpha) + o(\beta) & \text{otherwise.}
\end{cases}
\end{align*}
\]

Proof. (1) Let $\alpha = 0^m$ and $\beta = 0^n$ for some $m, n \in \omega$. Then

\[
o(\alpha 0 \beta) = o(0^{m+1+n}) = m + 1 + n = o(\alpha) + 1 + o(\beta).
\]

(2) Let $\alpha = 0^m$ for some $m \in \omega$ and $\beta = \beta_00\cdots 0\beta_n$, where all $\beta_j \in W_1$ and not all of them empty. Then

\[
o(\alpha 0 \beta) = o(0^{m+1}\beta_00\cdots 0\beta_n) = m + 1 + \omega^{o(\beta_0)} + \cdots + \omega^{o(\beta_n)} = o(\alpha) + 1 + o(\beta).
\]

(3) Similar for the case that $\beta = 0^n$ for some $n \in \omega$ and $\alpha = a_00\cdots a_m$, where all $\alpha_i \in W_1$ and not all of them empty.

(4) Let $\alpha = a_00\cdots 0\alpha_m$ and $\beta = \beta_00\cdots 0\beta_n$, where all $\alpha_j, \beta_j \in W_1$ and there are some $\alpha_i \neq \emptyset$ and $\beta_j \neq \emptyset$. Then

\[
o(\alpha 0 \beta) = o(a_00\cdots 0\alpha_m0\beta_00\cdots 0\beta_n) = o(\alpha) + o(\beta).
\]

The proof is now complete. □
This lemma will be used tacitly in the following. Let $\beta^{-n}$ be obtained from $\beta \in W_n$ by replacing every letter $m$ with $m-n$. $\beta^{+n}$ is similarly defined for $\beta \in W$. Below we write $o(\beta) < \omega$ for $o(\beta) = k$ for some $k \in \mathbb{N}$ and $o(\beta) \geq \omega$ otherwise.

**Theorem 35.** $o(\alpha[k]) = o(\alpha)[k]$ for all $\alpha \in W$.

**Proof.** There is nothing to prove if $\alpha = \emptyset$. If $\alpha = \beta 0$, then $\alpha[k] = \beta$ and

$$(o(\alpha))[k] = (o(\beta) + 1)[k] = o(\beta) = o(\alpha[k]).$$

Now let $\alpha = \beta m \gamma (n + 1)$, where $m \leq n$ and $\gamma \in W_{n+1}$. There are four cases to consider.

(i) $\gamma = \emptyset$ and $\beta m = \emptyset$, i.e. $\alpha = (n + 1)$ and $\alpha[k] = n^{k+1}$:

$$o(\alpha)[k] = o(\alpha)[k] = o(n(k + 1) = o(n^{k+1}) = o(\alpha[k]).$$

(ii) $\gamma = \emptyset$ and $\beta m \neq \emptyset$. We use an induction on $n$:

(a) $m$ is the minimum of the occurrences in $\beta m$.

$$o(\alpha) = o(\beta m(n + 1)) = o_m(o(\beta^{-m}0(n + 1 - m)))$$

$$= \begin{cases} o_m(o(\beta^{-m}) + o_{n+1-m}), & o(\beta^{-m}) \geq \omega, \\
o_m(o(\beta^{-m}) + 1 + o_{n+1-m}), & o(\beta^{-m}) < \omega. \end{cases}$$

And

$$o(\alpha)[k] = \begin{cases} o_m(o(\beta^{-m}) + o_{n-m}(k + 1)), & o(\beta^{-m}) \geq \omega, n > m, \\
o_m(o(\beta^{-m}) + k + 2), & o(\beta^{-m}) \geq \omega, n = m, \\
o_m(o(\beta^{-m}) + 1 + o_{n-m}(k + 1)), & o(\beta^{-m}) < \omega. \end{cases}$$

On the other hand,

$$o(\alpha[k]) = o(\beta mn^{k+1}) = o_m(o(\beta^{-m}0(n-m)^{k+1}))$$

$$= \begin{cases} o_m(o(\beta^{-m}) + o((n-m)^{k+1})), & o(\beta^{-m}) \geq \omega, n > m, \\
o_m(o(\beta^{-m}) + 1 + k + 1), & o(\beta^{-m}) \geq \omega, n = m, \\
o_m(o(\beta^{-m}) + 1 + o((n-m)^{k+1})), & o(\beta^{-m}) < \omega, \end{cases}$$

$$= \begin{cases} o_m(o(\beta^{-m}) + o_{n-m}(k + 1)), & o(\beta^{-m}) \geq \omega, n > m, \\
o_m(o(\beta^{-m}) + k + 2), & o(\beta^{-m}) \geq \omega, n = m, \\
o_m(o(\beta^{-m}) + 1 + o_{n-m}(k + 1)), & o(\beta^{-m}) < \omega. \end{cases}$$

Note that the case $n = 0$ is also proved since $m$ should then be 0.

(b) $n > 0$, $m > p$ and $\beta = \beta_0 p \cdots p \beta_{l+1}$, where all $\beta_i \in W_{p+1}$.

$$o(\alpha) = o(\beta_0 p \cdots p \beta_{l+1} m(n + 1))$$

$$= o_p(o(\beta_0^{-p}0 \cdots 0 \beta_{l+1}^p)(m-p)(n+1-p))$$

$$= o_p(o(\beta_0^{-p}0^+ \cdots 0^+ \beta_{l+1}^p)(m-p-1)(n-p)).$$

Since $o(\beta_{l+1}^{-p-1}(m-p-1)(n-p))$ is a limit, we have

$$o(\alpha)[k] = o_p(o(\beta_0^{-p-1}) + \cdots + o(\beta_{l}^{-p-1}) + o(\beta_{l+1}^{-p-1}(m-p-1)(n-p))[k])$$

$$= o_p(o(\beta_0^{-p-1}) + \cdots + o(\beta_{l}^{-p-1}) + o(\beta_{l+1}^{-p-1}(m-p-1)(n-p))[k])$$

$$= o_p(o(\beta_0^{-p-1}) + \cdots + o(\beta_{l}^{-p-1}) + o(\beta_{l+1}^{-p-1}(m-p-1)(n-p-1)+1)).$$
On the other hand,
\[ o(\alpha[k]) = o(\beta_0 p \cdots p \beta_{t+1} m n^{k+1}) = o(\omega p (\beta_0^{-p} \cdots \beta_{t+1}^{-p})(m - p)(n - p)^{k+1}) = o(\omega p (\omega^{\beta_0^{-p-1}} + \cdots + \omega^{\beta_{t+1}^{-p-1}})(m - p)(n - p)^{k+1}). \]

(iii) \( \gamma \neq \emptyset \) and \( \beta m = \emptyset \), i.e. \( \alpha = \gamma \langle n + 1 \rangle, \gamma \in W_{n+1} \):
\[ o(\alpha) = o_{n+1}(o(\gamma^{n+1} 0)) = o_{n+1}(o(\gamma^{n+1}) + 1). \]

Since \( o(\gamma^{n+1}) > 0 \), we have \( o(\alpha) = o_n(o(\gamma^{n+1}) \cdot (k + 1)) \). On the other hand,
\[ o(\alpha[k]) = o((\gamma n)^{k+1}) = o_n(o((\gamma n)^{k+1})) = o_n(o(\omega^{\gamma^{n+1}} \cdot (k + 1)). \]

(iv) \( \gamma \neq \emptyset \) and \( \beta m \neq \emptyset \). The claim will be shown by induction on \( n \):

(a) \( m \) is the minimum of the occurrences in \( \beta m \).
\[ o(\alpha) = o(\beta m \gamma \langle n + 1 \rangle) = o_m(o(\beta^{-m} \gamma^{-m} \langle n + 1 - m \rangle)) = \begin{cases} o_m(o(\beta^{-m}) + o_{n+1-m}(o(\gamma^{n+1} 0)), & o(\beta^{-m}) \geq \omega, \\ o_m(o(\beta^{-m}) + 1 + o_{n+1-m}(o(\gamma^{n+1} 0)), & o(\beta^{-m}) < \omega. \end{cases} \]

Since \( o(\gamma^{n+1}) > 0 \), we have
\[ o(\alpha) = \begin{cases} o_m(o(\beta^{-m}) + o_{n-m}(o(\gamma^{n+1}) \cdot (k + 1)), & o(\beta^{-m}) \geq \omega, \\ o_m(o(\beta^{-m}) + 1 + o_{n-m}(o(\gamma^{n+1}) \cdot (k + 1)), & o(\beta^{-m}) < \omega. \end{cases} \]

On the other hand,
\[ o(\alpha[k]) = o(\beta m (\gamma n)^{k+1}) = o_m(o(\beta^{-m} 0(n - m)\gamma^{-m} \langle n \rangle^{k+1})) = \begin{cases} o_m(o(\beta^{-m}) + o((\gamma^{-m})^{k+1} \langle n - m \rangle)), & o(\beta^{-m}) \geq \omega, \\ o_m(o(\beta^{-m}) + 1 + o((\gamma^{-m})^{k+1} \langle n - m \rangle)), & o(\beta^{-m}) < \omega. \end{cases} \]

Note that the case \( n = 0 \) is also proved since \( m \) should then be 0.

(b) \( n > 0, m > p \) and \( \beta = \beta_0 p \cdots p \beta_{t+1} \), where all \( \beta_i \in W_{p+1} \).
\[ o(\alpha) = o(\beta_0 p \cdots p \beta_{t+1} m \gamma \langle n + 1 \rangle) = o_p(o(\beta_0^{-p} \cdots p \beta_{t+1}^{-p})(m - p)(n + 1 - p)) = o_p(o(\omega^{\beta_0^{-p-1}} + \cdots + \omega^{\beta_{t+1}^{-p-1}} + \omega^{\beta_0^{-p-1} \cdots p^{-1} n - p})). \]
Since \( o(b_{i+1}^{-p-1}(m - p - 1)y^{-p-1}(n - p)) \) is a limit, we have

\[
o(\alpha)[k] = \omega_p(\omega^{o(\beta_0^{-p-1})} + \cdots + \omega^{o(\beta_i^{-p-1})} + \omega^{o(b_{i+1}^{-p-1}(m-p-1)y^{-p-1}(n-p))[k]})
\]

On the other hand,

\[
o(\alpha[k]) = o(\beta_0 p \cdots p \beta_{i+1} m(\gamma n)^{k+1})
\]

This completes the proof. \( \square \)

Remember that an ordinal \( \alpha = \omega^{\alpha_0} + \cdots + \omega^{\alpha_n} \) below \( \varepsilon_0 \) is said to be in Cantor normal form if \( \alpha > \alpha_0 \geq \cdots \geq \alpha_n \).

We can demand the same property from every \( x \in \varepsilon \) in Cantor normal form and an tree-ordinal which has the same tree figure, we can specify a set \( B \) of all tree-ordinals in so-called Cantor normal form. This implies in turn that \( B \) corresponds isomorphically to the set \( NF \subseteq S \) of all words in normal forms.

Let \( NF(W) \subseteq W \) be the set of all worms which are converses of a word in \( NF \). The worms in \( NF(W) \) are also said to be in Cantor normal form and the set \( NF(W) \) is isomorphic to \( \varepsilon_0 \).

**Lemma 36.** \( NF(W) \) can be characterized inductively as follows:

1. \( \emptyset \) and any worm of length 1 belong to \( NF(W) \);
2. assume that the length of the worm \( \alpha \) is larger than 1 and \( \alpha = \alpha_0 \cdots \alpha_n \), where all \( \alpha_i \in W_1 \). Then \( \alpha \in NF(W) \) if all \( \alpha_i \in NF(W) \) and \( o(\alpha_{j+1}) \leq o(\alpha_j) \) for all \( j < n \).

Note that \( o(\alpha) \in B \) for every \( \alpha \in NF(W) \), so we might talk about the linear ordering \( \prec \) of ordinals. It is also obvious that \( \alpha[k] \in NF(W) \) for all \( \alpha \in NF(W) \) and \( k \in \mathbb{N} \). Let \( \prec_0 \) be the well-ordering on \( NF(W) \) induced by the isomorphism \( o \).

**Lemma 37.** \( o \upharpoonright NF(W) : NF(W) \to B \) is an order-preserving isomorphism.

Having established an correspondence between \( W \) and \( T \) (or between \( NF(W) \) and \( B \)) it is now obvious that the Worm principle is the counterpart of the Hydra battle game on the tree-ordinals in \( T \) (resp. on the ordinals up to \( \varepsilon_0 \)).

On the other hand, the Hydra battle game has a direct connection to the Hardy–Wainer hierarchy. It is a folklore that the Hardy–Wainer hierarchy up to \( \varepsilon_0 \) features exactly the provably recursive functions in \( PA \). Fairtlough and Wainer [9] showed that an similar characterization of provably recursive function in \( PA \) is possible by using the tree-ordinals from \( T \). Furthermore, Weiermann [34] made a refinement in such a way that how fast heads of a hydra should be multiplied at cutting off the rightmost head, so that the Hydra battle game on the ordinals up to \( \varepsilon_0 \) remains unprovable in \( PA \). Using the same idea we show that an analogous process is possible with respect to the tree-ordinals from \( T \).

First we recall some well-known definitions and lemmata from subrecursive hierarchy theory based on the fundamental sequences defined in Definition 30. Let \( f, g \) range over unary arithmetical functions, \( k, n, x \) over \( \mathbb{N} \), and \( \alpha, \beta, \lambda, \epsilon \) etc. over \( \mathbb{T} \).

**Definition 38.** Let \( \lambda \in \text{Lim} \).

1. \( P^f \alpha := 0, P^f (\alpha + 1) := \alpha \) and \( P^f \lambda := P^f (\lambda[f(x)]) \).
2. \( Q^f \alpha := 0, Q^f (\alpha + 1) := \alpha \) and \( Q^f \lambda := \lambda[f(x)] \).
3. Let \( R \in \{ P, Q \} \).
   - \( R^{\upharpoonright}_\alpha := R^{\upharpoonright}_{\downarrow \alpha} \).
   - \( \alpha \succ^R f \beta \) if \( \beta = R^n f \cdots R^f_1 \).
   - \( \alpha \succ^R f \beta \) if \( \beta = R^n f \cdots R^f_1 \alpha \) for some positive \( n \).
• \( \alpha >_{f}^k \beta \) if \( \alpha >_{f}^R \beta \), where \( f \equiv k \).
• \( \alpha >_{f}^R \beta \) if \( \alpha >_{f}^R \beta \) or \( \alpha = \beta \).

(4) \( G_{1}(0) := 0, G_{1}(\alpha + 1) = G_{1}(\alpha) + 1 \) and \( G_{1}(\lambda) := G_{1}(\lambda[x]) \).

(5) \( H_{f}^{j}(x) := x, H_{f+1}^{j}(x) := H_{f}^{j}(x + 1) \) and \( H_{j}^{j}(x) := H_{f}^{j}(x) \).

(6) \( H_{\alpha} := H_{\alpha}^{0} \).

(7) \( mc(m) := m \) and \( mc(\alpha) := \max\{m_{1}, \ldots, m_{n}, mc(\alpha_{1}), \ldots, mc(\alpha_{n})\} \), where \( \alpha = \omega^{\alpha_{1}} \cdot m_{1} + \cdots + \omega^{\alpha_{n}} \cdot m_{n} \) such that \( \alpha_{i} > \alpha_{i+1} \) for each \( i < n \).

(8) \( \alpha_{0}(\beta) := \beta, \alpha_{n+1}(\beta) := \alpha_{n}(\beta) \) and \( \alpha_{n} := \alpha(1) \).

(9) \( \varepsilon_{0} := (\omega_{n+1})_{\alpha} \) and \( \varepsilon_{0}[k] := \omega_{k+1} \).

(10) \( H_{\varepsilon_{0}}(x) := H_{\varepsilon_{0}}[x](x) \).

Note that \( G_{1}(\omega_{n+1}) \geq 2_{n}(2) \).

**Theorem 39.** Let \( \alpha, \beta \in \mathbb{T} \).

(1) \( G_{\alpha} \) is increasing (strictly if \( \alpha \) infinite), and if \( \beta < \alpha[n] \), then \( G_{\beta}(n) < G_{\alpha}(n) \) for all \( n \) and \( G_{\alpha} \) eventually dominates \( G_{\beta} \).

(2) \( H_{\alpha} \) is strictly increasing, and if \( \beta < \alpha[n] \), then \( H_{\beta}(n) < H_{\alpha}(n) \) for all \( n \) and \( H_{\alpha} \) eventually dominates \( H_{\beta} \).

(3) \( H_{\alpha} \) is provably recursive in PA.

(4) Every provably recursive function in PA is dominated by \( H_{\alpha} \) for some \( \alpha \).

(5) \( H_{\varepsilon_{0}} \) is not provably recursive in PA.

**Proof.** See Fairtlough and Wainer [9]. \( \square \)

Given \( R \in \{P, Q\} \) set \( R_{x}^{(0)} \alpha := \alpha \) and \( R_{x}^{(i+1)} \alpha := R_{x}^{(i)} \alpha \). For \( \alpha \in \mathbb{T} \) and \( n \in \mathbb{N} \) let \( \alpha[\omega := n] \) be the natural number obtained by replacing every occurrence of \( \omega \) in \( \alpha \) with \( n \).

**Lemma 40.** Let \( R \in \{P, Q\} \).

(1) \( R_{x}(\alpha + \beta) := \alpha + R_{x} \beta \) for \( \beta \neq 0 \).

(2) \( \text{If } \alpha >_{x}^{P} \beta \text{ then } \alpha >_{x}^{Q} \beta \).

(3) \( \text{If } \beta >_{x}^{P} \gamma + \alpha \text{ then } \beta >_{x}^{Q} \gamma + \beta \).

(4) \( \text{If } \alpha >_{x}^{Q} \beta \text{ then } \alpha >_{x}^{Q} P_{x} \alpha \).

(5) \( \text{If } \lambda \text{ is a limit then } \lambda[x + 1] >_{Q}^{x} \lambda[x] \).

(6) \( \text{If } \lambda \text{ is a limit then } \lambda[x + 1] >_{Q}^{x+1} \lambda[x] + 1 \).

(7) \( \text{If } x > 0 \text{ then } \omega^{\alpha+1} >_{x} Q \omega^{\alpha} + \omega^{\alpha} \).

(8) \( \text{If } x > 0 \text{ then } \omega^{\alpha+1} >_{x} Q \omega^{\alpha} + 1 \).

(9) \( \text{If } x > 0 \text{ then } \omega_{n+1}(\alpha + 1) >_{x} Q \omega_{n+1}(\alpha) + \omega_{n+1} \).

(10) \( \text{If } \alpha > 0 \text{ then } \alpha >_{x+1}^{Q} P_{x} \alpha + 1 \).

(11) \( \alpha >_{x}^{Q} P_{x} \alpha \).

(12) \( \text{If } f, g \text{ are increasing, where } g(i) \leq f(i) \text{ for all } i, \text{ and } \alpha >_{f}^{R,m} \beta \text{, then } \alpha >_{f}^{R,n} \beta \text{ for some } n \geq m \).

(13) \( \text{If } \alpha >_{x}^{Q} P_{x}^{m} \gamma \text{ then } \alpha >_{x}^{P,n} \gamma \text{ for some } n \geq m \).

(14) \text{There are at most } G_{x+1}(\alpha) \text{ elements in } \{\beta < \alpha : mc(\beta) \leq x + 1\} \).

(15) \( \alpha[\omega := x + 1] \leq G_{x}(\alpha) \leq \alpha[\omega := x + 2] \).

(16) \( G_{x}(\alpha) = \min\{i : P_{x}^{(i)} \alpha = 0\} \).

(17) \( H_{\alpha}(x) = \min\{i : P_{x+i-1} \cdots P_{x} \alpha = 0\} + x, \text{ so } H_{\alpha}(x) = \min\{i \geq x : P_{i} \cdots P_{x} \alpha = 0\} + 1 \).

**Proof.** (1)–(16) are more or less obvious. (17) is proved in Fairtlough and Wainer [9]. \( \square \)

**Lemma 41.** Given \( n \in \mathbb{N} \) set \( g_{n}(i) := |i|_{n} \). Set also \( \beta := \omega_{n+1}(\lambda) + \omega_{n+1} \) where \( \lambda \in \mathbb{T} \) is a limit. Then there exists an \( i \geq H_{\alpha}(1) \) such that \( \beta >_{g_{n}}^{P_{x}} \omega_{n+1}(0) \).
Proof. Let $L := H_k(1) - 1 = \min\{i : P_i \cdots P_1 \lambda = 0\}$. By definition we have $g_n(i) \geq 1$ for all $i$. Further we obtain

$$
\begin{align*}
\beta &= \omega_{n+1}(\lambda) + \omega_{n+1} \\
&\geq_1^P \omega_{n+1}(\lambda) + P_1 \omega_{n+1} \\
&\geq_1^P \omega_{n+1}(\lambda) + P_1 P_1 \omega_{n+1} \\
&\geq_1^P \cdots 
\end{align*}
$$

Hence, there exists $i_0 \geq 2_n(2)$ such that $\beta >_1^{P,i_0} \omega_{n+1}(\lambda)$ since

$$\min\{n : P_1(n) \omega_{n+1} = 0\} = G_1 \omega_{n+1} \geq 2_n(2).$$

And for $i \geq i_0$ we have $2 \leq g_n(i)$. In addition, we have

$$
\begin{align*}
\omega_{n+1}(\lambda) &\geq_1^P \omega_{n+1}(P_1 \lambda + 1) \\
&\geq_2^P \omega_{n+1}(P_1 \lambda) + \omega_{n+1} \\
&\geq_2^P \omega_{n+1}(P_1 \lambda) + P_2 \omega_{n+1} \\
&\geq_2^P \omega_{n+1}(P_1 \lambda) + P_2 P_2 \omega_{n+1} \\
&\geq_2^P \cdots 
\end{align*}
$$

Therefore, there exists $i_0 \geq 2_n(2)$ such that $\beta >_1^{P,i_0} \omega_{n+1}(\lambda)$ since

$$\min\{k : P_2(k) \omega_{n+1} = 0\} = G_2 \omega_{n+1} \geq 3_n(3).$$

This process shows that given $k \leq L$ there is a sequence $\langle i_\ell \rangle_{\ell \leq k}$ such that

$$i_\ell \geq (\ell + 2)n(\ell + 2)$$

for all $\ell \leq k$ and $\beta >_1^{P,i_0+i_1+\cdots+i_k} \omega_{n+1}(P_k \cdots P_1 \lambda)$. The claim follows now from the fact that $i_0 + \cdots + i_L \geq L + 1 = H_k(1)$.

Lemma 42. Given $\alpha \in \mathbb{T} \cup \{e_0\}$ set $f_\alpha(i) := |i|_{H_\alpha^{-1}(i)}$. Then for every $\alpha := \omega_{n+1}(\alpha_n) + \omega_{n+1}$, $n \geq 2$, there is a $\delta \geq \omega_{n+1}(0)$ such that $\alpha >_1^{P,i} \delta$ for some $i \geq H_\alpha(1)$.

Proof. If $k \leq H_\alpha(1) := i_0$, then $H_{e_0}^{-1}(k) \leq H_{e_0}^{-1}(i_0) \leq n$. Hence $f_{e_0}(k) = |k|_{H_{e_0}^{-1}(k)} \geq |k| \alpha_n = g_n(k)$. By Lemma 41 there exists $\delta > \omega_{n+1}(0)$ such that $\alpha >_1^{P,i} \delta$. This implies that there is $i \geq i_0$ such that $\alpha >_1^{P,i} \delta$. \square

Theorem 43. Let $n \in \mathbb{N}$.

1. Let $g_n(i) := |i|_n$. Then PA $\not\vdash \forall k \exists m Q_1^{g_n} \cdots Q_m^{g_n} \omega_k = 0$.
2. Let $f(i) := |i|_{H_\alpha^{-1}(i)}$. Then PA $\not\vdash \forall k \exists m Q_1^f \cdots Q_m^f \omega_k = 0$.

Proof. It follows from the fact that the function $i \mapsto H_{\omega_n}(1)$ is not provably recursive in PA. Cf. Fairtlough and Wainer [9]. \square

The expected counterpart, i.e. provability, can be shown as follows.

Theorem 44. For $\alpha \in \mathbb{T}$ let $f_\alpha(i) := |i|_{H_\alpha^{-1}(i)}$. Then

$$\text{PRA} \vdash \forall k \exists m Q_1^{f_{\alpha,m}} \cdots Q_m^{f_{\alpha,m}} \omega_k = 0.$$

Proof. Assume that $k$ is large enough. How large $k$ should be will be obvious from the context. We claim that $Q_m^{f_{\alpha,m}} \cdots Q_1^{f_{\alpha,m}} \omega_k = 0$, where $m := 2_{H_{\omega_\alpha}(2)}(k)$. Assume otherwise. Since

$$mc(\omega_k[f_{\alpha}(1)] \cdots [f_{\alpha}(i)]) \leq f_{\alpha}(i) + 2 \leq f_{\alpha}(m) + 2$$


for every $i \leq m$ we have by Lemma 40(14) $m \leq G_{2+f_a(m)}(\omega_k)$. By Lemma 40(15)

$$m \leq (4 + |2H_{\omega^2}(k)|H_{\omega^2}(2H_{\omega^2}(2k)))_k(1)$$

$$\leq (4 + |2H_{\omega^2}(k)|H_{\omega^2}(k))_k(1)$$

$$= (5 + 2H_{\omega^2}(k) - H_{\omega^2}(k))_k(1)$$

$$< 2H_{\omega^2}(k) = m$$

for sufficiently large $k$. Contradiction! $\square$

Note that, for any $\alpha$, $\beta \in \mathbb{T}$, there is a $k \in \mathbb{N}$ such that

$$\min\{m : Q^f_{m\alpha} \cdots Q^f_{1\beta} = 0\} \leq \min\{m : Q^f_{m\alpha} \cdots Q^f_{1\omega_k} = 0\}.$$ 

The existence of such a $k$ is PA-provable.

**Theorem 45.** Let $\alpha \in \mathbb{T} \cup \{\varepsilon_0\}$.

1. **EWD**(inv) is PA-provable.
2. **EWD**(ga) is not PA-provable for $g_a(i) := |\cdot|_a$, $n \in \mathbb{N}$.
3. Let $f_a(i) := |\cdot|_{H^{-1}_\omega(\cdot)}$. Then **EWD**(f_a) is PA-provable iff $\alpha \in \mathbb{T}$.

**Proof.** Obvious by Theorems 35, 43 and 44. $\square$

It is now obvious to see that the same results hold for the worms in normal form. Define

$$\text{EWD}_{nf}(f) := \forall \alpha \in \text{NF}(W) \exists n (\alpha(f, n) = \emptyset).$$

**Theorem 46.** Let $\alpha \leq \varepsilon_0$ be an ordinal.

1. **EWD**_{nf}(inv) is PA-provable.
2. **EWD**_{nf}(ga) is not PA-provable for $g_a(i) := |\cdot|_a$, $n \in \mathbb{N}$.
3. Let $f_a(i) := |\cdot|_{H^{-1}_\omega(\cdot)}$. Then **EWD**_{nf}(fa) is PA-provable iff $\alpha < \varepsilon_0$.

### 3.3. Schütte and Simpson’s ordinal notation system

Another interesting ordinal notation system for $\varepsilon_0$ is introduced by Schütte and Simpson [27]. It is called $\pi_0(\omega)$ and a segment of $\pi(\omega)$ defined by letting out the addition and the function $\alpha \mapsto \omega^\omega$ in the construction of the ordinal notation system developed by Buchholz [5].

The new defined ordinal terms seem, at least for the author, so artificial that it would make no sense to say more about them other than their combinatorial property. Hence it is all the more meaningful to see that there is a canonical correspondence between them and worms.

In the following, we will proceed at first as in [27]. However with different access to the resulting notation system, i.e. we do not refer to the original collapsing functions any more. This seems to be somewhat more technical, but has the advantage that one can easily see the correspondence between Schütte and Simpson’s system and Beklemishev’s one.

In this section the small Greek letters $\alpha$, $\beta$, $\gamma$, . . . range over ordinals. We set $\Omega_0 := 0$ and, for $i > 0$, $\Omega_i$ the $i$-th infinite regular ordinal and $\Omega_{\omega_i} := \sup\{\Omega_i : i < \omega\}$.

**Definition 47.** We define $B^m_i(\alpha)$, $B_i(\alpha)$ and $\pi_i(\alpha)$ (by the main induction on $\alpha$ and the subsidiary induction on $m$):

- **(B1)** if $\gamma = 0$ or $\gamma < \Omega_i$, then $\gamma \in B^m_i(\alpha)$;
- **(B2)** if $i \leq j$, $\beta < \alpha$, $\beta \in B_j(\beta)$, and $\beta \in B^m_i(\alpha)$, then $\beta < B^m_i(\alpha)$;
- **(B3)** $B_1(\alpha) := \cup \{B^m_i(\alpha) : m < \omega\};$
- **(B4)** $\pi_i(\alpha) := \min\{\eta : \eta \notin B_i(\alpha)\}.$

**Lemma 48** (Schütte and Simpson [27]). (1) If $k < m$, then $B^k_i(\alpha) \subseteq B^m_i(\alpha)$.

(2) If $i \leq j$ and $\alpha \leq \beta$, then $B_i(\alpha) \subseteq B_j(\beta)$, $\pi_i(\alpha) \leq \pi_j(\beta)$. 
Let \( \pi \)

\( \Omega_1 \leq \pi_1 \alpha < \Omega_{i+1} \).

(4) If \( \gamma \in B_i(\alpha) \) and \( \gamma < \Omega_{i+1} \), then \( \gamma < \pi_1 \alpha \).

(5) If \( \alpha \in B_i(\alpha) \) and \( \alpha < \beta \), then \( \pi_1 \alpha < \pi_1 \beta \).

(6) If \( \alpha \in B_i(\alpha) \), \( \beta \in B_i(\beta) \), and \( \pi_1 \alpha = \pi_1 \beta \), then \( \alpha = \beta \).

**Definition 49.** \( \pi(\omega) \) is inductively defined as follows:

1. \( 0 \in \pi(\omega) \);
2. if \( \alpha \in \pi(\omega) \) and \( \alpha \in B_i(\alpha) \) then \( \pi_1 \alpha \in \pi(\omega) \).

To see that \( \pi(\omega) \) is a primitive recursive set we must be able to decide the relation \( \alpha \in B_i(\alpha) \) for \( \alpha \in \pi(\omega) \). For this we introduce an auxiliary concept of coefficients sets. The idea stems from Rathjen and Weiermann [23].

**Definition 50.** Inductive definition of a set of ordinals \( K_i(\alpha) \) for \( \alpha \in \pi(\omega) \).

1. \( K_i(0) := \emptyset \);
2. \( K_i(\pi_i(\alpha)) := \{ \alpha \} \cup K_i(\alpha) \) if \( i \leq j \),
   otherwise.

The following lemma can be shown by an simple induction.

**Lemma 51.** Let \( \alpha \in \pi(\omega) \). Then \( K_i(\alpha) \) is of the form \( \alpha \) if one of the following three cases holds:

\[ \alpha = 0 \text{ and } \beta \neq 0; \]
\[ \alpha = \pi_1 \delta, \beta = \pi_1 \gamma, \text{ and } i < j; \]
\[ \alpha = \pi_1 \delta, \beta = \pi_1 \gamma, \text{ and } \delta < \gamma. \]

**Proof.** (1) and (2) are obvious. (3) follows from Lemma 48.

Now it can be decided primitive recursively whether \( \alpha < \beta \), \( \alpha = \beta \), or \( \alpha > \beta \) for any \( \alpha, \beta \in \pi(\omega) \). In other words, \( \alpha < \beta \) can be read as a \( \Delta_0 \)-formula.

**Definition 53.** (1) We consider every element of \( \pi(\omega) \) as a term defined according to the induction and call it an ordinal term.

(2) \( \pi_0(\omega) \) is the set of ordinals from \( \pi(\omega) \) which are less than \( \Omega_1 \). That is,
\[ \pi_0(\omega) := \{ \alpha \in \pi(\omega) \mid \alpha = 0 \text{ or } \alpha = \pi_0 \beta \text{ for some } \beta \in \pi(\omega) \} = \pi_0 \Omega_\omega. \]

If we use the following abbreviations
\[ i_1 \cdots i_k 0 := \pi_{i_1} \cdots \pi_{i_k} 0, \]
then every \( \alpha \in \pi_0(\omega) \) is of the form \( \alpha = 0 \alpha_1 0 \cdots 0 \alpha_n 0^m 0 \) for some \( n, m \in \mathbb{N} \), where \( \alpha_i \in W_1 \). Note that, if \( n = 0 \), then \( \alpha = 0^m 0 \), hence \( \alpha = 0 \) if \( m = 0 \). The following lemma reveals something about the relationship between the elements of \( \pi_0(\omega) \) and \( NF(W) \).

**Lemma 54.** Let \( \alpha = 0 \alpha_1 0 \cdots 0 \alpha_n 0^m 0 \) be in \( \pi_0(\omega) \) with \( n > 1 \). If \( \alpha_i = \emptyset \) for some \( i, 1 \leq i < n \), then \( \alpha_{i+1} = \emptyset \).

**Proof.** Assume \( \alpha_i = \emptyset \) and \( \alpha_{i+1} \neq \emptyset \). Then \( \alpha \) has the form \( \pi_0 \cdots \pi_0 \pi_0 \pi_i \cdots 0 \) for some \( \ell > 0 \). However, this cannot be in \( \pi_0(\omega) \), since
\[ K_0(\pi_0 \pi_0 \pi_\ell \cdots 0) = \{ \pi_\ell \cdots \} \neq \pi_0 \pi_\ell \cdots 0. \]
Hence \( \alpha_{i+1} = \emptyset \).

\( \square \)
From now on we may assume for every $\alpha \in \pi_0(\omega)$ that $\alpha$ is of the form $\alpha = 0\alpha_10 \cdots 0\alpha_n0^m0$, where $\alpha_i \in W_1 \setminus \{0\}$ if $n \geq 1$.

**Definition 55.** (1) By a *functional* we mean a finite sequence $\gamma$ of natural numbers such that $\gamma 0 \in \pi(\omega)$.

(2) For $\alpha = i_1 \cdots i_n 0 \in \pi(\omega), n \geq 1$, define a functional $\bar{\alpha}$ by

$$\bar{\alpha} := \langle i_1 + 1 \rangle \cdots \langle i_n + 1 \rangle.$$ 

The following lemma can be proved by a simple induction.

**Lemma 56.** Let $\alpha, \beta > 0$ and $\gamma, \delta \in \pi_0(\omega)$.

1. $\bar{\alpha} \gamma \in \pi(\omega) \setminus \pi_0(\omega)$.
2. $\bar{\alpha} \gamma < \bar{\beta} \delta$ iff $\alpha < \beta$, or $\alpha = \beta$ and $\gamma < \delta$.
3. $K_{i+1}(\bar{\alpha} \gamma) < \bar{\beta} \gamma$ iff $K_i(\alpha) < \beta$.

**Lemma 57.** For every $\gamma \in \pi(\omega) \setminus \pi_0(\omega)$ there are uniquely determined $\alpha > 0$ and $\delta \in \pi_0(\omega)$ such that $\gamma = \bar{\alpha} \delta$. In fact, if $\gamma = \gamma_10\gamma_2$ with $\gamma_1 \in W_1$, then $\gamma = \bar{\alpha} 0\gamma_2$, where $\alpha := \gamma_1^{-1} 0$.

**Proof.** The uniqueness of $\alpha$ and $\delta$ follows from Lemma 56(2). It remains to show that $\alpha := \gamma_1^{-1} 0 \in \pi(\omega)$. We use the induction on the length of $\gamma_1$.

- $\gamma_1 = \langle i + 1 \rangle$. Then $\alpha = \pi i 0$ is obviously in $\pi_0(\omega)$.
- $\gamma_1 = \langle i + 1 \rangle \eta$, where $\eta \in \pi \setminus \pi_0(\omega)$ and $K_{i+1}(\eta) < \eta$. Then by I.H. $\beta := \eta^{-1} 0 \in \pi(\omega)$ with $\eta = \bar{\beta} 0 \gamma_2$, and $K_i(\beta) < \beta$ follows from $K_{i+1}(\eta) < \eta$. So $\alpha = i \beta \in \pi(\omega)$. \(\Box\)

**Lemma 58.** Let $\alpha, \beta > 0$ and $\delta \in \pi_0(\omega)$. If $K_0(\bar{\alpha} \delta) < \bar{\beta} \delta$, then $K_0(\alpha) < \beta$.

**Proof.** By induction on $\alpha$. If $\alpha = \pi 0 1$, then it is obvious since $\beta > 0$. Now let $\alpha = \pi i 1 \eta$ and $\eta > 0$. Then $\bar{\alpha} \delta = \pi_{i+1} \eta \delta$. Since $K_0(\bar{\alpha} \delta) < \bar{\beta} \delta$ we have $\eta \delta < \bar{\beta} \delta$ and $K_0(\bar{\eta} \delta) < \bar{\beta} \delta$. By I.H. $K_0(\eta) < \beta$, hence $K_0(\pi i 1 \eta) < \beta$. \(\Box\)

**Lemma 59.** Let $\alpha, \beta > 0$ such that $K_0(\alpha) < \beta$ and $\delta \in \pi_0(\omega)$. Then

$$K_0(\bar{\alpha} \delta) < \bar{\beta} \delta \iff \delta \in \pi_0 \bar{\beta} \delta.$$ 

**Proof.** If $K_0(\bar{\alpha} \delta) < \bar{\beta} \delta$, then $K_0(\delta) < \bar{\beta} \delta$. Hence $\delta \in B_0(\bar{\beta} \delta)$. By Lemma 48(4) we have $\delta \in \pi_0 \bar{\beta} \delta$. By Lemma 48(4) we have $\delta \in \pi_0 \bar{\beta} \delta$. Now assume $\delta < \pi_0 \bar{\beta} \delta$, i.e., $K_0(\delta) < \bar{\beta} \delta$. There are two cases.

- If $\alpha = \pi 0 1$, then $\bar{\alpha} \delta = \pi_{i+1} \delta$. Hence $K_0(\bar{\alpha} \delta) < \bar{\beta} \delta$.
- Let $\alpha = \pi i 1 \eta$, $\eta > 0$, and $\bar{\alpha} \delta = \pi_{i+1} \eta \delta$. Since $K_0(\alpha) < \beta$ then $\eta < \beta$ and $K_0(\eta) < \beta$. Hence $\eta \delta < \bar{\beta} \delta$ and $K_0(\bar{\eta} \delta) < \bar{\beta} \delta$ by I.H. \(\Box\)

**Definition 60.** Define $[\alpha]$ for $\alpha \in \pi_0(\omega)$ as follows.

1. $[0] := 0$;
2. if $\alpha = 0^n 1 0$ for some $m$, then $[\alpha] := 0 \bar{\alpha} = 0 1^{m+1}$;
3. if $\alpha = 0 \bar{\beta}$ with $\bar{\beta} \in \pi(\omega) \setminus \pi_0(\omega)$, then $[\alpha] := 0 \bar{\beta}$.

At first glance this definition seems to be somewhat different from the original one in [27]. But it is not because of Lemma 54.

**Lemma 61.** If $\alpha, \delta \in \pi_0(\omega)$, then $[\alpha] \delta \in \pi_0(\omega)$ iff $\delta < [\alpha] \delta$.

**Proof.** By induction on $\alpha$. If $\alpha = 0$, then $[\alpha] \delta = \pi 0 \delta$. Hence $[\alpha] \delta \in \pi_0(\omega)$ iff $K_0(\delta) < \delta$. This is exactly the case if $\delta < \pi_0 \delta = [\alpha] \delta$ because of Lemma 54.

Let $\alpha = \pi 0 \beta$, $K_0(\beta) < \beta$, and $[\alpha] \delta = \pi 0 \bar{\gamma} \delta$, where $\alpha = \gamma = 0^{n+1} 0$ for some $m$ by Lemma 54 if $\beta \in \pi_0(\omega)$, and $\gamma = \beta$ otherwise. In the first case, $\beta < \alpha$, hence $K_0(\beta) < \alpha$ and $K_0(\alpha) < \alpha$. Therefore, we have $K_0(\gamma) < \gamma$ in both cases. By Lemma 56(1) $\bar{\gamma} \delta \in \pi(\omega)$. Moreover, by Lemma 59 $K_0(\bar{\gamma} \delta) < \bar{\gamma} \delta$ iff $\delta < \pi_0 \bar{\gamma} \delta = [\alpha] \delta$. \(\Box\)

**Corollary 62.** $[\alpha]$ is a functional for every $\pi_0(\omega)$. 

The following characterization is obvious.

**Lemma 63.** Let \( \alpha, \beta, \gamma, \delta, [\alpha] \gamma \) and \([\beta] \delta \) be from \( \pi_0(\omega) \). Then \( [\alpha] \gamma < [\beta] \delta \) in exactly one of the following two cases:

\[ \alpha < \beta \quad \text{or} \quad (\alpha = \beta \text{ and } \gamma < \delta). \]

Note that if \( \alpha = 0\alpha_10\cdots0\alpha_n0^m0 \in \pi_0(\omega) \) and \( \alpha_i = 1\eta \), then \( \eta = 1^k \) for some \( k \). If not, we would have \( K_0(\alpha_10\cdots0\alpha_n0^m0) \neq \alpha_10\cdots0\alpha_n0^m0 \) which is not allowed. Hence the following lemma makes sense.

**Lemma 64.** For every \( \gamma \in \pi_0(\omega) \setminus \{0\} \) there are unique \( \alpha, \eta \in \pi_0(\omega) \) such that \( \gamma = [\alpha] \eta \). In fact, if \( \gamma = 0\beta0\delta \) and \( \beta \in W_1 \), then

\[ \gamma = \begin{cases} [0]0\delta & \text{if } \beta = \emptyset, \\ [\beta']0\delta & \text{otherwise}, \end{cases} \]

where \( \beta' := \begin{cases} \beta^-0 & \text{if } \beta = 1^k \text{ for some } k, \\ 0\beta^-0 & \text{if } \beta = j\eta \text{ for some } \eta \text{ and } j \geq 2. \end{cases} \)

**Proof.** The uniqueness follows from Lemma 63. Let

\[ \gamma' := \begin{cases} 0 & \text{if } \beta = \emptyset, \\ \beta' & \text{otherwise.} \end{cases} \]

We claim \( \gamma' \in \pi_0(\omega) \) and \( \gamma = [\gamma']0\delta \). In case of \( \beta = \emptyset \) it is obvious. Let \( \beta \neq \emptyset \). Since \( K_0(\beta0\delta) < \beta0\delta \), we have \( K_0(0\delta) < \beta0\delta \) and \( 0\delta < \beta0\delta \). Hence it holds that \( 0\beta^-0 \in \pi_0(\omega) \). If \( \beta = 1\eta \), then \( \beta^-0 = 0^m \) for some \( m \). So \( [\beta'] = [\beta^-0] = 0^m = 0\beta \). If \( \beta = j\eta \) for some \( \eta \) and \( j \geq 2 \), then \( [\beta'] = [0\beta^-0] = 0\beta^-+ = 0\beta \).

**Lemma 65.** Let \( n > 0 \) and \( \alpha = 0\alpha_10\cdots0\alpha_n0^m0 \) from \( \pi_0(\omega) \) with a non-empty \( \alpha_n \). Then \( \alpha' \geq \cdots \geq \alpha'_n \) and \( \alpha = [\alpha'_1] \cdots [\alpha'_n]0^m0 \).

**Proof.** The second claim is true by Lemma 64. For the first one, note that for every \( i < n \)

\[ [\alpha'_{i+1}] \cdots [\alpha'_n]0^m0 < [\alpha'_i] \cdots [\alpha'_n]0^m0 \]

by Lemma 61. The claim follows now by Lemma 63.

**Definition 66.** Define \( \hat{\sigma} : \pi_0(\omega) \to \varepsilon_0 \) by

\[ \hat{\sigma}(0\alpha_10\cdots0\alpha_n0^m) := \omega^\hat{\sigma}(\alpha'_1) + \cdots + \omega^\hat{\sigma}(\alpha'_n) + m, \]

where \( \alpha'_i \) is defined as in Lemma 64.

**Theorem 67.** \( \hat{\sigma} : \pi_0(\omega) \to \varepsilon_0 \) is an order-preserving isomorphism.

**Proof.** Define \( \tilde{g} : \varepsilon_0 \to \pi_0(\omega) \) by

\[ \tilde{g}(\omega^\alpha_1 + \cdots + \omega^\alpha_n + m) := 0\tilde{g}(\alpha'_1)0\cdots0\tilde{g}(\alpha'_n)0^m0, \]

where \( \alpha_1 \geq \cdots \geq \alpha_n > 0 \) and

\[ \beta'' := \begin{cases} (j + 1)\gamma^+ & \text{if } \beta = 0j\gamma0 \text{ and } j \geq 1, \\ 1^k & \text{if } \beta = 0^k0 \text{ for some } k. \end{cases} \]

Then we obviously have \( \tilde{g} \circ \hat{\sigma} = \hat{\sigma} \circ \tilde{g} = id \). Note only that \( (\alpha'')' = \alpha \) for every \( \alpha \in \pi_0(\omega) \) and \( (\beta'')'' = \beta \) for every \( \beta \in W_1 \). That \( \hat{\sigma} \) and \( \tilde{g} \) are order-preserving follows from Lemma 65.

\[ \square \]
Remark 68. It is somewhat interesting in the sense that the theorem above together with the definition of $\sigma$ gives simple and canonical order-preserving isomorphisms among $\pi_0(\omega)$, $NF(W)$, and $NF \subseteq S$. Indeed, $t_1: \pi_0(\omega) \rightarrow NF(W)$ and $t_2: \pi_0(\omega) \rightarrow NF$ are order-preserving isomorphisms:

$$t_1(0\alpha_10\cdots0\alpha_n0^m0) := \alpha_10\cdots0\alpha_n0^m$$

and

$$t_2(0\alpha_10\cdots0\alpha_n0^m0) := 0^m\alpha_n^*0\cdots0\alpha_1^*,$$

where $\beta^*$ is the converse of $\beta$.

4. A consequence of the structural equivalence

Having shown the structural equivalence, it is natural to expect that the behavior of each system in view of the slowly well-orderedness is the same. As in case of the Cantor system we need some norm functions. Let $lh(\alpha)$ be the length of the word $\alpha$ and $ht(\alpha)$ a maximal component of $\alpha^+$.

Definition 69. $^N\tilde{\eta}: \pi_0(\omega) \rightarrow \mathbb{N}$ and $^N\tilde{\eta}: NF(W) \rightarrow \mathbb{N}$ are defined as follows:

$$^N\tilde{\eta}(0\alpha_10\cdots0\alpha_n0^m0) := ^N\tilde{\eta}(\alpha_10\cdots0\alpha_n0^m) := m + n + 1 + \sum_{i=1}^{n} lh(\alpha_i) + \sum_{i=1}^{n} \sum_{k=0}^{n_i} a_{ik},$$

where $\alpha_i := a_{i0}\cdots a_{ini} \in W_1$.

Roughly speaking, $^N\tilde{\eta}\alpha$ and $^N\tilde{\eta}\alpha$ are the addition of the length of $\alpha$ and all of its components. Given $X \in \{\varepsilon, NF, NF(W), \pi_0(\omega)\}$ and $f: \mathbb{N} \rightarrow \mathbb{N}$ recall that $SWO(X, \subseteq X, f)$ is defined as follows:

for any $k$ there exists a constant $n$ which is so large that, for any finite sequence $\alpha_0, \ldots, \alpha_n$ from $X$ with $^N\tilde{\eta}\alpha_i \leq k + f(i)$ for all $i \leq n$, there exist indices $\ell < m \leq n$ satisfying $\alpha_\ell \subseteq X \alpha_m$.

Here $^N\tilde{\eta} \in \{N, ^N\tilde{\eta}, ^N\tilde{\eta}\}$ and $\subseteq X \in \{\leq, \leq_0, \leq\}$ depending on $X$.

Lemma 70. (1) Let $\alpha \in \pi_0(\omega)$. Then

$$N(\tilde{o}(\alpha)) \leq ^N\tilde{\eta}\alpha \quad \text{and} \quad ^N\tilde{\eta}(\alpha^+p) \leq (ht(\alpha) + p) \cdot N(\tilde{o}(\alpha)).$$

(2) Let $\alpha \in NF \cup NF(W)$. Then

$$N(\sigma(\alpha)) \leq ^N\tilde{\eta}\alpha \quad \text{and} \quad ^N\tilde{\eta}(\alpha^+p) \leq (ht(\alpha) + p) \cdot N(\sigma(\alpha)).$$

Proof. It suffices to show (2). We write just $N$ for $^N\tilde{\eta}$ without causing no confusions. Let $\alpha = 0^m\alpha_10\cdots0\alpha_n \in NF$. We show the claim by induction on the maximal component in $\alpha$. Note that $\sigma(\alpha) = \omega^{\sigma(\alpha^-)} + \cdots + \omega^{\sigma(\alpha^-)} + m$.

If $n = 0$ it is obvious. Now assume $n > 0$.

$$N(\sigma(\alpha)) = m + n + \sum_{i=1}^{n} N(\sigma(\alpha^-_i))$$

$$\leq m + n + \sum_{i=1}^{n} N\alpha^-_i \quad \text{(by I.H.)} \leq m + n - 1 + \sum_{i=1}^{n} N\alpha_i = N\alpha.$$

Further, we have

$$N\alpha^+p = N(\alpha^+_1p) + \cdots + N(\alpha^+_np) + p(m + n - 1) + (m + n - 1)$$

$$= N((\alpha^-_1)^+(p+1)) + \cdots + N((\alpha^-_n)^+(p+1)) + (p + 1)(m + n - 1)$$

$$\leq (ht(\alpha^-_1) + p + 1) \cdot N(\sigma(\alpha^-_i)) + \cdots + (ht(\alpha^-_n) + p + 1) \cdot N(\sigma(\alpha^-_i))$$

$$+ (p + 1)(m + n - 1) \quad \text{(by I.H.)}$$

$$\leq (ht(\alpha) + p)(N(\sigma(\alpha^-_1)) + \cdots + N(\sigma(\alpha^-_n))) + m + n - 1$$

$$= (ht(\alpha) + p) \cdot N(\sigma(\alpha)).$$

This completes the proof. □
This implies that the norm condition does not cause any essential difference in transformations between any two systems from $\varepsilon_0$, $\text{NF}$, $\text{NF}(W)$, or $\pi_0(\omega)$. Hence the following theorem is a direct consequence of Theorems 7 and 8.

**Theorem 71.** Let $\alpha \leq \varepsilon_0$.

1. $\text{SWO}(X, \sqsubseteq_X, f)$ is PRA-provable for $f(i) := |i| \cdot \text{inv}(i)$.
2. $\text{SWO}(X, \sqsubseteq_X, g_n)$ is not PA-provable for $g_n(i) := |i| \cdot |i|_n$.
3. Let $f_\alpha := |i| \cdot |i|_{H_\alpha(i)}$. Then $\text{SWO}(X, \sqsubseteq_X, f_\alpha)$ is PA-provable iff $\alpha < \varepsilon_0$.

Again let $X$ be one of the systems $\varepsilon_0$, $\text{NF}$, $\text{NF}(W)$, $\pi_0(\omega)$. Given a function $f : \mathbb{N} \rightarrow \mathbb{N}$ let $\text{EWD}(X, f)$ be defined as the Hydra game:

$$\text{EWD}(X, f) := \forall \alpha \in X \exists n \ (\alpha(f, n) = \emptyset).$$

**Theorem 72.** Let $\alpha \leq \varepsilon_0$.

1. $\text{EWD}(X, \text{inv})$ is PA-provable.
2. $\text{EWD}(X, g_n)$ is not PA-provable for $g_n(i) := |i|, n \in \mathbb{N}$.
3. Let $f_\alpha(i) := \cdot |i|_{H_\alpha(i)}$. Then $\text{EWD}(X, f_\alpha)$ is PA-provable iff $\alpha \in \varepsilon_0$.

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