Approximation and Fixed-Parameter Algorithms for Consecutive Ones Submatrix Problems

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Consecutive Ones Property (C1P)

A 0/1-matrix has the C1P if its columns can be permuted such that in each row the ones form a block.
Consecutive Ones Property (C1P)

Example for a matrix having the C1P:

\[
\begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 \\
\hline
1 & & & & 1 \\
1 & & & 1 & 1 \\
1 & 1 & & 1 & 1 \\
1 & 1 & 1 & 1 & \\
\end{array}
\]
Consecutive Ones Property (C1P)

Example for a matrix having the C1P:

\[
\begin{array}{ccccc}
1 & 2 & 3 & 4 & 5 \\
1 & 1 & & & 1 \\
1 & 1 & & & 1 \\
1 & 1 & & & 1 \\
\end{array}
\]

\[
\begin{array}{cccccc}
2 & 5 & 1 & 3 & 4 \\
1 & & 1 & & \\
1 & & 1 & & \\
1 & & 1 & & \\
\end{array}
\]
Consecutive Ones Property (C1P)

Examples for matrices **not** having the C1P:

1 | 1 | 0 | 0  
---|---|---|---
0 | 1 | 1 | 0  
1 | 0 | 1 | 1

1 | 1 | 0 | 0 | 0  
---|---|---|---|---
0 | 1 | 1 | 0 | 0  
0 | 0 | 1 | 1 | 0  
1 | 0 | 0 | 1 | 1

0 | 0 | 0 | 0 | 0  
---|---|---|---|---
0 | 0 | 0 | 1 | 1  
1 | 0 | 0 | 0 | 1

1 | 1 | 0 | 0 | 0  
---|---|---|---|---
0 | 1 | 1 | 0 | 0  
0 | 0 | 0 | 1 | 1  
1 | 0 | 1 | 0 | 1

Michael Dom, Universität Jena: Consecutive Ones Submatrix Problems
Consecutive Ones Property (C1P)

The Consecutive Ones Property...

- ...expresses “locality” of the input data.
- ...appears in many applications, e.g.
  - in railway system optimization
    [Ruf, Schöbel, Discrete Optimization, 2004; Mecke, Wagner, ESA '04],
  - bioinformatics
    [Christof, Oswald, Reinelt, IPCO '98; Lu, Hsu, J. Comp. Biology, 2003].

- ...can be recognized in polynomial time

- ...is subject of current research
Problem Definition

Min-COS-C (Min-COS-R)

*Given:* A matrix $M$ and a positive integer $k$.

*Question:* Can we delete at most $k$ columns (at most $k$ rows) such that the resulting matrix has the C1P?

Min-COS-C is NP-complete even on (2, 3)- and (3, 2)-matrices
Tan, Zhang, Algorithmica, 2007].

Min-COS-R is NP-complete even on (3, 2)-matrices
## Problem Overview

<table>
<thead>
<tr>
<th>(1’s per col, 1’s per row)</th>
<th>Max-COS-C</th>
<th>Min-COS-C</th>
</tr>
</thead>
<tbody>
<tr>
<td>(3, 2)</td>
<td>0.5-approx(^1)</td>
<td></td>
</tr>
<tr>
<td>(*, 2)</td>
<td>• No const. approx.(^1)</td>
<td></td>
</tr>
<tr>
<td>(*, (\Delta))</td>
<td>• No const. approx.(^1)</td>
<td></td>
</tr>
<tr>
<td>(2, 3)</td>
<td>0.8-approx(^1)</td>
<td></td>
</tr>
<tr>
<td>(2, *)</td>
<td>0.5-approx(^1)</td>
<td></td>
</tr>
<tr>
<td>((\Delta), *)</td>
<td></td>
<td></td>
</tr>
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\(^1\)[Tan, Zhang, Algorithmica, 2007]
## Problem Overview

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<tr>
<td>(*, 2)</td>
<td>● No const. approx.(^1)</td>
<td>● No 2,72-approx.</td>
</tr>
<tr>
<td></td>
<td>● W[1]-hard</td>
<td>● Problem kernel</td>
</tr>
<tr>
<td>(*, Δ)</td>
<td>● No const. approx.(^1)</td>
<td>● (Δ + 2)-approx.</td>
</tr>
<tr>
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<td>● (O((Δ + 2)^k \cdot Δ^O(Δ) \cdot</td>
</tr>
<tr>
<td>(2, 3)</td>
<td>0.8-approx(^1)</td>
<td></td>
</tr>
<tr>
<td>(2, *)</td>
<td>0.5-approx(^1)</td>
<td>● 6-approx</td>
</tr>
<tr>
<td></td>
<td></td>
<td>● (O(6^k \cdot \text{pol}(</td>
</tr>
<tr>
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\(^1\)[Tan, Zhang, Algorithmica, 2007]
## Problem Overview

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<td>(3, 2)</td>
<td>0.5-approx&lt;sup&gt;1&lt;/sup&gt;</td>
<td></td>
</tr>
</tbody>
</table>
| (∗, 2)                     | ● No const. approx.<sup>1</sup>  
● W[1]-hard               | ● No 2,72-approx.  
● Problem kernel          |
| (∗, ∆)                     | ● No const. approx.<sup>1</sup>  
● W[1]-hard               | ● (∆ + 2)-approx.  
● $O((\Delta + 2)^k \cdot \Delta^{O(\Delta)} \cdot |M|^{O(1)})$-alg. |
| (2, 3)                     | 0.8-approx<sup>1</sup> |           |
| (2, ∗)                     | 0.5-approx<sup>1</sup>  
|                           | ● 6-approx  
● $O(6^k \cdot \text{pol}(|M|))$-alg. |
| (∆, ∗)                     |           |           |

<sup>1</sup>[Tan, Zhang, Algorithmica, 2007]
Min-COS-C / Min-COS-R on \((\ast, \Delta)\)-Matrices

\[ M_{I_p}, \ p \geq 1 \]

\[ M_{II_p}, \ p \geq 1 \]

\[ M_{III_p}, \ p \geq 1 \]

\[ M_{IV} \]

\[ M_{V} \]

**Theorem:** A matrix has the C1P iff it contains none of the shown matrices.

[Tucker, Journal of Combinatorial Theory (B), 1972]
Min-COS-C / Min-COS-R on \((\ast, \triangle)-\text{Matrices}\)

\[
\begin{align*}
M_{I_p}, \ p \geq 1 & & \\
M_{II_p}, \ p \geq 1 & & \\
M_{III_p}, \ p \geq 1 & & \\
M_{IV} & & \\
M_{V} & & \\
\end{align*}
\]

Approach: Use a search tree algorithm.

Repeat:

1. Search for a “forbidden submatrix”.
2. Branch on which of its columns has to be deleted.
Min-COS-C / Min-COS-R on (∗, Δ)-Matrices

Search Tree Algorithm:

Finite size \( c \) of forbidden matrices \( \Rightarrow \) search tree of size \( O(c^k) \). (Alternatively: Factor-\( c \) approximation algorithm.)
Min-COS-C / Min-COS-R on \((*, \Delta)\)-Matrices

\[
\begin{array}{|c|c|}
\hline
p + 2 & p + 2 \\
\hline
1 & 1 & 0 & \cdots & 0 \\
0 & 1 & 1 & 0 & \cdots & 0 \\
& \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & 1 & 1 \\
1 & 0 & \cdots & 0 & 1 \\
\hline
\end{array}
\]

\(M_{I_p}, p \geq 1\)

\[
\begin{array}{|c|c|}
\hline
p + 3 & p + 3 \\
\hline
1 & 1 & 0 & \cdots & 0 \\
0 & 1 & 1 & 0 & \cdots & 0 \\
& \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & 1 & 1 \\
1 & 0 & \cdots & 1 & 0 \\
\hline
\end{array}
\]

\(M_{II_p}, p \geq 1\)

\[
\begin{array}{|c|c|}
\hline
p + 3 & p + 3 \\
\hline
1 & 1 & 0 & \cdots & 0 \\
0 & 1 & 1 & 0 & \cdots & 0 \\
& \cdots & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & 1 & 1 \\
0 & 1 & \cdots & 1 & 0 & 1 \\
\hline
\end{array}
\]

\(M_{III_p}, p \geq 1\)

\[
\begin{array}{|c|}
\hline
1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & 1 \\
\hline
\end{array}
\]

\(M_{IV}\)

\[
\begin{array}{|c|}
\hline
1 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 \\
1 & 1 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 & 1 \\
\hline
\end{array}
\]

\(M_{V}\)

A \((*, \Delta)\)-matrix can contain

- \(M_{I_p}\) with unbounded size,
- \(M_{II_p}\) with \(1 \leq p \leq \Delta - 2\),
- \(M_{III_p}\) with \(1 \leq p \leq \Delta - 1\),
- \(M_{IV}\), and \(M_{V}\).
Min-COS-C / Min-COS-R on ($\ast$, $\Delta$)-Matrices

Problem: Matrices $M_{I_p}$ of unbounded size can occur.
Min-COS-C / Min-COS-R on $(\ast, \Delta)$-Matrices

Problem: Matrices $M_{lp}$ of unbounded size can occur.

Idea: First destroy all “small” forbidden submatrices (search tree algorithm), and then see what happens...
Min-COS-C / Min-COS-R on \((\ast, \Delta)\)-Matrices

Algorithmic framework for Min-COS-C / Min-COS-R:

1. Destroy the submatrices from
\[
X := \{M_{I_p} \mid 1 \leq p \leq \Delta - 1\} \cup \{M_{II_p} \mid 1 \leq p \leq \Delta - 2\} \\
\cup \{M_{III_p} \mid 1 \leq p \leq \Delta - 1\} \cup \{M_{IV}, M_{V}\}.
\]

2. Destroy the remaining \(M_{I_p} (p \geq \Delta)\).

We show:

- We can find a submatrix from \(X\) in polynomial time.
- If a \((\ast, \Delta)\)-matrix \(M\) contains none of the matrices in \(X\) as a submatrix, then \(M\) can be divided into “independent” submatrices that have the “circular ones property (Circ1P)”.
- Min-COS-C / Min-COS-R can be solved in polynomial time on \((\ast, \Delta)\)-matrices with the Circ1P.
Min-COS-C / Min-COS-R on $(*, \Delta)$-Matrices

Algorithmic framework for Min-COS-C / Min-COS-R:

1. Destroy the submatrices from

$$X := \{M_{Ip} \mid 1 \leq p \leq \Delta - 1\} \cup \{M_{I\!I\!I}p \mid 1 \leq p \leq \Delta - 2\}$$
   $$\cup \{M_{I\!I\!I\!I}p \mid 1 \leq p \leq \Delta - 1\} \cup \{M_{IV}, M_{V}\}.$$  

2. Destroy the remaining $M_{Ip}$ ($p \geq \Delta$).

We show:

- **We can find a submatrix from $X$ in polynomial time.**
- If a $(*, \Delta)$-matrix $M$ contains none of the matrices in $X$ as a submatrix, then $M$ can be divided into “independent” submatrices that have the “circular ones property (Circ1P)”.
- Min-COS-C / Min-COS-R can be solved in polynomial time on $(*, \Delta)$-matrices with the Circ1P.
Min-COS-C / Min-COS-R on \((\ast, \Delta)-\text{Matrices}\)

Algorithmic framework for Min-COS-C / Min-COS-R:

1. Destroy the submatrices from

\[
X := \{M_{I_p} \mid 1 \leq p \leq \Delta - 1\} \cup \{M_{II_p} \mid 1 \leq p \leq \Delta - 2\} \\
\cup \{M_{III_p} \mid 1 \leq p \leq \Delta - 1\} \cup \{M_{IV}, M_V\}.
\]

2. Destroy the remaining \(M_{I_p} (p \geq \Delta)\).

We show:

- We can find a submatrix from \(X\) in polynomial time.

- If a \((\ast, \Delta)-\text{matrix}\) \(M\) contains none of the matrices in \(X\) as a submatrix, then \(M\) can be divided into “independent” submatrices that have the “circular ones property (Circ1P)”.

- Min-COS-C / Min-COS-R can be solved in polynomial time on \((\ast, \Delta)-\text{matrices}\) with the Circ1P.
Min-COS-C / Min-COS-R on ($\ast$, $\Delta$)-Matrices

If a ($\ast$, $\Delta$)-matrix $M$ contains none of the matrices in $X$ as a submatrix, then every component of $M$ has the *circular ones property* (Circ1P).

[Dom, Guo, Niedermeier, TAMC ’07]

Components of a matrix:

- $r_1$:
  - $c_1$: 1
  - $c_2$: 1
  - $c_3$: 0
  - $c_4$: 1
  - $c_5$: 0
  - $c_6$: 0
- $r_2$:
  - $c_1$: 0
  - $c_2$: 1
  - $c_3$: 0
  - $c_4$: 0
  - $c_5$: 0
  - $c_6$: 0
- $r_3$:
  - $c_1$: 0
  - $c_2$: 1
  - $c_3$: 1
  - $c_4$: 1
  - $c_5$: 0
  - $c_6$: 0
- $r_4$:
  - $c_1$: 0
  - $c_2$: 0
  - $c_3$: 0
  - $c_4$: 0
  - $c_5$: 0
  - $c_6$: 1

Graph representation of the components:

- $c_1$,
- $r_1$ connected to $c_4$,
- $c_5$,
- $r_4$ connected to $c_6$.
Min-COS-C / Min-COS-R on ($\ast, \Delta$)-Matrices

If a ($\ast, \Delta$)-matrix $M$ contains none of the matrices in $X$ as a submatrix, then every component of $M$ has the circular ones property ($\text{Circ1P}$).

[Dom, Guo, Niedermeier, TMC '07]

A 0/1-matrix $M$ has the Circ1P if its columns can be permuted such that in each row the 1's form a block when $M$ is wrapped around a vertical cylinder.
Min-COS-C / Min-COS-R on \((\ast, \Delta)\)-Matrices

If a \((\ast, \Delta)\)-matrix \(M\) contains none of the matrices in \(X\) as a submatrix, then every component of \(M\) has the \textit{circular ones property (Circ1P)}.  

[Dom, Guo, Niedermeier, TAMC ’07]

Proof by contraposition:

If a component \(B\) of a \((\ast, \Delta)\)-matrix does not have the Circ1P, then it contains one of the submatrices from \(X\).
Min-COS-C / Min-COS-R on \((\ast, \Delta)\)-Matrices

**Theorem:** Let \(M\) be a matrix and \(c_j\) be a column of \(M\). Form the matrix \(M'\) from \(M\) by complementing all rows with a 1 in column \(c_j\). Then \(M\) has the Circ1P iff \(M'\) has the C1P.

Min-COS-C / Min-COS-R on \((\ast, \Delta)\)-Matrices

**Theorem:** Let \(M\) be a matrix and \(c_j\) be a column of \(M\). Form the matrix \(M'\) from \(M\) by complementing all rows with a 1 in column \(c_j\). Then \(M\) has the Circ1P iff \(M'\) has the C1P.

Min-COS-C / Min-COS-R on \((*, \Delta)\)-Matrices

**Theorem:** Let \(M\) be a matrix and \(c_j\) be a column of \(M\). Form the matrix \(M'\) from \(M\) by complementing all rows with a 1 in column \(c_j\). Then \(M\) has the Circ1P iff \(M'\) has the C1P.

Min-COS-C / Min-COS-R on \((*, \Delta)\)-Matrices

Component \(B\) without circular ones property.
Min-COS-C / Min-COS-R on \((\ast, \Delta)\)-Matrices

Component \(B\) without circular ones property.
\[\Rightarrow \exists \text{ column } c \text{ such that } B' \text{ does not have the C1P.}\]
Min-COS-C / Min-COS-R on \((\ast, \Delta)\)-Matrices

Component \(B\) without circular ones property.

\[\Rightarrow \exists \text{ column } c \text{ such that } B' \text{ does not have the C1P}.\]

\[\Rightarrow \text{ There is a forbidden submatrix } A' \text{ in } B'.\]

\[
\begin{align*}
\text{Component } B & \quad \text{Component } B' \\
\begin{array}{c}
\text{A} \quad \text{A'} \\
\end{array} & \quad \text{c}
\end{align*}
\]
Min-COS-C / Min-COS-R on \((\ast, \Delta)\)-Matrices

Component \(B\) without circular ones property.
\[\Rightarrow \exists \text{ column } c \text{ such that } B' \text{ does not have the C1P}.\]
\[\Rightarrow \text{ There is a forbidden submatrix } A' \text{ in } B'.\]
\[\Rightarrow \text{ We can always find a submatrix from } X \text{ in } B.\]
Min-COS-C / Min-COS-R on \((\ast, \Delta)\)-Matrices

\[ M_{I_p}, p \geq 1 \]
\[ M_{II_p}, p \geq 1 \]
\[ M_{III_p}, p \geq 1 \]

\[ M_{IV} \]
\[ M_V \]

Case study 1: \(A'\) is an \(M_{IV}\), row 2 has been complemented.

Then we can find an \(M_V\) in \(B\).
Min-COS-C / Min-COS-R on \((\ast, \Delta)\)-Matrices

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{cccccc}
1 & 1 & 0 & \cdots & 0 \\
0 & 1 & 1 & 0 & \cdots \\
\vdots & & & & \\
0 & \cdots & 0 & 1 & 1 \\
1 & 0 & \cdots & 0 & 1
\end{array}
\end{array}
p + 2
\end{array}
\]
\(M_{I_p}, p \geq 1\)

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{cccccc}
1 & 1 & 0 & \cdots & 0 \\
0 & 1 & 1 & 0 & \cdots \\
\vdots & & & & \\
0 & \cdots & 0 & 1 & 1 \\
1 & \cdots & 1 & 0 & 1
\end{array}
\end{array}
\end{array}
p + 3
\end{array}
\]
\(M_{II_p}, p \geq 1\)

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{cccccc}
1 & 1 & 0 & \cdots & 0 \\
0 & 1 & 1 & 0 & \cdots \\
\vdots & & & & \\
0 & \cdots & 0 & 1 & 1 \\
0 & 1 & \cdots & 1 & 0 & 1
\end{array}
\end{array}
\end{array}
p + 3
\end{array}
\]
\(M_{III_p}, p \geq 1\)

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{cccc}
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0
\end{array}
\end{array}
\end{array}
\]
\(M_{IV}\)

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{cccc}
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \\
1 & 1 & 1 & 0
\end{array}
\end{array}
\end{array}
\]
\(M_{V}\)

Case study 2: \(A'\) is an \(M_{V}\), row 3 has been complemented.

\[
\begin{array}{ccccccc}
6 & 1 & 2 & 3 & 4 & 5 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 \\
2 & 0 & 0 & 0 & 1 & 1 & 0 \\
3 & 1 & 0 & 0 & 0 & 0 & 1 \\
4 & 0 & 1 & 0 & 1 & 0 & 1
\end{array}
\]

\[
\begin{array}{ccccccc}
0 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 1 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 1
\end{array}
\]

Then we can find an \(M_{IV}\) in \(B\).
Min-COS-C / Min-COS-R on \((\ast, \Delta)\)-Matrices

Most complicated case: \(A'\) is an \(M_{I_p}\) with \(p \geq \Delta\).

Then we can find an \(M_{III_1}\) or an \(M_{IV}\) in \(B\).
Min-COS-C / Min-COS-R on \((\ast, \Delta)\)-Matrices

Algorithmic framework for Min-COS-C / Min-COS-R:

1. Destroy the submatrices from

\[
X := \{ M_{lp} \mid 1 \leq p \leq \Delta - 1 \} \cup \{ M_{lp} \mid 1 \leq p \leq \Delta - 2 \} \\
\cup \{ M_{lp} \mid 1 \leq p \leq \Delta - 1 \} \cup \{ M_{IV}, M_{V} \}.
\]

2. Destroy the remaining \( M_{lp} \ (p \geq \Delta) \).

We show:

\begin{itemize}
  \item We can find a submatrix from \( X \) in polynomial time.
  \item If a \((\ast, \Delta)\)-matrix \( M \) contains none of the matrices in \( X \) as a submatrix, then \( M \) can be divided into “independent” submatrices that have the “circular ones property (Circ1P)”.
  \item Min-COS-C / Min-COS-R can be solved in polynomial time on \((\ast, \Delta)\)-matrices with the Circ1P.
\end{itemize}
From Circ1P to C1P

C1P: 1’s blockwise after column permutations
Circ1P: 1’s blockwise on a cylinder after column permutations
strong C1P: 1’s blockwise *without* column permutations
strong Circ1P: 1’s blockwise on a cylinder *without* column permutations

(Circ1P/C1P means: Strong Circ1P/strong C1P can be obtained by column permutations.)
From Circ1P to C1P

We imagine the matrices as wrapped around a vertical cylinder.

Strong Circ1P:

Strong C1P:
From Circ1P to C1P

We imagine the matrices as wrapped around a vertical cylinder.

Strong Circ1P:

Strong C1P:

Strong C1P =
Strong Circ1P + “cut”
From Circ1P to C1P

Our task:

\[
\begin{array}{c}
\text{strong Circ1P} \\
\text{column deletions} \\
\text{strong Circ1P + C1P}
\end{array}
\]
From Circ1P to C1P

Our task:

\[ \text{strong Circ1P} \rightarrow \text{column deletions} \rightarrow \text{strong Circ1P} + \text{C1P} \]

First consider this task:

\[ \text{strong Circ1P} \rightarrow \text{column deletions} \rightarrow \text{strong Circ1P} + \text{strong C1P} \]
From Circ1P to C1P

Our task:

\[
\text{strong Circ1P} \quad \rightarrow \quad \text{column deletions} \quad \rightarrow \quad \text{strong Circ1P + C1P}
\]

First consider this task:

\[
\text{strong Circ1P} \quad \rightarrow \quad \text{column deletions} \quad \rightarrow \quad \text{strong Circ1P + strong C1P}
\]

\textit{Obs.}: Deleting a consecutive set of columns is always optimal.
From Circ1P to C1P

Our task:

\[ \text{strong Circ1P} \implies \text{column deletions} \implies \text{strong Circ1P + C1P} \]

First consider this task: Easy!!!

\[ \text{strong Circ1P} \implies \text{column deletions} \implies \text{strong Circ1P + strong C1P} \]

We hope: Does “strong Circ1P + C1P” imply “strong C1P”?
From Circ1P to C1P

Conjecture: If a matrix has the strong Circ1P and the C1P, then it has also the strong C1P.
Conjecture: If a matrix has the strong Circ1P and the C1P, then it has also the strong C1P.

Counterexample:
**Conjecture**: If a matrix has the strong Circ1P and the C1P, then it has also the strong C1P.

Counterexample:
From Circ1P to C1P

Conjecture: If a matrix has the strong Circ1P and the C1P, then it has also the strong C1P.

Counterexample:

New conjecture: If a matrix with $\geq 2\Delta - 1$ columns has the strong Circ1P and the C1P, then it has also the strong C1P.
From Circ1P to C1P

*To be proven:* If a matrix with $\geq 2\Delta - 1$ columns has the strong Circ1P and the C1P, then it has also the strong C1P.

Very helpful:

*Theorem:* Let $M$ have the strong Circ1P. Then every column permutation that also yields the strong Circ1P can be obtained by a series of circular module reversals.

From Circ1P to C1P

*To be proven:* If a matrix with $\geq 2\Delta - 1$ columns has the strong Circ1P and the C1P, then it has also the strong C1P.

Very helpful:

*Theorem:* Let $M$ have the strong Circ1P. Then every column permutation that also yields the strong Circ1P can be obtained by a series of circular module reversals.

From Circ1P to C1P

To be proven: If a matrix with $\geq 2\Delta - 1$ columns has the strong Circ1P and the C1P, then it has also the strong C1P.

Very helpful:
Theorem: Let $M$ have the strong Circ1P. Then every column permutation that also yields the strong Circ1P can be obtained by a series of circular module reversals.

From Circ1P to C1P

To be proven: If a matrix with $\geq 2\Delta - 1$ columns has the strong Circ1P and the C1P, then it has also the strong C1P.

Very helpful:

Theorem: Let $M$ have the strong Circ1P. Then every column permutation that also yields the strong Circ1P can be obtained by a series of circular module reversals.

From Circ1P to C1P

*Now to be proven:* Let $M$ be a matrix with at least $2\Delta - 1$ columns that has the strong Circ1P and the strong C1P. Reversing an arbitrary circular module of $M$ does not affect these properties.
From Circ1P to C1P

Algorithm for Min-COS-C on matrices with Circ1P:

1. Permute the columns to get the strong Circ1P.
2. Search for a set of *consecutive* consecutive columns whose deletion yields the strong C1P.

[Dom, Niedermeier, ACiD '07]
Results for Min-COS-C and Min-COS-R

**FPT algorithm:**
Running time:
\[
(\left| \text{submatrix} \right|)^k \cdot (\text{search} + \text{“Circ1P→C1P” time})
\]
\[
(\Delta + 2)^k \cdot (n^{O(1)} + O(\Delta mn))
\]

**Approximation algorithm:**
Approximation factor: \( |\text{submatrix}| \)
Running time: \( k \cdot (\text{search} + \text{“Circ1P→C1P” time}) \)
Open Question

How can a matrix that has the (strong) Circ1P be modified by deleting a minimum number of 1-entries such that the resulting matrix has the C1P?
## More Open Questions

<table>
<thead>
<tr>
<th>(1’s per col, 1’s per row)</th>
<th>Max-COS-C</th>
<th>Min-COS-C</th>
</tr>
</thead>
<tbody>
<tr>
<td>(3, 2)</td>
<td>0.5-approx(^1)</td>
<td></td>
</tr>
</tbody>
</table>
| (*, 2)                     | ● No const. approx.\(^1\)  
 ● W[1]-hard               | ● No 2.72-approx.  
 ● Problem kernel         |           |
| (*, \(\Delta\))           | ● No const. approx.\(^1\)  
 ● W[1]-hard               | ● \((\Delta + 2)\)-approx.  
 ● \(O((\Delta + 2)^k \cdot \Delta^{O(\Delta)} \cdot |M|^{O(1)})\)-alg. |           |
| (2, 3)                     | 0.8-approx\(^1\) |           |
| (2, *)                     | 0.5-approx\(^1\)  
 | 6-approx  
 | \(O(6^k \cdot \text{pol}(|M|))\)-alg. |           |
| (\(\Delta, *\))           | ?         | ?         |

\(^1\)[Tan, Zhang, Algorithmica, 2007]
Jena, Germany
Min-COS-C on \((\ast, 2)\)-Matrices

Min-COS-C is equivalent to Induced Disjoint Paths Subgraph (IDPS).

**Induced Disjoint Paths Subgraph (IDPS)**

*Given:* A graph \(G\) and a positive integer \(k\).

*Question:* Can we delete at most \(k\) vertices of \(G\) such that the resulting graph is a vertex-disjoint disjoint union of paths?

\[
\begin{array}{cccc}
C_1 & C_2 & C_3 & C_4 \\
1 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 \\
0 & 1 & 1 & 0 \\
0 & 1 & 0 & 1 \\
\end{array}
\]
Min-COS-C on (\(\ast, 2\))-Matrices

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\]

\[
\begin{array}{c}
C_1 \\
C_2 \\
C_3 \\
C_4
\end{array}
\]
Problem Kernel for Min-COS-C on (∗, 2)-Matrices

**Problem Kernel:**
Given a parameterized problem instance \((X, k)\).
Transform it in polynomial time into an instance \((X', k')\)
with \(|X'| \leq f(k)\) and \(k' \leq k\).
**Problem Kernel for Min-COS-C on (∗, 2)-Matrices**

**Theorem:** IDPS with parameter $k$ admits a problem kernel with $O(k^2)$ vertices and $O(k^2)$ edges.

Data reduction rules:

1. If a degree-two vertex $v$ has two degree-at-most-two neighbors $u, w$ with $\{u, w\} \not\in E$, remove $v$ from $G$ and connect $u, w$ by an edge.

2. If a vertex $v$ has more than $k + 2$ neighbors, then remove $v$ from $G$, add $v$ to the solution, and decrease $k$ by one.
Problem Kernel for Min-COS-C on (∗, 2)-Matrices

- At most $k$ red vertices.
Problem Kernel for Min-COS-C on $(\ast, 2)$-Matrices

- At most $k$ red vertices.
- They have at most $k \cdot (k + 2)$ blue neighbors.
Problem Kernel for Min-COS-C on $(\ast, 2)$-Matrices

- At most $k$ red vertices.
- They have at most $k \cdot (k + 2)$ blue neighbors.
- At least every third blue vertex must be a neighbor of a red vertex.
Problem Kernel for Min-COS-C on $(\ast, 2)$-Matrices

- At most $k$ red vertices.
- They have at most $k \cdot (k + 2)$ blue neighbors.
- At least every third blue vertex must be a neighbor of a red vertex.

$\Rightarrow k + 3 \cdot k \cdot (k + 2)$ vertices.
$\Rightarrow k \cdot (k + 2) + 3 \cdot k \cdot (k + 2) - 1$ edges.