

One-Switch Independence for Multiattribute Utility Functions

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Assessment of multiattribute utility functions is significantly simplified if it is possible to decompose the function into more manageable pieces. Utility independence is a powerful property that serves well for this purpose, but if it is not appropriate in a given situation, what options does the analyst have? We review some possibilities and propose a new independence assumption based on the one-switch property. We argue that it is a natural generalization of utility independence and show how it leads to tractable multiattribute utility functions.

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1. Introduction

Although the early uses of utility functions in the sense of von Neumann and Morgenstern (1947) focused on money, interest in multiattribute utility functions, in which outcomes are described by multiple objectives, quickly grew. It was soon realized that the assessment of a two-attribute utility function, $u(x, y)$, can be a challenge if there is no decomposition structure that can be assumed. Utility independence (Keeney and Raiffa 1976) is an assumption that allows $u(x, y)$ to be decomposed into single-attribute functions. An attribute X is *utility independent* of another, Y , if preferences for gambles over X , for a fixed level of Y , do not depend on that fixed level.

If X is utility independent of Y , then for any fixed value of Y , say y_1 , there exist functions $f(y)$ and $g(y) > 0$ such that

$$u(x, y) = f(y) + g(y)u(x, y_1).$$

This holds because any two utility functions of X that have the same rank order for gambles must be linearly related.

Let X be relevant over a range x^0 to x^* and Y over a range y^0 to y^* and assume that $u(x^*, y) > u(x^0, y)$ for all y and $u(x, y^*) > u(x, y^0)$ for all x . It will prove convenient to define a conditional utility function

$$u(x | y) = \frac{u(x, y) - u(x^0, y)}{u(x^*, y) - u(x^0, y)} \quad (1)$$

and similarly $u(y | x)$, distinguishing between the two by the order of the attributes.

If X is utility independent of Y then $u(x | y)$ does not depend on the particular value of y . We may write this condition using (1) as

$$u(x, y) = u(x^0, y) + [u(x^*, y) - u(x^0, y)]u(x | y^0). \quad (2)$$

If X and Y are mutually utility independent, then

$$u(x, y) = u(x, y^0) + u(x^0, y) + cu(x, y^0)u(x^0, y), \quad (3)$$

for some constant, c , or alternatively,

$$u(x, y) = a_1u(x | y^0) + a_2u(y | x^0) + (1 - a_1 - a_2)u(x | y^0)u(y | x^0), \quad (4)$$

where $a_1 = u(x^*, y^0)$ and $a_2 = u(x^0, y^*)$. Mutual utility independence therefore reduces the assessment into two single-attribute functions and some scaling constants.

But what if utility independence does not hold? Several authors have discussed this issue and presented methods to incorporate preference dependence between the attributes (see, for example, Fishburn 1974, 1977; Farquhar 1977; Bell 1979; Kirkwood 1976; Abbas and Howard 2005; and Abbas 2009). Fishburn (1974) suggested a decomposition (“Model 5”) of the form

$$u(x, y) = f(x) + g(y) + h(x)j(y) \quad (5)$$

and a “generalized multiplicative form” (Fishburn 1977) of

$$u(x, y) = f(x)j(y) + g(y)h(x). \quad (6)$$

Bell (1979) suggested an approximation of the conditional utility function based on interpolation,

$$u(x | y) = f(y)u(x | y^0) + (1 - f(y))u(x | y^*) \quad (7)$$

and

$$u(y | x) = g(x)u(y | x^0) + (1 - g(x))u(y | x^*). \quad (8)$$

These two equations, if simultaneously appropriate, yield the form

$$\begin{aligned} u(x, y) = & a_1 u(x | y^0) + a_2 u(y | x^0) - k u(x | y^0) u(y | x^0) \\ & + (k - a_1) u(x | y^0) u(y | x^*) + (k - a_2) u(y | x^0) \\ & \cdot u(x | y^*) + (1 - k) u(x | y^*) u(y | x^*), \end{aligned} \quad (9)$$

where a_1, a_2 are as defined above and k is an arbitrary constant.

The appeal of (5), (6), (7), (8), and (9) is evident to a quantitatively trained analyst, but the charm of utility independence is that it is also readily understood by decision makers.

Of course, one can develop independence tests for (5)–(9) (e.g., (5) is “bilateral independence” and (7) is “interpolation independence”), but these tests are likely to be difficult for lay decision makers to answer with much confidence.

In this paper we introduce a new independence assumption that we believe is (i) a natural generalization of utility independence in terms of the number of preference switches that may occur, (ii) should be understandable by decision makers, and (iii) is easily assessed. We also show how it provides a new interpretation for the functional forms (5)–(9).

2. Utility Independence and the Zero-Switch Condition

Bell (1988) introduced the notion of categorizing single-attribute utility functions according to the maximum number of times that two gambles can switch in preference as the decision maker’s wealth increases. For example, if there are two wealth levels $w_1 < w_2$ such that for two gambles \tilde{x}_1 and \tilde{x}_2 we have

$$\begin{aligned} \tilde{x}_1 > \tilde{x}_2 \quad w < w_1 \\ \tilde{x}_1 < \tilde{x}_2 \quad w_1 < w < w_2 \\ \tilde{x}_1 > \tilde{x}_2 \quad w > w_2, \end{aligned} \quad (10)$$

then \tilde{x}_1 and \tilde{x}_2 switch twice. An n -switch utility function is one that permits a maximum of n switches.

The linear and exponential utility functions are the only zero-switch utility functions, that is, the decision maker ranks all gambles independently of wealth (Pfanzagl 1959).

It is the purpose of this paper to suggest that the n -switch concept may be useful in characterizing preference relations *between* attributes and thereby deriving the functional form of the multiattribute utility function. For a general two-attribute utility function $u(x, y)$, we will say that X is n -switch independent of Y if two gambles \tilde{x}_1 and \tilde{x}_2 can

switch in preference at most n times as Y progresses from its lowest to its highest value.

2.1. Conditional Utility Independence and the Zero-Switch Condition

As we mentioned in the introduction, if X is utility independent of Y , then the utility function can be decomposed as

$$u(x, y) = u(x^0, y) + [u(x^*, y) - u(x^0, y)]u(x | y^0). \quad (11)$$

We now make the following definition.

DEFINITION 1. An attribute X exhibits *zero-switch independence* from another attribute Y if preference for gambles on X is unchanged as the fixed level of Y increases.

It should be clear that X exhibits zero-switch independence from Y if and only if it is utility independent of Y , because both conditions forbid changes in preference over gambles on X as Y varies. It is this equivalence that forms the basis for our claim that the one-switch condition is a “natural generalization of utility independence.”

We have discussed the zero-switch independence implications of Equation (11) for attribute X , but what about its implications for attribute Y ? Refer back to Equation (11) and consider two lotteries $\tilde{y}_1 > \tilde{y}_2$ at a fixed x^1 . How do preferences between these lotteries over Y change as we vary X ?

The difference in expected utility is

$$\begin{aligned} & u(x^1, \tilde{y}_1) - u(x^1, \tilde{y}_2) \\ & = [u(x^0, \tilde{y}_1) - u(x^0, \tilde{y}_2)] \\ & \quad + \{ [u(x^*, \tilde{y}_1) - u(x^0, \tilde{y}_1)] - [u(x^*, \tilde{y}_2) - u(x^0, \tilde{y}_2)] \} \\ & \quad \cdot u(x^1 | y^0). \end{aligned}$$

The right-hand side depends on x^1 only through $u(x^1 | y^0)$. If $u(x | y^0)$ is a monotonic function of x , then the left-hand side can equal zero at most once as we vary x^1 . Therefore, with some monotonicity conditions, the utility independence of X on Y implies *both* that preferences for lotteries over X do not change as we vary Y *and* that preferences for lotteries over Y change at most once as we vary X . We have proved the following result.

PROPOSITION 2.1. *If marginal preferences are monotone, then zero-switch independence of X from Y implies one-switch independence of Y from X .*

This proposition is actually true in general; n -switch independence of X from Y implies $(n + 1)$ -switch independence of Y from X with some monotonicity conditions.

3. Utility Dependence and the One-Switch Condition

Think about a young couple buying their first home. In selecting among available options they consider many

factors, some relating to their likely enjoyment of a particular home (comfort level after settling in, having good neighbors, the quality of the schools, construction plans in the area, . . . etc.), and some relating to the affordability of such a home (future income, mortgage payments, maintenance costs, and so on). We can represent their concerns with two high-level attributes, “Home Quality” (X), a constructed scale that accounts for all benefits associated with owning a home, and “Wealth” (Y), a measure of the financial capacity of the couple (current assets plus future income potential). For most people, a home purchase does represent a gamble: we don’t know how much we will like the home we buy, what it will cost to maintain, and how reliable our income will be.

As a first step in assessing a utility function over X and Y , we ask whether X is utility independent of Y . It seems likely that the answer would be no. Assuming that several houses are in the consideration set, most couples will buy a different house if they are wealthy than if they are poor.

Consider two particular houses, A and B. Our couple thinks A is a nicer house than B, but it is also more expensive. If they can afford it, they would definitely prefer to buy A, but they may have to settle for B depending on their financial situation. As a thought experiment, we ask them to consider their relative preference for A and B as a function of their wealth level, assumed to be known with certainty. Is it the case that there is some critical wealth level such that below that level they would prefer B, and above that they would prefer A? If this is the case, and if this condition would hold for all pairs of houses (although with different critical wealth levels), then we have established that X is one-switch independent of Y .

Note that we do not require that house preferences be stable with respect to wealth over the couple’s lifetime. It might be that later in life our couple might be looking to downsize and have entirely different preferences than they did earlier.

DEFINITION 2. An attribute X exhibits *one-switch independence* from another attribute Y , written as X IS Y , if preference between any pair of gambles on X can switch at most once as the level of Y increases.

Note that this definition implies, for convenience, that if X exhibits zero-switch independence of Y , then it also exhibits one-switch independence of Y , because the maximum number of switches is what matters. The following result characterizes the utility functions that obey one-switch independence.

THEOREM 1. X is one-switch independent of Y if and only if

$$u(x, y) = g_0(y) + f_1(x)g_1(y) + f_2(x)g_2(y), \tag{12}$$

where $g_1(y) \neq 0$ has constant sign, and $g_2(y) = g_1(y)\phi(y)$ for some monotonic function ϕ .

The functions g_0, g_1, g_2, f_1, f_2 can be expressed directly in terms of a normalized multiattribute utility function as per the following proposition.

PROPOSITION 3.1. X is one-switch independent of Y if and only if

$$u(x, y) = g_0(y) + g_1(y)[f_1(x) + f_2(x)\phi(y)], \tag{13}$$

where

$$g_0(y) = u(x^0, y), \quad g_1(y) = [u(x^*, y) - u(x^0, y)] > 0,$$

$$f_1(x) = u(x | y^0), \quad f_2(x) = u(x | y^*) - u(x | y^0)$$

and

$$\phi(y) \triangleq \frac{u(x | y) - u(x | y^0)}{u(x | y^*) - u(x | y^0)} \tag{14}$$

is a monotone function of y and independent of x .

We will discuss the assessment of ϕ in the next subsection. Viewed as a functional approximation, (13) may be a little intimidating, but the *decision maker* need only introspect about the following fairly simple question.

Test for One-Switch Independence. X is one-switch independent of Y , if, for any two gambles on X , either (i) one is preferred to the other for all values of Y , (ii) they are indifferent for all levels of Y , or (iii) there exists a value of Y, y^* such that one gamble is always preferred for values less than y^* and the other is always preferred for values of Y greater than y^* .

The one-switch independence property extends to lotteries involving more than one attribute by thinking of the variables of the lottery as a partition. For example, if preferences between pairs of joint lotteries over X, Y can change only once as we change a third attribute Z , then we may use the previous results to write

$$u(x, y, z) = g_0(z) + g_1(z)[f_1(x, y) + f_2(x, y)\phi(z)]. \tag{15}$$

Similarly, we may envision various combinations of one-switch independence assumptions over a variety of subsets.

PROPOSITION 3.2. If X is n -switch independent of Y , then there exist some functions f_i, g_i such that

$$u(x, y) = g_0(y) + \sum_{i=1}^{n+1} f_i(x)g_i(y).$$

3.1. Assessment of a One-Switch Independent Utility Function

The assessment of a one-switch independent utility function requires five single-attribute utility assessments, two scaling constants, and two verification tests. As we now show, all of the assessments require no greater level of cognitive complexity than those of the case of utility independence; only single-attribute utility functions such as $u(y | x^0)$ and

some scaling constants, obtained using indifference assessments, are involved. See Keeney and Raiffa (1976) for more information on assessment techniques for these types of single-attribute functions and these scaling constants.

The functions $f_1(x) = u(x | y^0)$, $f_2(x) = u(x | y^*) - u(x | y^0)$ in Proposition 3.1. require only two normalized conditional utility functions, $u(x | y^0)$ and $u(x | y^*)$. The functions $g_0(y) = u(x^0, y)$, $g_1(y) = [u(x^*, y) - u(x^0, y)]$ require two normalized conditional utility functions, $u(y | x^0)$ and $u(y | x^*)$, and two normalizing constants, $a_1 = u(x^*, y^0)$ and $a_2 = u(x^0, y^*)$ because

$$g_0(y) \stackrel{\Delta}{=} u(x^0, y) = a_2 u(y | x^0) \quad \text{and} \quad (16)$$

$$u(x^*, y) = a_1 + (1 - a_1)u(y | x^*). \quad (17)$$

To assess $\phi(y)$, we can pick any intermediate value of x that suits the decision maker, say \hat{x} , and assess the function $u(\hat{x} | y)$, which also gives us the parameters $u(\hat{x} | y^0)$ and $u(\hat{x} | y^*)$ which we substitute into (14). If the decision maker finds it more convenient to assess $u(y | \hat{x})$ instead of $u(\hat{x} | y)$ (although both are simply a collection of indifference judgments), then we may use the two equations

$$u(\hat{x}, y) = u(x^0, y) + [u(x^*, y) - u(x^0, y)]u(\hat{x} | y) \quad (18)$$

and

$$u(\hat{x}, y) = u(\hat{x}, y^0) + [u(\hat{x}, y^*) - u(\hat{x}, y^0)]u(y | \hat{x}) \quad (19)$$

to convert $u(y | \hat{x})$ into $u(\hat{x} | y)$ using the additional scaling constants $u(\hat{x}, y^0)$, $u(\hat{x}, y^*)$.

Note further that (18) may be expressed as $u(\hat{x}, y) = g_0(y) + g_1(y)u(\hat{x} | y)$, which when equated with (19) relates the two assessments of $u(y | \hat{x})$ and $u(\hat{x} | y)$ in terms of the previously assessed functions $g_0(y)$, $g_1(y)$ and the normalizing constants $u(\hat{x}, y^*)$, $u(\hat{x}, y^0)$.

Two verification tests are now required: (i) we need to verify that the assessed $g_1(y)$ does not change sign, and (ii) that $\phi(y)$ is monotonic. Of course, $\phi(y)$ is monotonic if $u(\hat{x} | y)$ is, because the remaining terms in $\phi(y)$ are constants.

EXAMPLE. The following example illustrates a stylized assessment of a one-switch utility function. Consider two attributes X, Y that are relevant over the ranges $x^0 = y^0 = 0$, $x^* = y^* = 1$, and suppose we pick the intermediate point $\hat{x} = \frac{1}{2}$. For simplicity, let us suppose that $u(x | y^0) = x$, $u(x | y^*) = x^2$, and $u(y | x^0) = y$, $u(y | x^*) = y^2$. Suppose that we also assess $a_1 = u(x^*, y^0) = 0.4$ and $a_2 = u(x^0, y^*) = 0.3$. Then we may calculate the functions $g_0(y) = 0.3y$ and $g_1(y) = 0.4 + 0.6y^2 - 0.3y$, both of which are positive on the interval $[0, 1]$. Finally, suppose we assess $u(\hat{x} | y) = 0.5 - 0.25y$, which implies, in particular, that $u(\hat{x} | y^0) = 0.5$, $u(\hat{x} | y^*) = 0.25$. We may calculate $\phi(y) = ([0.5 - 0.25y] - 0.5)/(0.25 - 0.5) = y$, which is monotonic, leading to the one-switch independent utility function, using (13) and (14), $u(x, y) = 0.3y + [0.4 + 0.6y^2 - 0.3y][x + (x^2 - x)y]$.

Using (18) and (19), we may calculate that $u(y | \hat{x}) = 2y + 15y^2 - 6y^3/11$, a function that looks a little intimidating (compared to our stylized assessments) but that is in fact monotonic and well-behaved on the interval $[0, 1]$.

4. Monotone Dependence

Recall that the conditional utility function $u(x | y)$ is a rescaling of $u(x, y)$, where $u(x^* | y) = 1$ and $u(x^0 | y) = 0$ for all y . If X is utility independent of Y , then $u(x | y)$ is independent of y and thus a constant function as y varies. If X is not utility independent of Y , it might be reasonable to think that the next level of complexity in the relationship between X and Y is that $u(x | y)$ depends monotonically on y , i.e., it is some monotonic function of y . In this section we investigate the relationship between the various independence conditions we have discussed and this desirable notion of monotonicity.

DEFINITION 3. We will say that X is *monotonically dependent* on Y if $u(x | y)$ is strictly monotonic (or strictly constant) in y for each x .

The following definition applies monotonicity to interpolation.

DEFINITION 4. We will say that X is *monotonically interpolation independent* of Y if it is interpolation independent of Y , i.e., $u(x | y) = f(y)u(x | y^*) + (1 - f(y))u(x | y^0)$ and f is a monotonic function.

PROPOSITION 4.1. If X is *monotonically interpolation independent* of Y , then X is *monotonically dependent* on Y .

PROPOSITION 4.2. If X is *one-switch independent* of Y , then X is *monotonically dependent* on Y .

One-switch independence is a condition about differences in the expected utility of uncertain lotteries, which is to say it is concerned with terms like $\text{Eu}(\tilde{x}_1 | y) - \text{Eu}(\tilde{x}_2 | y)$, which are not necessarily monotonic in y even if X is monotonically dependent on Y . We now state the following definition.

DEFINITION 5. X is *monotonically trade-off dependent* on Y if, for all gambles \tilde{x}_1 and \tilde{x}_2 , $\text{Eu}(\tilde{x}_1 | y) - \text{Eu}(\tilde{x}_2 | y)$ is strictly monotonic (or strictly constant) in y .

Clearly if X is monotonically trade-off dependent on Y then X must also be one-switch independent of Y .

The following surprising result is central to our claim that our one-switch condition is a natural generalization of utility independence.

THEOREM 2. The following are equivalent:

- (i) X is *one-switch independent* of Y ;
- (ii) X is *monotonically trade-off dependent* on Y ; and
- (iii) X is *monotonically interpolation independent* of Y .

Theorem 2 illustrates a few important results. First, one-switch independence is equivalent to the interpolation independence condition when the function f is monotonic. Second, one-switch independence is also equivalent to the notion of monotone trade-off dependence. This result provides an interpretation of Definitions 4 and 5 in terms of the new notion of one-switch independence. It also provides a necessary and sufficient condition for them to hold.

A practical value of these properties also stems from the implications if they do *not* hold. If X is not monotonically trade-off dependent on Y , this means that preferences for X depend on Y in quite a complicated way, in particular, complicated enough that some gambles will switch twice as Y varies. This might suggest that the relationship between X and Y is too complex for the decision maker to give coherent judgments about, and may also be too complex for the analyst to capture. This may be a signal that the analyst needs to reformulate the attributes (Keeney 1992).

5. Mutual One-Switch Independence

Of course it may be that one-switch independence makes sense in both directions: X 1S Y and Y 1S X , which we write as X MIS Y .

Because X 1S Y is equivalent to X monotonically interpolation independent of Y (when the function ϕ in (14) is monotonic), then X MIS Y implies that X is mutually interpolation independent of Y , and thus can be decomposed as (9), together with monotonicity conditions in both directions. The following theorem provides a functional characterization of X MIS Y in terms of the required utility assessments.

THEOREM 3. *X MIS Y if and only if Equation (9) holds and the following ratios are monotonic:*

$$\phi_y(y) \triangleq \frac{(k - a_2)u(y | x^0) + (1 - k)u(y | x^*)}{a_1 + (1 - a_1)u(y | x^*) - a_2u(y | x^0)} \quad (20)$$

and

$$\phi_x(x) \triangleq \frac{(k - a_1)u(x | y^0) + (1 - k)u(x | y^*)}{a_2 + (1 - a_2)u(x | y^*) - a_1u(x | y^0)} \quad (21)$$

For an easy-to-construct example of mutual one-switch independence, suppose, as before, that $u(x | y^0) = x$ and $u(x | y^*) = x^2$ on a domain of interest $0 < x < 1$. Then (21) is $(k - a_1)x + (1 - k)x^2 / (a_2 + (1 - a_2)x^2 - a_1x)$, which is monotonic if the numerator of its derivative

$$(a_2 + (1 - a_2)x^2 - a_1x)[(k - a_1) + 2(1 - k)x] - ((k - a_1)x + (1 - k)x^2)[2(1 - a_2)x - a_1]$$

does not change sign. This simplifies to $a_2(k - a_1) + 2a_2(1 - k)x - [k - k(a_1 + a_2) + a_1a_2]x^2$ not changing sign over the interval $0 < x < 1$, which is satisfied when $a_1 < k < 2a_2$. If we make similar assumptions in the other direction, then we also require $a_2 < k < 2a_1$. If the assessed constants do not satisfy these constraints, then X MIS Y will not be satisfied. The decision maker would need to reconsider that assumption or reconsider the assessment of the constants.

Two special cases of X MIS Y are worth highlighting.

THEOREM 4. *The following decompositions satisfy mutual one-switch independence:*

$$u(x, y) = f_1(x) + f_2(y) + \phi_1(x)\phi_2(y); \quad (22)$$

$$u(x, y) = u_1(x)u_2(y)[1 + \phi_1(x)\phi_2(y)], \quad (23)$$

where u_1, u_2 have constant sign and ϕ_1 and ϕ_2 are strictly monotonic.

These are Fishburn’s decompositions (compare (12) and (5), (13) and (6)), but with added monotonicity requirements.

In general, whereas utility independence permits an attribute to be represented by one marginal utility function, one-switch independence requires two. For utility functions involving many attributes, it is still desirable that utility independence holds between most of the attributes, but one-switch independence is available if for one or two attributes the preference relations are more complex.

6. Discussion

For assessment purposes, once it has become clear that X is not utility independent of Y , we offer one-switch independence as a logical next step of complexity. If utility independence of X on Y does not hold, then it must be that $(\tilde{x}_1, y_1) > (\tilde{x}_2, y_1)$ and $(\tilde{x}_1, y_2) < (\tilde{x}_2, y_2)$ for some $\tilde{x}_1, \tilde{x}_2, y_1$, and y_2 . We now ask the decision maker whether further increases in Y above y_2 might cause a reversal of preference back to \tilde{x}_1 over \tilde{x}_2 . Of course it is not our suggestion that the decision maker be asked to consider this issue explicitly for all possible choices of \tilde{x}_1 and \tilde{x}_2 . Just as we would check for utility independence by means of a general question such as “As you think about your preferences for gambles on X , do they depend on the particular level of Y ?” so for one-switch independence we propose a question such as “As you think about your preferences between pairs of gambles on X do you think it possible that your preference could vary back and forth as Y varies?” If the decision maker says “Yes, of course,” then we are bound to wonder whether the attributes are defined in the most useful way (Keeney 1992, 1979; Keeney and Gregory 2005).

In an ideal world, all attributes will be mutually utility independent, but we see one-switch independence as a useful next step when utility independence does not hold. If one-switch independence does not hold, our thought is that the step after that is not two-switch independence, but rather, a reformulation of the attributes.

Appendix

In the following proofs we adopt the following notations: X 1S Y means that X is one-switch independent of Y , and, more generally, X nS Y for n -switch. X UI Y will represent utility independence of X from Y . X MD Y will

stand for monotonic dependence, X MII Y for monotonic interpolation independence, and X MTD Y for monotonic trade-off dependence.

PROOF OF THEOREM 1.

Necessity: We start the proof by considering two discrete uncertain lotteries, \tilde{x}_A and \tilde{x}_B , having the same probabilities but with different values of the outcomes. We derive the one-switch implications for these lotteries and then generalize it to the case where the probabilities do not need to be the same. Assume that the two lotteries are

$$\langle p_1, x_{A1}; p_2, x_{A2}; \dots; p_m, x_{Am} \rangle \quad \text{and}$$

$$\langle p_1, x_{B1}; p_2, x_{B2}; \dots; p_m, x_{Bm} \rangle.$$

The difference in expected utility of two lotteries over X when another attribute (or parameter), y , is fixed, can be written as

$$\begin{aligned} \text{Eu}_A(y) - \text{Eu}_B(y) &= \sum_{i=1}^m p_i [u(x_{Ai}, y) - u(x_{Bi}, y)] \\ &\stackrel{\Delta}{=} \theta(y) = \sum_{i=1}^m p_i V_i(y), \end{aligned}$$

where $V_i(y) = u(x_{Ai}, y) - u(x_{Bi}, y)$.

For fixed values y_0, y_1, \dots, y_m , we can write in matrix form

$$\begin{pmatrix} V_1(y_0) & V_2(y_0) & \dots & V_m(y_0) \\ V_1(y_1) & V_2(y_1) & \dots & V_m(y_1) \\ \dots & \dots & \dots & \dots \\ V_1(y_m) & V_2(y_m) & \dots & V_m(y_m) \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ \dots \\ p_m \end{pmatrix} = \begin{pmatrix} \theta(y_0) \\ \theta(y_1) \\ \dots \\ \theta(y_m) \end{pmatrix}$$

X IS $Y \Rightarrow$ there does not exist $y_0 < y_1 < y_2$ such that $\theta(y_0) = +ve$, $\theta(y_1) = -ve$, $\theta(y_2) = +ve$.

For this to occur, any 3×3 portion of the V matrix above must be singular, which implies that

$$V_i(y) = \alpha V_j(y) + \beta V_k(y), \quad \forall i, j, k, \quad (24)$$

where α, β cannot depend on y (or else the columns would not be linearly dependent), but can depend on the particular instances of $x_{A_i}, x_{B_i}, x_{A_j}, x_{B_j}, x_{A_k}, x_{B_k}$.

Referring back to the definition of $V_i(y) = u(x_{Ai}, y) - u(x_{Bi}, y)$ gives

$$\begin{aligned} u(x_{Ai}, y) - u(x_{Bi}, y) &= \alpha [u(x_{Aj}, y) - u(x_{Bj}, y)] \\ &\quad + \beta [u(x_{Ak}, y) - u(x_{Bk}, y)]. \end{aligned} \quad (25)$$

Equation (25) applies to any arbitrary fixed values $x_{A_i}, x_{B_i}, x_{A_j}, x_{B_j}, x_{A_k}, x_{B_k}$ and relates the six functional assessments of y at these instantiations. Denote

$$\begin{aligned} x_{Bi} &= x_0, \quad x_{Ai} = x, \quad u(x_0, y) = g_0(y), \\ [u(x_{Aj}, y) - u(x_{Bj}, y)] &= g_1(y), \\ [u(x_{Ak}, y) - u(x_{Bk}, y)] &= g_2(y). \end{aligned}$$

Rearranging gives

$$\begin{aligned} u(x, y) &= \alpha(x, x_{A_j}, x_{B_j}, x_{A_k}, x_{B_k}) g_1(y) \\ &\quad + \beta(x, x_{A_j}, x_{B_j}, x_{A_k}, x_{B_k}) g_2(y) + g_0(y). \end{aligned} \quad (26)$$

Further define for arbitrary fixed values $x_{A_j}, x_{B_j}, x_{A_k}, x_{B_k}$,

$$\begin{aligned} f_1(x) &= \alpha(x, x_{A_j}, x_{B_j}, x_{A_k}, x_{B_k}), \\ f_2(x) &= \beta(x, x_{A_j}, x_{B_j}, x_{A_k}, x_{B_k}). \end{aligned}$$

Substituting into (26) gives

$$u(x, y) = f_1(x) g_1(y) + f_2(x) g_2(y) + g_0(y). \quad (27)$$

Define $\phi(y) \stackrel{\Delta}{=} g_2(y)/g_1(y)$, $g_1(y) \neq 0$. Substituting into (15) gives

$$\begin{aligned} \text{Eu}_A(y) - \text{Eu}_B(y) &= g_1(y) ([f_1(\tilde{x}_A) - f_1(\tilde{x}_B)] + [f_2(\tilde{x}_A) - f_2(\tilde{x}_B)] \phi(y)). \end{aligned} \quad (28)$$

We distinguish two cases: Case (i) $g_1(y)$ does not change sign: for (28) to change sign only once, with two given arbitrary lotteries \tilde{x}_A and \tilde{x}_B , and constant terms $f_1(\tilde{x}_A), f_1(\tilde{x}_B), f_2(\tilde{x}_A), f_2(\tilde{x}_B)$, then $\phi(y)$ must be monotonic, or else we may have different points of indifference for arbitrary lotteries. (ii) $g_1(y)$ does change sign: This would imply that the term $([f_1(\tilde{x}_A) - f_1(\tilde{x}_B)] + [f_2(\tilde{x}_A) - f_2(\tilde{x}_B)] \phi(y))$ either does not change sign or it changes sign at the same value of y for which $g_1(y)$ changes sign. This is impossible for arbitrary functions nonconstant f_1, f_2 and arbitrary lotteries.

Sufficiency: For two arbitrary lotteries, \tilde{x}_A, \tilde{x}_B and a monotonic $\phi(y)$ we have only one possible indifference point that occurs at y_1

$$\begin{aligned} \text{Eu}(\tilde{x}_A, y) - \text{Eu}(\tilde{x}_B, y) &= g_1(y_1) ([f_1(\tilde{x}_A) - f_1(\tilde{x}_B)] + [f_2(\tilde{x}_A) - f_2(\tilde{x}_B)] \phi(y_1)) = 0, \\ \phi(y_1) &= \frac{f_1(\tilde{x}_B) - f_1(\tilde{x}_A)}{f_2(\tilde{x}_B) - f_2(\tilde{x}_A)}, \end{aligned}$$

below and after which the rank order of the lotteries must reverse.

We have shown that when the lotteries have the same probabilities but different values of the variables, the form (27) is necessary and sufficient for one-switch independence. Because X IS Y must apply for lotteries having either the same or different probabilities, the general solution for X IS Y must be contained by this solution (because it needs to satisfy both constraints; same and different probabilities). It is straightforward to show that the form (27) does indeed satisfy one-switch preferences for any two lotteries even if they have different probabilities. Consequently, the functional form (27) must be the general solution for X IS Y .

PROOF OF PROPOSITION 3.1. Without loss of generality, when $u(x^*, y) \neq u(x^0, y)$, we have for any two-attribute utility function

$$u(x, y) = u(x^0, y) + [u(x^*, y) - u(x^0, y)]u(x | y),$$

where

$$g_0(y) = u(x^0, y), g_1(y) = [u(x^*, y) - u(x^0, y)].$$

In the proof of Theorem 1 we defined

$$g_0(y) = u(x_{Bi}, y)$$

$$g_1(y) = u(x_{Aj}, y) - u(x_{Bj}, y).$$

This must be true also for the special case where the lottery B is deterministic in X (i.e., when $x_{Bi} = x_{Bj} = x_0$). Setting $x_{Aj} = x^*$ gives

$$g_0(y) = u(x^0, y)$$

$$g_1(y) = u(x^*, y) - u(x^0, y).$$

Comparing this expression to the one-switch expression, where $\phi(y)$ is monotonic, gives

$$u(x, y) = g_0(y) + g_1(y)(f_1(x) + f_2(x)\phi(y)).$$

Hence,

$$u(x | y) = (f_1(x) + f_2(x)\phi(y)). \tag{29}$$

Without loss of generality, we can also assume that $\phi(y^0) = 0, \phi(y^*) = 1$ because any scale or shift parameters can be added to the functions $f_1(x), f_2(x)$.

By direct substitution into (27), we have

$$f_1(x) = u(x | y^0).$$

Therefore,

$$f_2(x) = u(x | y^*) - u(x | y^0).$$

Hence,

$$\phi(y) = \frac{u(x | y) - u(x | y^0)}{u(x | y^*) - u(x | y^0)},$$

which must be monotonic in y if $u(x | y)$ is monotonic in y , and (from Theorem 1) cannot depend on x .

PROOF OF PROPOSITION 3.2. The proof follows the same necessity steps of Theorem 1. For 2-switch, we need any 4×4 portion of the matrix to be singular, and $X \text{ nS } Y \Rightarrow$ there does not exist $y_0 < y_1 < \dots < y_{n+1}$ such that $\text{sign}[\theta(y_i)] = (-1)^i$. For this to occur, any $(n + 2) \times$

$(n + 2)$ portion of the V matrix above must be singular, which implies that

$$V_{n+2}(y) = \sum_{i=1}^{n+1} \alpha_i(x_{A1}, x_{A1}, \dots, x_{A(n+2)}, x_{B(n+2)})V_i(y).$$

Referring back to the definition of

$$V_{n+2}(y) = u(x_{A(n+2)}, y) - u(x_{B(n+2)}, y),$$

gives

$$u(x_{A(n+2)}, y) - u(x_{B(n+2)}, y) = \sum_{i=1}^{n+1} \alpha_i(x)V_i(y). \tag{30}$$

Denote

$$x_{B(n+2)} = x_0, x_{Ai} = x_0 + t_i, u(x_0, y) = g_0(y), V_i(y) = g_i(y).$$

Rearranging gives

$$u(x_0 + t_{n+2}, y) = \sum_{i=1}^{n+1} \alpha_i(t_1, \dots, t_{n+2})g_i(y) + g_0(y).$$

Further define

$$x = x_0 + t_{n+2}, \delta^i_{t_1, \dots, t_{n+1}}(t_{n+2}) = \alpha_i(t_1, \dots, t_{n+2}),$$

$$f_i(x) = \delta^i_{t_1, \dots, t_{n+1}}(x - x_0).$$

Substituting gives

$$u(x, y) = \sum_{i=1}^{n+1} f_i(x)g_i(y) + g_0(y).$$

PROOF OF PROPOSITION 4.1. $[X \text{ MII } Y] \Rightarrow [X \text{ MD } Y]$. If $f(y)$ is monotone in y , then $u(x | y^*) + f(y)[u(x | y^*) - u(x | y^0)]$ must be monotone in y .

PROOF OF PROPOSITION 4.2. $[X \text{ 1S } Y] \Rightarrow [X \text{ MD } Y]$. From (29), $X \text{ 1S } Y$ implies $u(x | y) = (f_1(x) + f_2(x)\phi(y))$, which is monotonic in y since $\phi(y)$ is monotonic.

PROOF OF THEOREM 2. First we show that $X \text{ 1S } Y \Leftrightarrow X \text{ MTD } Y$. It is evident that $X \text{ MTD } Y \Rightarrow X \text{ 1S } Y$. Now we show the other direction. If $X \text{ 1S } Y$, then $u(x | y) = (f_1(x) + f_2(x)\phi(y))$. This implies that

$$\text{Eu}(\tilde{x}_1 | y) - \text{Eu}(\tilde{x}_2 | y) = \bar{f}_1 + \bar{f}_2\phi(y),$$

where $\bar{f}_i = E(f_i(\tilde{x}_1) - f_i(\tilde{x}_2))$. Because $\phi(y)$ is monotonic, then $\text{Eu}(\tilde{x}_1 | y) - \text{Eu}(\tilde{x}_2 | y)$ is monotonic with y . Hence, $X \text{ 1S } Y \Rightarrow X \text{ MTD } Y$, and therefore $X \text{ 1S } Y \Leftrightarrow X \text{ MTD } Y$.

We now show that $X \text{ MII } Y \Leftrightarrow X \text{ 1S } Y$. From (29), if $X \text{ 1S } Y$, we have

$$\begin{aligned} u(x | y) &= f_1(x) + f_2(x)\phi(y) \\ &= u(x | y^0) + [u(x | y^*) - u(x | y^0)]\phi(y) \\ &= u(x | y^0)[1 - \phi(y)] + u(x | y^*)\phi(y), \end{aligned}$$

which is by definition the notion of X MII Y . The reverse direction is straightforward, and thus the two formulations are equivalent.

PROOF OF THEOREM 3. First we recall that one-switch independence in any direction is equivalent to monotone interpolation independence in that direction. Mutual one-switch independence is by definition equivalent to mutual monotonic interpolation independence. Bell (1979) shows that mutual interpolation independence holds if and only if Equation (9) holds. Note that this equation can also be written as

$$u(x, y) = g_0(y) + g_1(y)[f_1(x) + f_2(x)\phi(y)],$$

where

$$g_0(y) \triangleq u(x^0, y) = a_2u(y | x^0);$$

$$g_1(y) \triangleq u(x^*, y) - u(x^0, y) \\ = [a_1 + (1 - a_1)u(y | x^*) - a_2u(y | x^0)];$$

$$f_1(x) \triangleq u(x | y^0); \quad f_2(x) \triangleq u(x | y^*) - u(x | y^0).$$

Moreover,

$$u(x | y) \triangleq \frac{u(x, y) - u(x^0, y)}{u(x^*, y) - u(x^0, y)} \\ = u(x | y^0)(1 - \phi(y)) + u(x | y^*)\phi(y) \\ = u(x | y^0) \left\{ \frac{a_1 - ku(y | x^0) + (k - a_1)u(y | x^*)}{a_1 + (1 - a_1)u(y | x^*) - a_2u(y | x^0)} \right\} \\ + u(x | y^*) \left\{ \frac{(k - a_2)u(y | x^0) + (1 - k)u(y | x^*)}{a_1 + (1 - a_1)u(y | x^*) - a_2u(y | x^0)} \right\}.$$

Hence,

$$\phi(y) = \frac{(k - a_2)u(y | x^0) + (1 - k)u(y | x^*)}{a_1 + (1 - a_1)u(y | x^*) - a_2u(y | x^0)}$$

must be monotonic in y for X IS Y to hold. Similarly, writing out the expression for $u(y | x)$ shows that the function

$$\frac{(k - a_1)u(x | y^0) + (1 - k)u(x | y^*)}{a_2 + (1 - a_2)u(x | y^*) - a_1u(x | y^0)}$$

must be monotonic in x for Y IS X to hold.

PROOF OF THEOREM 4.

Sufficiency: If $u(x, y) = f_1(x) + f_2(y) + \phi_1(x)\phi_2(y)$, then

$$\text{Eu}(\tilde{x}_A, y) - \text{Eu}(\tilde{x}_B, y) = f_1(\tilde{x}_A) - f_1(\tilde{x}_B) \\ + [\phi_1(\tilde{x}_A) - \phi_1(\tilde{x}_B)]\phi_2(y).$$

This equals zero only once for arbitrary lotteries if and only if $\phi_2(y)$ is monotonic. The same applies for lotteries over y . Similarly, if $u(x, y) = u_1(x)u_2(y)[1 + \phi_1(x)\phi_2(y)]$, then $\text{Eu}(\tilde{x}_A, y) - \text{Eu}(\tilde{x}_B, y) = u_2(y)[u_1(\tilde{x}_A) - u_1(\tilde{x}_B)] + [u_1(\tilde{x}_A)\phi_1(\tilde{x}_A) - u_1(\tilde{x}_B)\phi_1(\tilde{x}_B)]\phi_2(y)$, which equals zero only once if $u_2(y)$ does not change sign and $\phi_2(y)$ is monotonic. The same applies for lotteries over y .

References

- Abbas, A. E. 2009. Multiattribute utility copulas. *Oper. Res.* **57**(6) 1367–1383.
- Abbas, A. E., R. A. Howard. 2005. Attribute dominance utility. *Decision Anal.* **2**(4) 185–206.
- Bell, D. E. 1979. Consistent assessment procedures using conditional utility functions. *Oper. Res.* **27**(5) 1054–1066.
- Bell, D. E. 1988. One-switch utility functions and a measure of risk. *Management Sci.* **34**(12) 1416–1424.
- Farquhar, P. H. 1977. A survey of multiattribute utility theory and applications. M. K. Starr, M. Zeleny, eds. *Multiple Criteria Decision Making*. TIMS Studies in the Management Sciences, Vol. 6. North-Holland Publishing Company, Amsterdam, 59–89.
- Fishburn, P. C. 1974. von Neumann-Morgenstern utility functions on two attributes. *Oper. Res.* **22**(1) 35–45.
- Fishburn, P. C. 1977. Approximations of two-attribute utility functions. *Math. Oper. Res.* **2**(1) 30–44.
- Keeney, R. L. 1979. Evaluation of pumped storage sites. *Oper. Res.* **27**(1) 48–64.
- Keeney, R. L. 1992. *Value-Focused Thinking: A Path to Creative Decision Making*. Harvard University Press, Cambridge, MA.
- Keeney R. L., R. Gregory. 2005. Selecting attributes to measure the achievement of objectives. *Oper. Res.* **53**(1) 1–11.
- Keeney, R. L., H. Raiffa. 1976. *Decisions with Multiple Objectives*. Wiley, New York.
- Kirkwood, C. W. 1976. Parametrically dependent preferences for multiattributed consequences. *Oper. Res.* **24**(1) 92–103.
- Pfanzagl, J. 1959. A general theory of measurement: Applications to utility. *Naval Res. Logist. Quart.* **6**(4) 283–294.
- von Neumann, J., O. Morgenstern. 1947. *The Theory of Games and Economic Behavior*, 2nd ed. Princeton University Press, Princeton, NJ.