Robustness Design of Nonlinear Dynamic Systems via Fuzzy Linear Control

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Abstract—This study introduces a fuzzy linear control design method for nonlinear systems with optimal $H^\infty$ robustness performance. First, the Takagi and Sugeno fuzzy linear model is employed to approximate a nonlinear system. Next, based on the fuzzy linear model, a fuzzy controller is developed to stabilize the nonlinear system, and at the same time the effect of external disturbance on control performance is attenuated to a minimum level. Thus based on the fuzzy linear model, $H^\infty$ performance design can be achieved in nonlinear control systems. In the proposed fuzzy linear control method, the fuzzy linear model provides rough control to approximate the nonlinear control system, while the $H^\infty$ scheme provides precise control to achieve the optimal robustness performance. Linear matrix inequality (LMI) techniques are employed to solve this robust fuzzy control problem. In the case that state variables are unobservable, a fuzzy observer-based $H^\infty$ control is also proposed to achieve a robust optimization design for nonlinear systems. A simulation example is given to illustrate the performance of the proposed design method.

Index Terms—$H^\infty$ robust control, linear matrix inequality, nonlinear fuzzy observer, Takagi–Sugeno fuzzy control.

I. INTRODUCTION

The control design of nonlinear systems is a difficult process, and in practical control systems the plants are always nonlinear. Thus many nonlinear control methods have been developed for nonlinear systems to overcome the difficulty in controller design for real systems. However, in these control system designs, the nonlinear systems must have some predictable behaviors. For example, the system must be minimum-phase, it must be sufficiently smooth, and the parameters must be exactly known in order for the feedback linearization method to be applied. Furthermore, these control schemes are so complicated that they are not suitable for practical application.

Recently, the nonlinear $H^\infty$ control schemes [19], [20] have been introduced to deal with the robust performance design problem of nonlinear systems. However, the designer has to solve a Hamilton–Jacobi equation, which is a nonlinear partial differential equation. Only some very special nonlinear systems have a closed-form solution. In general, conventional nonlinear $H^\infty$ control schemes are not suitable for practical control system design.

In the past few years, there has been rapidly growing interest in fuzzy control of nonlinear systems, and there have been many successful applications. Despite the success, it has become evident that many basic issues remain to be addressed. The most important issue for fuzzy control systems is how to get a system design with the guarantee of stability and control performance, and recently there have been significant research efforts on the issue of stability in fuzzy control systems [2], [5], [6], [22]–[24]. In [6], an approach was given for the stability of fuzzy design issues of nonlinear systems. In other studies, a nonlinear plant was approximated by a Takagi–Sugeno fuzzy linear model [1], and then a model-based fuzzy control was developed to stabilize the Takagi–Sugeno fuzzy linear model.

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To deal with this fuzzy observer-based control design, it needs to simultaneously solve multiple quadratic matrix inequalities. These quadratic matrix inequalities can be transformed to equivalent linear matrix inequalities (LMI's), and these LMI's are combined into a standard LMI problem (LMIP). This standard LMIP can be solved to complete the optimal $H^\infty$ fuzzy control design. Finally, the optimal $H^\infty$ design of fuzzy control system is formulated as a so-called eigenvalue problem (EVP) to minimize the maximum eigenvalue of a matrix that depends on a variable, subject to the LMI constraints.

The main contribution of this paper is twofold: i) robust design of nonlinear systems is dealt with via a fuzzy observer-based linear controller and ii) both the stability of the fuzzy observer-based control system and the optimal $H^\infty$ attenuation of the effect of the external disturbance on the control performance are guaranteed.

A simulation example is provided to illustrate the design procedure and performance of the proposed methods. In the proposed hybrid fuzzy control method, the fuzzy linear method provides rough tuning to approximate the nonlinear control system, while the optimal $H^\infty$ scheme provides precise tuning to achieve an optimal robustness performance. The simulation results show that the optimal robustness performance can be achieved by the proposed method.

The paper is organized as follows: the problem description is presented in Section II. Robust stabilization and optimal $H^\infty$ performance design via fuzzy linear control are described in Section III. In Section IV, a fuzzy observer-based $H^\infty$ control with robustness optimization is introduced. In Section V, a simulation example is provided to demonstrate the effectiveness of fuzzy $H^\infty$ control design and to confirm the robust performance. Finally, concluding remarks are made in Section VI. Note that this paper is a modified version of the conference paper in [26].

II. PROBLEM DESCRIPTION

Consider the following nonlinear system:

$$\dot{x}(t) = f(x(t)) + g(x(t))u(t) + w(t)$$

(1)

where

$$x(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T \in R^{n \times 1}$$

denotes the state vector,

$$u(t) = [u_1(t), u_2(t), \ldots, u_m(t)]^T \in R^{m \times 1}$$

denotes the control input,

$$w(t) = [w_1(t), w_2(t), \ldots, w_v(t)]^T \in R^{v \times 1}$$

denotes the unknown disturbances with a known upper bound $u_{wb} = ||w(t)||$, and $f(x)$ and $g(x)$ depend on $x$.

Definition I [17]: The solutions of a dynamic system are said to be uniformly ultimately bounded (UUB) if there exist positive constants $\beta$ and $\kappa$, and for every $\delta \in (0, \kappa)$ there is a positive constant $T = T(\delta)$, such that

$$||x(t_0)|| < \delta \Rightarrow ||x(t)|| \leq \beta, \forall t \geq t_0 + T.$$

A fuzzy dynamic model has been proposed by Takagi and Sugeno [1] to represent local linear input/output relations of nonlinear systems. This fuzzy linear model is described by fuzzy If–Then rules and will be employed here to deal with the control design problem of the nonlinear system (1). The $i$th rule of this fuzzy model for the nonlinear system (1) is of the following form [1], [2], [6]:

Plant Rule $i$:

If $z_1(t)$ is $F_{i1}$ and $\cdots$ and $z_g(t)$ is $F_{ig}$,

Then $\dot{x}(t) = A_i x(t) + B_i u(t) + w(t)$

(2)

for $i = 1, 2, \ldots, L$ where $F_{ij}$ is the fuzzy set, $A_i \in R^{n \times n}$, $B_i \in R^{n \times m}$, $L$ is the number of If–Then rules, and $z_1(t), z_2(t), \ldots, z_g(t)$ are the premise variables.

The overall fuzzy system is inferred as follows [1], [6], [16]:

$$\dot{x}(t) = \sum_{i=1}^{L} \mu_i(z(t)) (A_i x(t) + B_i u(t)) + w(t)$$

(3)

$$\sum_{i=1}^{L} \mu_i(z(t)) = 1$$

(4)

for all $t$. Therefore, we get [2], [6]

$$h_i(z(t)) \geq 0$$

(5)

for $i = 1, 2, \ldots, L$ and

$$\sum_{i=1}^{L} h_i(z(t)) = 1.$$
where
\[
\begin{align*}
\left\{ \left( f(x) - \sum_{i=1}^{L} h_i(z(t))A_i x(t) \right) \\
+ \left( g(x) - \sum_{i=1}^{L} h_i(z(t))B_i u(t) \right) \right\}
\end{align*}
\]

denotes the approximation error between the nonlinear system (1) and the fuzzy model (3).

Suppose the following fuzzy controller is employed to deal with the above control system design:

**Control Rule j:**
- If \( z_1(t) \) is \( F_{j1} \) and \( \cdots \) and \( z_g(t) \) is \( F_{jg} \),
- Then \( u(t) = K_{j} x(t) \) (7)

for \( j = 1, 2, \cdots, L \). Hence, the overall fuzzy controller is given by

\[
u(t) = \frac{\sum_{j=1}^{L} \mu_j(z(t))(K_j x(t))}{\sum_{j=1}^{L} \mu_j(z(t))} = \sum_{j=1}^{L} h_j(z(t))K_j x(t) \quad (8)
\]

where \( h_j(z(t)) \) is defined in (4) and (5) and \( K_j \) are the control parameters (for \( j = 1, 2, \cdots, L \)).

Substituting (8) into (6) yields the closed-loop nonlinear control system as follows:

\[
\dot{x}(t) = f(x(t)) + g(x(t))u(t) + w(t)
\]

\[
\begin{align*}
&= \sum_{i=1}^{L} \sum_{j=1}^{L} h_i(z(t))h_j(z(t))(A_i + B_i K_j)x(t) \\
&\quad + \left( f(x) - \sum_{i=1}^{L} h_i(z(t))A_i x(t) \right) \\
&\quad + \sum_{i=1}^{L} h_i(z(t)) \sum_{j=1}^{L} h_j(z(t))(g(x) - B_i K_j x(t) + u(t) \\
&= \sum_{i=1}^{L} \sum_{j=1}^{L} h_i(z(t))h_j(z(t))(A_i + B_i K_j)x(t) \\
&\quad + \Delta f + \Delta g + u(t) \\
&\quad (9)
\end{align*}
\]

where

\[
\Delta f = \left( f(x) - \sum_{i=1}^{L} h_i(z(t))A_i x(t) \right) \quad (10)
\]

\[
\Delta g = \sum_{i=1}^{L} h_i(z(t)) \sum_{j=1}^{L} h_j(z(t))(g(x) - B_i K_j x(t). \quad (11)
\]

Suppose that there exist bounding matrices \( \Delta A_i \) and \( \Delta B_i \) such that

\[
||\Delta f|| \leq \left\| \sum_{i=1}^{L} h_i(z(t)) \Delta A_i x(t) \right\| \quad (12)
\]

and

\[
||\Delta g|| \leq \left\| \sum_{i=1}^{L} h_i(z(t)) \sum_{j=1}^{L} h_j(z(t)) \Delta B_i K_j x(t) \right\| \quad (13)
\]

for all trajectory \( x(t) \) and the bounding matrices \( \Delta A_i \) and \( \Delta B_i \) can be described by

\[
\left[ \begin{array}{c} \Delta A_i \\ \Delta B_i \end{array} \right] = \left[ \begin{array}{c} \Delta \xi_i A_p \\ \eta_i B_p \end{array} \right] \quad (14)
\]

where \( ||\xi_i|| \leq 1 \) and \( ||\eta_i|| \leq 1 \), for \( i = 1, 2, \cdots, L \) [15].

**Remark 1:**

1) If we assume \( g = n \) and \( z_1(t) = x_1(t), z_2(t) = x_2(t), \cdots, z_n(t) = x_n(t) \), then the plant rule can be represented as

**Plant Rule \( \bar{i} \):**
- If \( x_{\bar{i}}(t) \) is \( F_{\bar{i}1} \) and \( \cdots \) and \( x_{\bar{n}}(t) \) is \( F_{\bar{m}} \),
- Then \( \dot{x}(t) = A_{\bar{i}} x(t) + B_{\bar{i}} u(t) + w(t) \) (15)

2) Obviously, according to assumption above

\[
\left\| \sum_{i=1}^{L} h_i(z(t)) \Delta A_i x(t) \right\| (\geq ||\Delta f||)
\]

and

\[
\left\| \sum_{i=1}^{L} h_i(z(t)) \sum_{j=1}^{L} h_j(z(t)) \Delta B_i K_j x(t) \right\| (\geq ||\Delta g||)
\]

are the worst case representations for the approximation error if there exist \( \Delta A_i (= \xi_i A_p) \) and \( \Delta B_i (= \eta_i B_p) \) such that (12) and (13) hold for some \( ||\xi_i|| \leq 1 \) and \( ||\eta_i|| \leq 1 \) (for \( i = 1, 2, \cdots, L \)).

According to assumption above, we get

\[
(\Delta f)^T(\Delta f)
\]

\[
= \left( f(x) - \sum_{i=1}^{L} h_i(z(t))A_i x(t) \right)^T \left( f(x) - \sum_{i=1}^{L} h_i(z(t))A_i x(t) \right)
\]

\[
\leq \left( \sum_{i=1}^{L} h_i(z(t)) \Delta A_i x(t) \right)^T \left( \sum_{i=1}^{L} h_i(z(t)) \Delta A_i x(t) \right)
\]

\[
= \left( \sum_{i=1}^{L} h_i(z(t))A_i x(t) \right)^T \left( \sum_{i=1}^{L} h_i(z(t))A_i x(t) \right)
\]

\[
\leq (A_{\bar{i}} x(t))^T(A_{\bar{i}} x(t)) \quad (16)
\]
and

\[(\Delta g)^T(\Delta g)\]
\[= \left( \sum_{i=1}^{L} h_i(z(t)) \sum_{j=1}^{L} h_j(z(t))(g(x) - B_j)K_jx(t) \right)^T\]
\[\cdot \left( \sum_{i=1}^{L} h_i(z(t)) \sum_{j=1}^{L} h_j(z(t))(g(x) - B_i)K_jx(t) \right)\]
\[\leq \left( \sum_{i=1}^{L} h_i(z(t)) \sum_{j=1}^{L} h_j(z(t))\Delta B_iK_jx(t) \right)^T\]
\[\cdot \left( \sum_{i=1}^{L} h_i(z(t)) \sum_{j=1}^{L} h_j(z(t))\Delta B_jK_i x(t) \right)\]
\[= \left( \sum_{i=1}^{L} h_i(z(t)) \sum_{j=1}^{L} h_j(z(t))\eta B_p K_j x(t) \right)^T\]
\[\cdot \left( \sum_{i=1}^{L} h_i(z(t)) \sum_{j=1}^{L} h_j(z(t))\eta B_p K_i x(t) \right)\]
\[\leq \left( \sum_{i=1}^{L} h_i(z(t))B_p K_j x(t) \right)^T \left( \sum_{j=1}^{L} h_j(z(t))B_p K_j x(t) \right)\]
\[\tag{17}\]

i.e., the approximation error in the closed-loop nonlinear system is bounded by the specified structured bounding matrices \(A_p\) and \(B_p\).

Stability is the most important issue in the control systems. Obviously, it is appealing for control engineers to specify control parameters \(K_j\) in the fuzzy controller (8) such that the stability for the closed-loop nonlinear system (9) can be guaranteed.

In this study, we assume that \(u(t)\) is unknown but bounded. However, the effect of \(u(t)\) will deteriorate the control performance of fuzzy control system. Therefore, how to eliminate the effect of \(u(t)\) to guarantee the control performance is an important issue in the control systems. Since \(H^\infty\) control is the most important control design to efficiently eliminate the effect of \(u(t)\) on the control system, it will be employed to deal with the robust performance control in (9). Let us consider the following \(H^\infty\) control performance [7], [13]:

\[\int_0^{t_f} x^T(t)Qx(t)\,dt < \rho^2\]
\[\tag{18}\]

or

\[\int_0^{t_f} x^T(t)Qx(t)\,dt < \rho^2 \int_0^{t_f} u^T(t)u(t)\,dt\]
\[\tag{19}\]

where \(t_f\) denotes the terminal time of the control, \(\rho\) is a prescribed value which denotes the worst case effect of \(u(t)\) on \(x(t)\), and \(Q\) is a positive-definite weighting matrix. The physical meaning of (19) is that the effect of \(u(t)\) on \(x(t)\) must be attenuated below a desired level \(\rho\) from the viewpoint of energy, no matter what \(u(t)\) is, i.e., the \(L_2\) gain from \(u(t)\) to \(x(t)\) must be equal to or less than a prescribed value \(\rho^2\).

In general, \(\rho\) is chosen as a positive small value less than 1 for attenuation of \(u(t)\).

The inequality in (19) can be seen as bounded-disturbance and bounded-state but with a prescribed gain \(\rho\). If the initial condition is also considered, the inequality (19) can be modified as

\[\int_0^{t_f} x^T(t)Qx(t)\,dt < x^T(0)Px(0) + \rho^2 \int_0^{t_f} u^T(t)u(t)\,dt\]
\[\tag{20}\]

where \(P\) is some symmetric positive-definite weighting matrix.

From the analysis above, the design purpose of the proposed fuzzy control system is to specify a linear fuzzy control (8) such that both the stability of fuzzy linear control system and the \(H^\infty\) control performance in (20) with a prescribed attenuation level \(\rho\) are guaranteed.

The robustness optimization is to achieve a minimum \(\rho^2\) in (20) to obtain maximum elimination of the effect of \(u(t)\). For nonlinear system (1), this design problem is how to specify a stabilizable fuzzy control in (8) to minimize \(\rho^2\) subject to the constraint (20).

III. \(H^\infty\) CONTROL DESIGN VIA FUZZY LINEAR CONTROL

From the description in the above section, the design purpose of this study is how to specify a fuzzy linear control law in (8) for the nonlinear system in (9) with the guaranteed \(H^\infty\) control performance in (20).

Let us choose a Lyapunov function for the system (9) as

\[V(t) = x^T(t)Px(t)\]
\[\tag{21}\]

where the weighting matrix \(P = P^T > 0\).

The time derivative of \(V(t)\) is

\[\dot{V}(t) = x^T(t)P\dot{x}(t) + x^T(t)P\dot{x}(t)\]
\[\tag{22}\]

By substituting (9) into (22), we get

\[\dot{V}(t) = \sum_{i=1}^{L} h_i(z(t)) \sum_{j=1}^{L} h_j(z(t))\cdot \{x^T(t)(P(A_i + B_i K_j) + (A_i + B_j K_i)^T P)x(t)\}
+ (\Delta f)^T P\dot{x}(t) + x^T(t)(\Delta g)^T + (\Delta g)^T P\dot{x}(t)
+ x^T(t)P(\Delta g) + x^T(t)P\dot{x}(t) + u^T(t)P\dot{x}(t)\]
\[\leq \sum_{i=1}^{L} h_i(z(t)) \sum_{j=1}^{L} h_j(z(t))\cdot \{x^T(t)(P(A_i + B_i K_j) + (A_i + B_j K_i)^T P)x(t)\}
+ (\Delta f)^T (\Delta f) + x^T(t)PP\dot{x}(t) + (\Delta g)^T (\Delta g)
+ x^T(t)PP\dot{x}(t) + x^T(t)P\dot{x}(t) + u^T(t)P\dot{x}(t)\]
\[
\begin{align*}
\sum_{i=1}^{L} \sum_{j=1}^{L} h_i(z(t)) h_j(z(t)) & \cdot \left\{ x^T(t)(A_i^T P + PA_i + PB_i K_j + K_j^T B_i^T P + A_i^T A_i + (B_i K_j)^T (B_i K_j) + 2PP)x(t) + \frac{1}{\rho^2} x^T(t)PPx(t) \right\} \\
& + x^T(t)Pw(t) + w^T(t)Px(t)
\end{align*}
\]

Then, we get the following result.

**Theorem 1:** If the fuzzy controller (8) is employed in the nonlinear system (1) and there exists a positive-definite matrix \( P = P^T > 0 \) such that the following matrix inequalities:

\[
(24)
\]

are satisfied for \( i, j = 1, 2, \ldots, L \) then the closed-loop nonlinear system (9) is UUB and the control performance of (20) is guaranteed for a prescribed \( \rho^2 \).

**Proof:** From (23), we get

\[
\dot{V}(t) \leq \sum_{i=1}^{L} \sum_{j=1}^{L} h_i(z(t)) h_j(z(t)) \frac{1}{\rho^2} x^T(t)PPx(t) + \frac{1}{\rho^2} x^T(t)PPx(t) + \rho^2 w^T(t)w(t)
\]

\[
= \sum_{i=1}^{L} \sum_{j=1}^{L} h_i(z(t)) h_j(z(t)) \frac{1}{\rho^2} x^T(t)PPx(t) + \frac{1}{\rho^2} x^T(t)PPx(t) + \rho^2 w^T(t)w(t)
\]

From \( (24) \), we get

\[
\dot{V}(t) < -x^T(t)Qx(t) - \frac{1}{\rho^2} x^T(t)PPx(t) + \frac{1}{\rho^2} x^T(t)PPx(t) + \rho^2 w^T(t)w(t)
\]

\[
= -x^T(t)Qx(t) + \rho^2 w^T(t)w(t).
\]

Since \( \|w(t)\| \leq w_{kd} \), we get

\[
\dot{V}(t) < -x^T(t)Qx(t) + \rho^2 w_{kd}^2
\]

\[
\leq -c_1 x^T(t)x(t) + \rho^2 w_{kd}^2
\]

From (21), we get

\[
\int_0^{t_f} x^T(t)Pw(t) dt + \rho^2 \int_0^{t_f} w^T(t)w(t) dt.
\]

This is (20) and the \( H^\infty \) control performance is achieved with a prescribed \( \rho^2 \).

**Corollary 1:** In the case of \( w(t) = 0 \), if the fuzzy controller (8) is employed in the closed-loop nonlinear system (9) and there exists a positive-definite matrix \( P = P^T > 0 \) such that the matrix inequalities in (24) are satisfied, then the closed-loop system (9) is quadratically stable.

**Proof:** In the case of \( w(t) = 0 \), from (28) we get

\[
\dot{V}(t) < -x^T(t)Qx(t) < 0.
\]
Therefore, the closed-loop system (9) is quadratically stable. This completes the proof. □

In general, it is not easy to analytically determine a common solution \( P = P^T > 0 \) for (24). Furthermore, the solution may not be unique. Fortunately, (24) can be reformulated into the linear matrix inequality problem (LMIP) \([15]\). The LMIP can be solved in a computationally efficient manner using a technique such as the interior point method. First, the matrix inequalities in (24) are transformed to the equivalent LMI’s by the following procedures.

By introducing new variables \( W = P^{-1} \) and \( Y_j = K_j W \), (24) is equivalent to the following matrix inequalities:

\[
WA^T_j + A_jW + B_jY_j + Y_j^TB^T_j + WA^T_pA_pW
\]

\[
+ (B_pY_j)^T(B_pY_j) + \left(2 + \frac{1}{\rho^2}\right)I + WQW < 0. \tag{32}
\]

By the Schur complements \([15]\), (32) is equivalent to the following LMI’s:

\[
\begin{bmatrix}
WA^T_j + A_jW + B_jY_j + Y_j^TB^T_j + \left(2 + \frac{1}{\rho^2}\right)I \\
(B_pY_j)^T & W \\
-I & 0 \\
0 & -\{A^T_pA_p + Q\}^{-1}
\end{bmatrix} < 0 \tag{33}
\]

for \( j = 1, 2, \ldots, L \).

If the LMI’s in (33) have a positive-definite solution for \( W \), then the closed-loop system is stable and the \( H^\infty \) control performance in (20) is guaranteed for a prescribed \( \rho \).

Therefore, the \( H^\infty \) design optimization for fuzzy control system of (1) is formulated as the following constrained optimization problem:

minimize \( \rho^2 \)

subject to \( W = W^T > 0 \) and (33), \( \tag{34} \)

This problem is also called eigenvalue problem (EVP) and can be solved very efficiently by convex optimization algorithm. More details will be discussed in the next section.

IV. FUZZY OBSERVER-BASED \( H^\infty \) CONTROL

In the previous sections, we assumed that all the state variables are available. In practice, this assumption often does not hold. In this situation, we need to estimate state vector \( x \) from output \( y \) for state feedback control. Suppose the nonlinear system to be controlled is of the following form:

\[
\dot{x}(t) = f(x(t)) + g(x(t))u(t) + w(t) \tag{35}
\]

where \( y(t) \) denotes the output of the system.

The \( i \)-th rule of the fuzzy model for the nonlinear system (35) is of the following form:

Plant Rule \( i: \)

If \( z_i(t) = F_{i1} \) and \( \cdots \) and \( z_g(t) = F_{ig} \)

Then \( \hat{x}(t) = \Lambda_i x(t) + B_i u(t) + w(t) \)

\[
y(t) = C_i x(t) \tag{36}
\]

for \( i = 1, 2, \ldots, L \) where \( C_i \in \mathbb{R}^{n\times n} \).

The state dynamic of fuzzy system is the same as (3) and the output of the fuzzy system is inferred as follows:

\[
y(t) = \sum_{i=1}^{L} h_i(z(t))C_i x(t), \tag{37}
\]

Therefore, the output of the nonlinear system in (35) can be rearranged as the following equivalent system:

\[
y(t) = h(x(t))
\]

\[
= \sum_{i=1}^{L} h_i(z(t))C_i x(t) + h(x(t)) - \sum_{i=1}^{L} h_i(z(t))C_i x(t)
\]

\[
= \sum_{i=1}^{L} h_i(z(t))C_i x(t) + \Delta h \tag{38}
\]

where

\[
\Delta h = h(x(t)) - \sum_{i=1}^{L} h_i(z(t))C_i x(t) \tag{39}
\]

denotes the approximation error between the output of nonlinear uncertain system (35) and the fuzzy output (37).

Suppose the following fuzzy linear observer is proposed to deal with the state estimation of nonlinear system (1):

Observer Rule \( i: \)

If \( z_1(t) = F_{i1} \) and \( \cdots \) and \( z_g(t) = F_{ig} \)

Then \( \hat{x}(t) = \Lambda_i \hat{x}(t) + B_i u(t) + L_i (\hat{y}(t) - y(t)) \) \( \tag{40} \)

where \( \hat{y}(t) = \sum_{i=1}^{L} h_i(z(t))C_i \hat{x}(t) \) and \( L_i \) are the observer parameters (for \( i = 1, 2, \ldots, L \)).

The overall fuzzy observer can be defined as follows:

\[
\dot{\hat{x}}(t) = \sum_{i=1}^{L} h_i(z(t))\{ (\Lambda_i \hat{x}(t) + B_i u(t)) + L_i (\hat{y}(t) - y(t)) \}
\]

\[
= \sum_{i=1}^{L} h_i(z(t)) \sum_{j=1}^{L} h_j(z(t)) \cdot (\Lambda_i \hat{x}(t) + B_i u(t) + L_i C_j (x(t) - \hat{x}(t)) + L_i \Delta h). \tag{41}
\]

Remark 2: In this situation, the estimated state variables may be specified as the premise variables, i.e., \( z(t) = \hat{x}(t) \). □

Suppose the following fuzzy controller is employed to deal with the above control system design:

Control Rule \( j: \)

If \( z_1(t) = F_{j1} \) and \( \cdots \) and \( z_g(t) = F_{jg} \)

Then \( u(t) = K_j \hat{x}(t), \quad j = 1, 2, \ldots, L. \) \( \tag{42} \)

Hence, the overall fuzzy controller is given by

\[
u(t) = \sum_{j=1}^{L} h_j(z(t))K_j \hat{x}(t) \tag{43}
\]

where \( K_j \) are the control parameters (for \( j = 1, 2, \ldots, L \)).

Let us denote the estimation errors as

\[
\epsilon(t) = x(t) - \hat{x}(t). \tag{44}
\]
By differentiating (44), we get

\[
\dot{\delta}(t) = \dot{\delta}(t) - \dot{\delta}(t) = f(x(t)) + g(x(t))u(t) - \sum_{i=1}^{L} h_i(z(t)) \sum_{j=1}^{L} h_j(z(t)) \\
\cdot (A_i \dot{x}(t) + B_i \dot{u}(t) + L_i C_j \dot{x}(t) + L_i \Delta h) + w(t) \\
= \sum_{i=1}^{L} h_i(z(t)) \sum_{j=1}^{L} h_j(z(t)) (A_i \dot{x}(t) + B_i K_j \dot{x}(t)) \\
+ \Delta f + \Delta g - \sum_{i=1}^{L} h_i(z(t)) \sum_{j=1}^{L} h_j(z(t)) \\
\cdot (A_i \dot{x}(t) + B_i \dot{u}(t) + L_i C_j \dot{x}(t) + L_i \Delta h) + w(t) \\
= \sum_{i=1}^{L} h_i(z(t)) \sum_{j=1}^{L} h_j(z(t)) \\
\cdot \left( (A_i - L_i C_j) \dot{x}(t) + L_i \Delta h \right) + \Delta f + \Delta g + w(t) \\
\] (45)

where \( \Delta h \) is defined as (39), \( \Delta f \) is the same as (10), and \( \Delta g \) is modified as

\[
\Delta g = \sum_{i=1}^{L} h_i(z(t)) \sum_{j=1}^{L} h_j(z(t)) (g(x) - B_i K_j \dot{x}(t)).
\]

Assume that a bounding matrix \( \Delta C_i = C_p \varphi_i \) exists where \( \| \varphi_i \| \leq 1 \) such that

\[
\left\| \sum_{i=1}^{L} h_i(z(t)) L_i \Delta h \right\| \leq \left\| \sum_{i=1}^{L} h_i(z(t)) L_i \Delta C_i x(t) \right\|
\] (46)

for all trajectories.

Then, the augmented system is equivalent to the following form:

\[
\left[ \begin{array}{c}
\dot{x}(t) \\
\dot{\delta}(t)
\end{array} \right] = \\
\left[ \begin{array}{c}
\sum_{i=1}^{L} h_i(z(t)) \sum_{j=1}^{L} h_j(z(t)) (A_i \dot{x}(t)) \\
+ B_i u(t) + L_i C_j \dot{x}(t) + L_i \Delta h)
\end{array} \right] \\
\left\{ \begin{array}{c}
\sum_{i=1}^{L} h_i(z(t)) \sum_{j=1}^{L} h_j(z(t)) ((A_i - L_i C_j) \\
\cdot \left( \dot{x}(t) + L_i \Delta h \right) + \Delta f + \Delta g + w(t)
\end{array} \right\}
\]

(47)

After manipulation, (47) can be expressed as the following form:

\[
\left[ \begin{array}{c}
\dot{x}(t) \\
\dot{\delta}(t)
\end{array} \right] = \\
\sum_{i=1}^{L} \sum_{j=1}^{L} h_i(z(t)) h_j(z(t)) \\
\cdot \left[ \begin{array}{c}
(A_i + B_i K_j) \\
0
\end{array} \right] \left[ \begin{array}{c}
\dot{x}(t) \\
\dot{\delta}(t)
\end{array} \right] \\
+ \left[ \begin{array}{c}
\sum_{i=1}^{L} h_i(z(t)) L_i \Delta h \\
\sum_{i=1}^{L} h_i(z(t)) L_i \Delta h
\end{array} \right] + \left[ \begin{array}{c}
0 \\
0
\end{array} \right] \\
\cdot \left[ \begin{array}{c}
\Delta f \\
\Delta g + w(t)
\end{array} \right].
\] (48)

Let us denote

\[
\dot{x}(t) = \left[ \begin{array}{c}
\dot{x}(t) \\
\dot{\delta}(t)
\end{array} \right] \\
\Delta \dot{h} = \sum_{i=1}^{L} h_i(z(t)) L_i \Delta h \\
\Delta \dot{\delta} = \left[ \begin{array}{c}
\Delta f \\
\Delta g
\end{array} \right] \\
\Delta \ddot{h} = \sum_{i=1}^{L} h_i(z(t)) L_i \Delta h
\]

(49)

Therefore, the augmented system defined in (48) can be expressed as the following form:

\[
\dot{x}(t) = \sum_{i=1}^{L} h_i(z(t)) \sum_{j=1}^{L} h_j(z(t)) A_{ij} \dot{x}(t) + \Delta \dot{h} \\
+ \Delta \dot{f} + \Delta \dot{\delta} + \dot{w}(t)
\] (50)

where

\[
(\Delta \dot{f})^T(\Delta \dot{f}) = (\Delta f)^T(\Delta f) \\
\leq (A_p \dot{x}(t))^T(A_p \dot{x}(t)) \\
= (A_p \dot{x}(t) + A_p \dot{e}(t))^T(A_p \dot{x}(t) + A_p \dot{e}(t)) \\
= ([A_p \dot{x}(t)]^T[A_p \dot{x}(t)]) \\
= (\Phi \dot{x}(t))^T(\Phi \dot{x}(t))
\] (51)

where \( \Phi = [A_p \quad A_p] \) and

\[
(\Delta \dot{\delta})^T(\Delta \dot{\delta}) = (\Delta \dot{g})^T(\Delta \dot{g}) \\
\leq \left( \sum_{i=1}^{L} h_i(z(t)) B_p K_j \dot{x}(t) \right)^T \left( \sum_{i=1}^{L} h_i(z(t)) B_p K_j \dot{x}(t) \right) \\
= \left( \sum_{i=1}^{L} h_i(z(t)) B_p K_j 0 \right) \dot{x}(t) \\
\cdot \left( \sum_{i=1}^{L} h_i(z(t)) B_p K_j 0 \right) \dot{x}(t) \\
= \left( \sum_{i=1}^{L} h_i(z(t)) \Omega_J \dot{x}(t) \right)^T \left( \sum_{i=1}^{L} h_i(z(t)) \Omega_J \dot{x}(t) \right)
\] (52)

where \( \Omega_j = [B_p K_j 0] \) for \( j = 1, \ldots, L \) and
where \( \Xi_i = [L_i C_p \ L_i C_p] \) for \( i = 1, \ldots, L \).

Therefore, the \( H^\infty \) control performance can be modified as follows:

\[
\int_0^{t_f} \dot{x}^T(t) \dot{Q} \ddot{x}(t) \, dt \\
\leq \dot{x}^T(0) \dot{P} \ddot{x}(0) + \rho^2 \int_0^{t_f} \ddot{u}^T(t) \ddot{u}(t) \, dt \tag{54}
\]

where \( t_f \) denotes the terminal time of the control, \( \rho \) is a prescribed value which denotes the worst case effect of \( \ddot{u}(t) \) on \( \dot{x}(t) \), and \( \dot{P} \) and \( \dot{Q} \) are some positive-definite weighting matrix.

Define the Lyapunov \( V(t) \) for the augmented system in (50) as

\[
V(t) = \dot{x}^T(t) \dot{P} \ddot{x}(t). \tag{55}
\]

The time derivative of \( V(t) \) is

\[
\dot{V}(t) = \sum_{i=1}^{L} \sum_{j=1}^{L} h_i(z(t)) h_j(z(t)) \ddot{x}^T(t) (\dot{A}_{ij} \dot{P} + \dot{P} \dot{A}_{ij}) \ddot{x}(t)
+ (\Delta \dot{h})^T \dot{P} \ddot{x}(t) + \ddot{x}^T(t) \dot{P} (\Delta \dot{h}) + (\Delta \dot{h})^T \dot{P} \ddot{x}(t)
\]

\[
+ \ddot{x}^T(t) \dot{P} \Delta \dot{h} + (\Delta \dot{h})^T \dot{P} \ddot{x}(t) + \ddot{x}^T(t) \dot{P} \Delta \dot{h} + \ddot{x}^T(t) \dot{P} \Delta \dot{h}
\]

\[
\leq \sum_{i=1}^{L} \sum_{j=1}^{L} h_i(z(t)) h_j(z(t)) \ddot{x}^T(t) (\dot{A}_{ij} \dot{P} + \dot{P} \dot{A}_{ij}) \ddot{x}(t)
+ (\Delta \dot{h})^T (\Delta \dot{h}) + \ddot{x}^T(t) \dot{P} \ddot{x}(t) + (\Delta \dot{h})^T (\Delta \dot{h})
\]

\[
+ \ddot{x}^T(t) \dot{P} \ddot{x}(t) + (\Delta \dot{h})^T (\Delta \dot{h}) + \ddot{x}^T(t) \dot{P} \ddot{x}(t)
\]

\[
\leq \sum_{i=1}^{L} \sum_{j=1}^{L} h_i(z(t)) h_j(z(t)) \ddot{x}^T(t) (\dot{A}_{ij} \dot{P} + \dot{P} \dot{A}_{ij})
+ 2 \Xi_i^T \dot{Q} + \Phi^T \Phi + \Omega_j^T \Omega_j + 3 \dot{P} \dot{P} \ddot{x}(t)
\]

\[
+ \ddot{x}^T(t) \dot{P} \ddot{x}(t) + \ddot{x}^T(t) \dot{P} \ddot{x}(t), \tag{56}
\]

Then we get the following result.

**Theorem 2:** Suppose the fuzzy control law (43) is employed in the nonlinear system (50), and \( \dot{P} = \dot{P}^T > 0 \) is the common solution of the following matrix inequalities:

\[
\dot{A}_{ij} \dot{P} + \dot{P} \dot{A}_{ij} + 2 \Xi_i^T \dot{Q} + \Phi^T \Phi + \Omega_j^T \Omega_j + 3 \dot{P} \dot{P} \ddot{x}(t)
+ (3 + \frac{1}{\rho^2}) \dot{P} \dot{P} + \dot{Q} < 0 \tag{57}
\]

for \( i, j = 1, 2, \ldots, L \) then the closed-loop system (50) is UUB and the \( H^\infty \) control performance of (54) is guaranteed for a prescribed \( \rho^2 \).

**Proof:** From (56), we get

\[
\dot{V}(t) \leq \sum_{i=1}^{L} \sum_{j=1}^{L} h_i(z(t)) h_j(z(t)) (\dot{x}^T(t) (\dot{A}_{ij} \dot{P} + \dot{P} \dot{A}_{ij})
+ 2 \Xi_i^T \dot{Q} + \Phi^T \Phi + \Omega_j^T \Omega_j + 3 \dot{P} \dot{P} \ddot{x}(t)
\]

\[
+ \ddot{x}^T(t) \dot{P} \ddot{x}(t) + \ddot{x}^T(t) \dot{P} \ddot{x}(t) - \rho^2 \ddot{u}^T(t) \ddot{u}(t)
\]

\[
- \frac{1}{\rho^2} \dot{x}^T(t) \dot{P} \dot{P} \ddot{x}(t)
\]

\[
+ \frac{1}{\rho^2} \ddot{x}^T(t) \dot{P} \ddot{x}(t) + \rho^2 \ddot{u}^T(t) \ddot{u}(t)
\]

\[
\leq \sum_{i=1}^{L} \sum_{j=1}^{L} h_i(z(t)) h_j(z(t)) (\dot{x}^T(t) (\dot{A}_{ij} \dot{P} + \dot{P} \dot{A}_{ij})
+ 2 \Xi_i^T \dot{Q} + \Phi^T \Phi + \Omega_j^T \Omega_j + 3 \dot{P} \dot{P} \ddot{x}(t)
\]

\[
+ \ddot{x}^T(t) \dot{P} \ddot{x}(t) + \ddot{x}^T(t) \dot{P} \ddot{x}(t) + \rho^2 \ddot{u}^T(t) \ddot{u}(t).
\]

From (57), we get

\[
\dot{A}_{ij} \dot{P} + \dot{P} \dot{A}_{ij} + 2 \Xi_i^T \dot{Q} + \Phi^T \Phi + \Omega_j^T \Omega_j + 3 \dot{P} \dot{P} \ddot{x}(t)
\]

\[
< - \frac{1}{\rho^2} \dot{P} \dot{P} - \dot{Q}. \tag{59}
\]

From (58) and (59), we get

\[
\dot{V}(t) \leq \sum_{i=1}^{L} \sum_{j=1}^{L} h_i(z(t)) h_j(z(t)) \ddot{x}^T(t) \left[ -\dot{Q} - \frac{1}{\rho^2} \dot{P} \dot{P} \right] \ddot{x}(t)
\]

\[
+ \frac{1}{\rho^2} \ddot{x}^T(t) \dot{P} \ddot{x}(t) + \rho^2 \ddot{u}^T(t) \ddot{u}(t).
\]

From the properties of \( h_i(z(t)) \) in (4) and (5), the above inequality can imply the following inequality:

\[
\dot{V}(t) \leq - \ddot{x}^T(t) \dot{Q} \ddot{x}(t) - \frac{1}{\rho^2} \ddot{x}^T(t) \dot{P} \ddot{x}(t)
\]

\[
+ \frac{1}{\rho^2} \ddot{x}^T(t) \dot{P} \ddot{x}(t) + \rho^2 \ddot{u}^T(t) \ddot{u}(t)
\]

\[
= - \ddot{x}^T(t) \dot{Q} \ddot{x}(t) + \rho^2 \ddot{u}^T(t) \ddot{u}(t). \tag{61}
\]

Since \( ||\ddot{u}(t)|| = ||u(t)|| \leq u_{bd} \), we get

\[
\dot{V}(t) < - \ddot{x}^T(t) \dot{Q} \ddot{x}(t) + \rho^2 u_{bd}^2
\]

\[
\leq - \delta_1 \ddot{x}^T(t) \dot{Q} \ddot{x}(t) + \rho^2 u_{bd}^2 \tag{62}
\]

where \( \delta_1 = \lambda_{\text{max}}(\dot{Q}) \).
Whenever \(|\dot{z}(t)| > p_u u_d/(\sqrt{c_2})\), \(\dot{V}(t) < 0\). By the argument as in Theorem 1, this demonstrates that the trajectories of the closed-loop system (50) are UUB.

Integrating (61) from \(t = 0\) to \(t = t_f\) yields

\[
V(t_f) - V(0) \leq -\int_0^{t_f} \dot{z}^T(t)\dot{z}(t)\,dt + \rho^2 \int_0^{t_f} \dot{w}^T(t)\dot{w}(t)\,dt.
\]  

(63)

From (55), we get

\[
\int_0^{t_f} \dot{z}^T(t)\dot{z}(t)\,dt \leq \dot{x}^T(0)\dot{P}\dot{x}(0) + \rho^2 \int_0^{t_f} \dot{w}^T(t)\dot{w}(t)\,dt.
\]  

(64)

This is (54) and the \(H^\infty\) control performance is achieved with a prescribed \(\rho^2\).

**Corollary 2:** In the case of \(\dot{w}(t) = 0\), suppose the fuzzy control law (43) is employed and there exists a common solution \(\hat{P} = \hat{P}^T > 0\) such that the matrix inequalities in (57) are satisfied then the closed-loop system (50) is quadratically stable.

**Proof:** The proof is trivial.

From the above analysis, the most important task to deal with the fuzzy observer-based state feedback problem is to solve a common solution \(\hat{P}\) from the matrix inequalities (57).

For the convenience of design, \(\hat{P}\) and \(\hat{Q}\) are chosen in the following form:

\[
\hat{P} = \begin{bmatrix} \hat{P}_{11} & 0 \\ 0 & \hat{P}_{22} \end{bmatrix}, \quad \hat{Q} = \begin{bmatrix} \hat{Q}_{11} & 0 \\ 0 & \hat{Q}_{22} \end{bmatrix}
\]  

(65)

where \(\hat{P}_{11} > 0\), \(\hat{P}_{22} > 0\), \(\hat{Q}_{11} > 0\), and \(\hat{Q}_{22} > 0\).

By substituting (65) into (57), we get

\[
\begin{align*}
\hat{A}^T_j\hat{P} + \hat{P}\hat{A}_j + \sum \Xi_i + \Psi^T \Phi + \Omega^T \Omega_j \\
+ \left(3 + \frac{1}{\rho^2}\right)\hat{P}\hat{P} + \hat{Q}
\end{align*}
\]

\[
= \begin{cases} 
\hat{P}_{11}(A_i + B_i K_j) + (A_i + B_i K_j)^T \hat{P}_{11} \\
+ 2(L_i C_p)^T(L_i C_p) + A_p^T A_p + \hat{Q}_{11} \\
+ (B_p K_j)^T(B_p K_j) + \left(3 + \frac{1}{\rho^2}\right)\hat{P}_{11} \hat{P}_{11} \\
\end{cases}
\]

\[
\{C_j^T \hat{L} \hat{P}_{11} + 2(L_i C_p)^T (L_i C_p) + A_p^T A_p \} 
\]

\[
\{\hat{P}_{11} L_i C_j + 2(L_i C_p)^T (L_i C_p) + A_p^T A_p \} 
\]

\[
\{\hat{P}_{22}(A_i - L_i C_j) + (A_i - L_i C_j)^T \hat{P}_{22} \\
+ 2(L_i C_p)^T (L_i C_p) + A_p^T A_p \\
+ \left(3 + \frac{1}{\rho^2}\right)\hat{P}_{22} \hat{P}_{22} + \hat{Q}_{22} \}
\]

(66)

By introducing a new matrix

\[
W = \begin{bmatrix} W_{11} & 0 \\ 0 & I \end{bmatrix} = \begin{bmatrix} \hat{P}_{11}^{-1} & 0 \\ 0 & I \end{bmatrix}
\]

where \(W_{11} = \hat{P}_{11}^{-1}\) and multiplying it into (66), we get

\[
\begin{bmatrix} 
\hat{P}_{11}(A_i + B_i K_j) + (A_i + B_i K_j)^T \hat{P}_{11} \\
+ 2(L_i C_p)^T(L_i C_p) + A_p^T A_p + \hat{Q}_{11} \\
+ (B_p K_j)^T(B_p K_j) + \left(3 + \frac{1}{\rho^2}\right)\hat{P}_{11} \hat{P}_{11} \\
\end{bcases}
\]

\[
\{C_j^T \hat{L} \hat{P}_{11} + 2(L_i C_p)^T (L_i C_p) + A_p^T A_p \} 
\]

\[
\{\hat{P}_{11} L_i C_j + 2(L_i C_p)^T (L_i C_p) + A_p^T A_p \} 
\]

\[
\{\hat{P}_{22}(A_i - L_i C_j) + (A_i - L_i C_j)^T \hat{P}_{22} \\
+ 2(L_i C_p)^T (L_i C_p) + A_p^T A_p \\
+ \left(3 + \frac{1}{\rho^2}\right)\hat{P}_{22} \hat{P}_{22} + \hat{Q}_{22} \}
\]

(68)

Therefore, (68) is equivalent to

\[
\begin{bmatrix} 
(A_i + B_i K_j) W_{11} + W_{11}(A_i + B_i K_j)^T \\
+ 2W_{11}(L_i C_p)^T(L_i C_p)W_{11} + W_{11} A_p^T A_p W_{11} \\
+ (B_p K_j) W_{11} W_{11} + \left(3 + \frac{1}{\rho^2}\right) I \\
\end{bmatrix}
\]

\[
\begin{bmatrix} 
W_{11} \hat{Q}_{11} W_{11} \\
\end{bmatrix}
\]

\[
\{L_i C_j + W_{11}(L_i C_p)^T + A_p^T A_p \} 
\]

\[
\{\hat{P}_{22}(A_i - L_i C_j) + (A_i - L_i C_j)^T \hat{P}_{22} \\
+ 2(L_i C_p)^T (L_i C_p) + A_p^T A_p \\
+ \left(3 + \frac{1}{\rho^2}\right)\hat{P}_{22} \hat{P}_{22} + \hat{Q}_{22} \}
\]

(69)

With \(Y_j = K_j W_{11}\), and \(Z_i = \hat{P}_{22} L_i\), the matrix inequalities (69) can be rearranged as the following form:

\[
\begin{bmatrix} 
M_{11}^W & W_{11} \\
W_{11} & M_{22} \\
B_p Y_j & 0 & -I \\
0 & 0 & 0 \\
{M^W_{11}} & 0 & 0 & {M^W_{44}} \\
0 & 0 & 0 & L_i C_p & -1/2I
\end{bmatrix}
\]

(70)

where

\[
M_{11}^W = A_i W_{11} + W_{11} A_p^T + B_i Y_j + (B_i Y_j)^T + \left(3 + \frac{1}{\rho^2}\right) I
\]

\[
M_{22}^W = C_j^T \hat{L} \hat{P}_{11} + 2(L_i C_p)^T(L_i C_p) + A_p^T A_p W_{11}
\]

\[
M_{44}^W = - (2(L_i C_p)^T(L_i C_p) + A_p^T A_p + \hat{Q}_{11})^{-1}
\]

\[
M_{44}^W = \hat{P}_{22}(A_i - L_i C_j) + (A_i - L_i C_j)^T \hat{P}_{22} \\
+ 2(L_i C_p)^T (L_i C_p) + A_p^T A_p \\
+ \left(3 + \frac{1}{\rho^2}\right)\hat{P}_{22} \hat{P}_{22} + \hat{Q}_{22}.
\]

The analysis above shows that when dealing with the fuzzy
observer-based fuzzy control system, the most important task is to solve common solutions $W_{11} = W_{11}^T > 0$ and $\hat{P}_{22} = \hat{P}_{22} > 0$ from the matrix inequalities (70). Since the variables $L_i$ and $Z_i; (=\hat{P}_{22} L_i)$ are cross-coupled, there are no effective algorithms for solving this matrix inequalities as yet. However, we easily check that the matrix inequalities (70) imply $M_{44}^* < 0$, i.e.,

$$\hat{P}_{22} A_i + A_i^T \hat{P}_{22} - Z_i C_j - C_j^T Z_i^T + A_i^T A_p + \left(3 + \frac{1}{\rho^2}\right) \hat{P}_{22} \hat{P}_{22} + \hat{Q}_{22} < 0 \quad (71)$$

for $i, j = 1, 2, \ldots, L$.

By the Schur complements, (71) can be transformed into the following linear matrix inequalities (LMI’s) for a prescribed $\rho$:

$$\begin{bmatrix}
\{\hat{P}_{22} A_i + A_i^T \hat{P}_{22} - Z_i C_j \\ C_j^T Z_i^T + A_i^T A_p + \hat{Q}_{22}\}
- \left(3 + \frac{1}{\rho^2}\right)^{-1} \hat{I}
\end{bmatrix} < 0, \quad (72)$$

Note that solving $\hat{P}_{22}$ and $Z_i$ (thus $L_i = \hat{P}_{22}^{-1} Z_i$) from (72) is a standard linear matrix inequality problem (LMI). By solving the LMIP in (72) and substituting $\hat{P}_{22}$, $Z_i$, and $L_i$ into (70), (70) become standard linear matrix inequalities (LMI’s). Then, solve the LMIP in (70) to obtain $W_{11}$ and $Y_j$ (thus $K_j = Y_j W_{11}^{-1}$).

If the LMI’s in (70) and (72) have a positive-definite solution for $W_{11}$ and $\hat{P}_{22}$, respectively, then the closed-loop system (50) is stable and the $H^\infty$ control performance in (54) is achieved for a prescribed $\rho$.

Therefore, the $H^\infty$ optimization design for fuzzy observer-based control system of (50) is formulated as the following constrained optimization problem:

$$\begin{align*}
\text{minimize} & \quad \rho^2 \\
\text{subject to} & \quad W_{11} = W_{11}^T > 0, \quad \hat{P}_{22} = \hat{P}_{22}^T > 0, \\
& \quad (70) \text{ and } (72). \quad (73)
\end{align*}$$

This problem can be solved by decreasing $\rho^2$ until $W_{11} > 0$ and $\hat{P}_{22} > 0$ cannot be found in (70) and (72).

The $H^\infty$ optimization design procedures of the fuzzy control systems are summarized as follows.

**Design Procedures:**

Step 1) Select fuzzy plant rules and membership function for nonlinear system (1).

Step 2) Select weighting matrix $Q_{11}$, $Q_{22}$ and bounding matrices $\Delta A_i (= \delta_i A_p)$ and $\Delta B_i (= \eta_i B_p)$ and $\Delta C_i (= \varphi_i C_p)$.

Step 3) Select the attenuation level $\rho^2$ and solve the LMIP in (72) to obtain $\hat{P}_{22}$ and $Z_i$ (thus $L_i = \hat{P}_{22}^{-1} Z_i$ can also be obtained).

Step 4) Substitute $\hat{P}_{22}$, $Z_i$, and $L_i$ into (70) and then solve the LMIP in (70) to obtain $W_{11}$ and $Y_j$ (thus $K_j = Y_j W_{11}^{-1}$ can also be obtained).

Step 6) Decrease $\rho^2$ and repeat Steps 3–5 until $W_{11}$ and $\hat{P}_{22}$ cannot be found.

Step 7) Check the assumptions of

$$||\Delta f|| \leq \left|\sum_{i=1}^{L} h_i(z(t)) \delta_i A_p x(t)\right|$$

$$||\Delta g|| \leq \left|\sum_{i=1}^{L} h_i(z(t)) \sum_{j=1}^{L} h_j(z(t)) \eta_k B_p \varphi_j x(t)\right|$$

and

$$\left|\sum_{i=1}^{L} h_i(z(t)) L_i \Delta h_i\right| \leq \left|\sum_{i=1}^{L} h_i(z(t)) L_i C_p \varphi_i x(t)\right|$$

If they are not satisfied, adjust (expand) the bounds for all elements in $\Delta A_i$, $\Delta B_i$, and $\Delta C_i$ and then repeat Steps 3–6.

Step 8) Construct the fuzzy observer in (41).

Step 9) Obtain fuzzy control rule in (43).

**Remark 3:**

1) The procedures for determining $\delta_i$, $\eta_i$, $\varphi_i$, $A_p$, $B_p$, and $C_p$ are described by the following simple example.

Assuming that the possible bounds for all elements in $\Delta A_i$, $\Delta B_i$, and $\Delta C_i$ are

$$\Delta A_i = \begin{bmatrix} \Delta a_{11} & \Delta a_{12} \\ \Delta a_{21} & \Delta a_{22} \end{bmatrix}$$

$$\Delta B_i = \begin{bmatrix} \Delta b_{11} & \Delta b_{12} \\ \Delta b_{21} & \Delta b_{22} \end{bmatrix}$$

and

$$\Delta C_i = \begin{bmatrix} \Delta c_{11} & \Delta c_{12} \\ \Delta c_{21} & \Delta c_{22} \end{bmatrix}$$

where $-c_{ars} \leq \Delta a_{rs} \leq c_{ars}$, $-c_{rk} \leq \Delta b_{rk} \leq c_{rk}$, and $-c_{cq} \leq \Delta c_{qk} \leq c_{cq}$ for some $c_{ars}$, $c_{rk}$, and $c_{cq}$ and $r, k, q = 1, 2$.

One possible description for the bounding matrices $\Delta A_i$, $\Delta B_i$, and $\Delta C_i$ is

$$\Delta A_i = \begin{bmatrix} \delta_{11} & 0 \\ 0 & \delta_{22} \end{bmatrix} \begin{bmatrix} c_{a11} & c_{a12} \\ c_{a21} & c_{a22} \end{bmatrix} = \delta_i A_p$$

$$\Delta B_i = \begin{bmatrix} \eta_{11} & 0 \\ 0 & \eta_{22} \end{bmatrix} \begin{bmatrix} c_{b11} & c_{b12} \\ c_{b21} & c_{b22} \end{bmatrix} = \eta_k B_p$$

$$\Delta C_i = \begin{bmatrix} c_{c11} & c_{c12} \\ 0 & c_{c22} \end{bmatrix} = C_p \varphi_i$$

where $-1 \leq \delta_{11}, \delta_{22}, \eta_{11}, \eta_{22} \leq 1$, and $-1 \leq c_{rs}, c_{qk} \leq 1$ for $r, k, q = 1, 2$. 

![Fig. 1. Fuzzy sets of $x_i$.](image-url)
Fig. 2. The trajectories of states $x_1$ and $x_2$ (including estimated states $\hat{x}_1$ and $\hat{x}_2$). ($x_1$: solid line; $x_2$: dashed line; $\hat{x}_1$: dotted line; $\hat{x}_2$: dash–dot line.)

Note that $\delta_i$, $\eta_i$, and $\varphi_i$ can be chosen by other forms as long as $\|\delta_i\| \leq 1$, $\|\eta_i\| \leq 1$, and $\|\varphi_i\| \leq 1$. Then follow the design procedures and check (12), (13), and (46) in the simulation. If they are not satisfied, adjust (expand) the bounds for all elements in $\Delta A_i$, $\Delta B_i$, and $\Delta C_i$, and repeat the design procedures until (12), (13), and (46) hold.

2) In general, it is not easy to solve the EVP or LMIP analytically. Fortunately, the EVP or LMIP can be solved very efficiently using convex optimization techniques such as interior point algorithm [15]. Software packages such as LMI optimization toolbox in Matlab are developed for this purpose [27].

V. SIMULATION EXAMPLE

To illustrate the fuzzy linear control approach, a balancing problem of an inverted pendulum on a cart is considered. For this example, the state equations of the inverted pendulum is given by

$$
\begin{align*}
\dot{x}_1 &= x_2, \\
\dot{x}_2 &= \frac{1.0}{\left((M+m)(J + ml^2) - (ml \cos x_1)^2\right)} \\
&\cdot \left(-f_1(M+m)x_2 - (mlx_2)^2 \sin x_1 \cos x_1 \right) \\
&\quad + (M+m)mg \sin x_1 - ml \cos x_1 u) + d \\
y &= x_1
\end{align*}
$$

(74)

where $x_1$ denotes the angle (rad) of the pendulum from the vertical, $x_2$ is the angular velocity (rad/s), $g = 9.8$ m/s$^2$ is the gravity constant, $m$ is the mass (kg) of the pendulum, $M$ is the mass (kg) of the cart, $f_1$ is the friction factor (N/rad/s) of the pendulum, $l$ is the length (m) from the center of mass of the pendulum to the shaft axis, $J$ is the moment of inertia (kg$\cdot$m$^2$) of the pendulum, $d$ is external disturbance, and $u$ is the force (N) applied the cart. The pendulum parameters are chosen as $m = 0.22$ (kg), $M = 10$ (kg), $l = 0.304$ (m), $J = 0.004063$ (kg$\cdot$m$^2$), and $f_1 = 0.007056$ (N/rad/s).

Now, following the design procedure in the preceding section, the robust performance design is given by the following steps.

Steps 1 and 2: To use the fuzzy linear control approach, we must have a fuzzy model which represents the dynamics of the nonlinear plant. Therefore, we first represent the system (74) by a Takagi–Sugeno fuzzy model. To minimize the design effort and complexity, we try to use as few rules as possible. Hence, we approximate the system by the following four-rule fuzzy model:

**Rule 1:** If $x_1$ is about 0

THEN $\dot{x} = A_1 x + B_1 u + w$, $y = C_1 x$

**Rule 2:** If $x_1$ is about $\pm \pi/9$

THEN $\dot{x} = A_2 x + B_2 u + w$, $y = C_2 x$

**Rule 3:** If $x_1$ is about $\pm 2\pi/9$

THEN $\dot{x} = A_3 x + B_3 u + w$, $y = C_3 x$

**Rule 4:** If $x_1$ is about $\pm \pi/3$

THEN $\dot{x} = A_4 x + B_4 u + w$, $y = C_4 x$
Fig. 3. The control input.

where

\[
A_1 = \begin{bmatrix} 0 & 1 \\ 0 & -0.2839 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ -0.2633 \end{bmatrix},
\]

\[
A_2 = \begin{bmatrix} 0 & 1 \\ 0 & -0.2833 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ -0.2469 \end{bmatrix},
\]

\[
A_3 = \begin{bmatrix} 0 & 1 \\ 0 & -0.2818 \end{bmatrix}, \quad B_3 = \begin{bmatrix} 0 \\ -0.2002 \end{bmatrix},
\]

\[
A_4 = \begin{bmatrix} 0 & 1 \\ 0 & -0.2802 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 0 \\ -0.1299 \end{bmatrix},
\]

and the bounding matrices are chosen as

\[
A_p = \begin{bmatrix} 0.1076 & 0.0030 \\ 0 & 0.0014 \end{bmatrix}, \quad B_p = \begin{bmatrix} 0 \\ 0 \end{bmatrix},
\]

\[
w = [0 \ d]^T, \quad C_i = [1 \ 0], \quad \delta_i = I, \quad \eta_i = I \text{ (for } i = 1, 2, \cdots, 4).\]

Membership functions for Rules 1–4 are shown in Fig. 1. Select

\[
\tilde{Q}_{11} = \tilde{Q}_{22} = \begin{bmatrix} 0.01 & 0 \\ 0 & 0.01 \end{bmatrix}.
\]

Steps 3–6: The optimal \( \rho^2 = 0.6084 \) is found after several iterations using the LMI optimization toolbox in Matlab. In this case, we obtain the common solution for (70) and (72) as follows:

\[
W_{11} = \begin{bmatrix} 16.1564 & -80.0150 \\ -80.0150 & 434.0087 \end{bmatrix},
\]

\[
\tilde{R}_{22} = \begin{bmatrix} 1.9318 & -0.2116 \\ -0.2116 & 0.2038 \end{bmatrix}.
\]

Step 7: The assumptions of

\[
\left\| f(x) - \sum_{i=1}^{4} h_i(z(t))A_i x(t) \right\| \leq \sum_{i=1}^{4} h_i(z(t))A_i x(t)
\]

and

\[
\left\| (g(x) - \sum_{i=1}^{4} h_i(z(t))B_i x(t)) \right\| \leq \sum_{i=1}^{4} h_i(z(t))B_i x(t)
\]

are satisfied (refer to Figs. 4 and 5).

Step 8: Then, we construct the observer as

\[
\dot{x}(t) = \sum_{i=1}^{4} h_i(x_1(t))[(A_i \dot{x}(t) + B_i u(t)) + L_i (\gamma(t) - \tilde{y}(t))].
\]
Fig. 4. The plots of $\left\| f(x(t)) - \sum_{i=1}^{l} h_i(x_1(t)) A x(t) \right\|$ (dashed line) and $\left\| \sum_{i=1}^{l} h_i(x_1(t)) \Delta A x(t) \right\|$ (solid line).

Fig. 5. The plots of $\left\| \sum_{i=1}^{l} h_i(x_1(t)) \sum_{j=1}^{k} h_j(x_1(t)) g(x(t)) - B_i K_0 \dot{x}(t) \right\|$ (dashed line) and $\left\| \sum_{i=1}^{l} h_i(x_1(t)) \sum_{j=1}^{k} h_j(x_1(t)) \Delta B_i K_0 \dot{x}(t) \right\|$ (solid line).
Step 9: Therefore, we obtain the control law
\[ u(t) = \sum_{j=1}^{4} h_j(x_2(t))K_j\hat{\theta}(t). \]

Figs. 2–5 present the simulation results. Initial condition is assumed to be \((x_2(0), \dot{x}_2(0), \ddot{x}_2(0)) = (\pi/4, 0, 0)^T\), and the external disturbance \(d\) is assumed to be periodically square wave with amplitude ±0.2 in the simulations. Fig. 2 shows the trajectories of the states \(x_2\) and \(\dot{x}_2\) (including estimated states \(\ddot{x}_2\) and \(\ddot{x}_2\)). The control input is presented in Fig. 3. The simulation results show that the fuzzy observer-based controller can balance the inverted pendulum with large external disturbance and the \(H^\infty\) performance can be achieved. Due to persistent periodical external disturbance, state variables \(x_4\) and \(x_5\) still have small vibration around zero at steady state.

VI. CONCLUSION

In this paper a fuzzy linear control technique and an \(H^\infty\) attenuation technique have provided a rough tuning and a precise tuning, respectively, and are combined to achieve robust performance for nonlinear dynamic systems. If the state variables are unavailable, a fuzzy observer-based control scheme has been also proposed to achieve \(H^\infty\) performance. Furthermore, the stability of the fuzzy control systems is also guaranteed in this work.

Actually, the proposed fuzzy linear control can be applied to any robust control design of nonlinear systems. With the aid of any linear fuzzy approximation algorithm and LMI technique, the robust \(H^\infty\) control design can be extended from linear systems toward nonlinear systems. A robust optimization technique is also developed. By employing the \(H^\infty\) attenuation technique, the performance of linear fuzzy control design for nonlinear systems can be significantly improved. Furthermore, the robust fuzzy control scheme is also developed to eliminate as much as possible the effect of the external disturbance. Therefore, the proposed design algorithm is appropriate for practical control design of mechanical systems with bounded external disturbances. The proposed design method is simple and the number of membership functions for the proposed control law can be extremely small. However, because of the use of fuzzy approximation technique and \(H^\infty\) control scheme, the results are less conservative than the other robust control methods. Based on the interior point optimization technique, a design procedure is proposed for the fuzzy observer-based control to achieve the robust optimization design of the nonlinear systems. Simulation results indicate that the desired robust performance for nonlinear systems can be achieved via the proposed method.

REFERENCES


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