Sensor placement by maximal projection on minimum eigenspace for linear inverse problem

Chaoyang Jiang, Yeng Chai Soh, and Hua Li

Abstract—This paper presents a new greedy sensor placement algorithm, named maximal projection on minimum eigenspace (MPME), for linear inverse problem. For each sensing location, the projection of its observation vector onto the eigenspace associated with the minimum eigenvalue of the dual observation matrix is shown to be negative correlated with the worst case error variance (WCEV) of the estimated parameters. We individually select the sensing location whose observation vector has a maximum projection onto the eigenspace of the minimum eigenvalue of the current dual observation matrix. The proposed MPME is shown computationally more efficient than the state-of-the-art. The Monte-Carlo simulation shows that the MPME outperforms the convex relaxation method [1], the SparSenSe [2], and the FrameSense [3] in terms of the WCEV and the mean square error (MSE) of the estimated parameters, especially when the number of sensor nodes is slightly more than the dimension of the estimated vector.

Index Terms—Inverse problem, sensor placement, greedy algorithm, rank-one modification.

I. INTRODUCTION

SENSOR networks are widely used in monitoring temporal-spatial physical fields. Each sensor node can observe the field intensity (e.g., temperature, humidity, and concentration of contaminant, etc.) at a particular location, but with a network of sparse sensor observations, a physical field of interest may be reconstructed by solving a linear inverse problem [3]–[8]. In physical field estimation, the number and the spatial locations of sensor nodes are closely related to the coverage, cost, battery energy consumption, and even the error of the estimated physical field. Therefore, the determination of the number and locations of sensor nodes is critical in sensor network design.

For a linear inverse problem, the sensor placement is to seek a group of sensing locations with the minimum number from a known domain such that the estimation accuracy can meet the requirement. Specifically, assuming that the observation models of all potential sensing locations are known, we select the minimum number required with which the physical field of interest can be recovered within a predefined accuracy. Obviously, one straightforward method is to evaluate the performance of all possible combinations of all possible sizes of the candidate sensing locations, and then find the one with fewest sensor nodes satisfying the required estimation accuracy. Such a combinatorial approach is computationally intractable. In practice, direct enumeration is impossible if the number of potential sensing locations is large. Apart from the enumeration method, the optimal solution can also be obtained by the branch-and-bound methods [9], [10], which unfortunately do take a very long time, even for a modest scale problem [1]. Consequently, in recent years the sensor placement for the linear inverse problem has attracted increasing attention to find a good suboptimal solution [1]–[8], [11]–[15].

A. Related Prior Work

Heuristics have been proposed to reduce the cost of exhaustive search. The simplest one is to place sensor nodes at the spatial maxima and minima of the proper orthogonal components of the physical field of interest [5]. This method is simple but only suitable for some special cases [6]. Other heuristics include genetic algorithms [13], particle swarm optimizer [14], tabu search [14], and cross-entropy optimization [15]. They all involve a prohibitive computational cost and the solutions have no optimality guarantee.

Joshi and Boyd [1] formulated the sensor placement problem as an elegant nonconvex optimization problem, and approximated it into one convex optimization problem based on the relaxation of the nonconvex Boolean constraints, representing the sensor placement, to a convex box set. This convex relaxation then was used in many work [2], [11], [16]–[19]. The sensing locations can be easily determined based on the solution of the convex optimization problem, but the sensor placement has no optimality guarantees due to the gap between the nonconvex and the convex optimization problems. Such a result has been shown to be no better than other work [3], [18], [19]. However, the authors in [1] provided a local optimization technique to improve the result, and many numerical examples showed that with the local optimization, the convex relaxation method can provide very good results.

The sensor placement problem was also solved by some greedy algorithms in which the sensor locations are individually determined by optimizing some proxies of the error of the estimated physical field, such as the determinant of Fisher information matrix [12], and the condition number [6]–[8] and the frame potential [3] of the observation matrix. The η-confidence ellipsoid of the estimation error depends on the determinant of the Fisher information matrix [1], which was optimized using one greedy method in [12], but it is shown to be no better than other methods in the examples in [3].
For the sensor placement problem, the minimum requirement of the solution is that the observation model should be well-conditioned. Therefore, some researchers determined the sensor locations by minimizing the condition number of the observation matrix. However, the condition number is a concept for nonsingular matrix, and we need to firstly determine a group of sensing locations to guarantee that the observation matrix is nonsingular, which is unfortunately a combinatorial problem. Additionally, the minimum condition number of the observation matrix does not mean the minimum estimation error except when the observation vector of each sensing location has the same norm because the sensing energy should be considered, which corresponds to the signal-noise-ratio. Recently, Ranieri et al. provided a novel greedy algorithm by minimizing the frame potential of the observation matrix. This method is computationally efficient but: 1) like the condition number minimization, it is only effective for the case that all the sensing locations have the same sensing energy; 2) it cannot guarantee that the observation matrix is well-conditioned.

All the above mentioned work focused on the case that the number of sensor nodes is fixed. One sparse-promoting technique has been used to minimize the number of required sensor nodes by adding a sparsity-promoting penalty term to the cost function. The convex relaxation technique is applied to relax the noncovex sparsity-aware sensor placement problem into a convex one. Such an operation has a drawback which will be detailed later.

Besides the sensor placement for the linear inverse problem, many other excellent sensor placement work focused on the continuous system, nonlinear model, energy saving, state estimation for dynamic system, and Gaussian process interpolation.

**B. Our Contributions**

In this paper, we propose a new greedy algorithm to minimize the number of required sensor nodes and determine their locations for a linear inverse problem such that the estimation error meets the requirement. We determine the sensing location individually until the estimation accuracy is satisfied by maximizing the projection of each observation vector onto the eigenspace of the minimum eigenvalue of the current dual observation matrix. It is shown that such a projection is negatively correlated with the worst case error variance (WCEV) and the mean square error (MSE).

Compared with the state-of-the-art, the proposed greedy algorithm which we call the maximal projection on minimum eigenspace (MPME), has the following advantages:

- The MPME outperforms the convex relaxation method, the SparSenSe, and the FrameSense in terms of the WCEV and the MSE of the estimated vector.
- The MPME can guarantee that the observation matrix is well-conditioned but the convex relaxation method, the SparSenSe and the FrameSense cannot, especially when the number of sensor nodes is near the dimension of the estimated vector.
- With the local optimization, the MPME still outperforms the convex relaxation method, the SparSenSe and the FrameSense.
- The proposed MPME is one of the most computationally efficient sensor placement algorithms.

**C. Outline and Notations**

The rest of this paper is organized as follows. In Section II, we introduce the linear inverse problem and the sensor placement problem, and briefly review three current methods. In Section III, we present the MPME algorithm. We provide two examples to compare the effectiveness of MPME with current methods via Monte-Carlo simulations in Section IV. In Section V, we analyze the computational cost of the MPME algorithm and compare it with current methods. The conclusions are given in Section VI.

This paper uses the following notations: Upper (lower) bold letters as \( \mathbf{A}(\alpha) \) or \( \mathbf{\Phi}(\varphi) \) indicate matrices (column vectors), and the corresponding lower normal letters indicate the entries of the matrices (column vectors). \( (\cdot)^T \) denotes transposition. \( \mathbf{I} \) represents an identify matrix with proper dimension whose \( i \)-th column vector is denoted by \( \mathbf{e}_i \). \( \mathbf{1} \) is a vector with all entries one in a proper dimension. \( \mathbb{E}(\cdot), \text{tr}(\cdot), \text{span}(\cdot), \det(\cdot), \text{dim}(\cdot), \text{rank}(\cdot) \) are the expectation, trace, spanned space, determinant, dimension, and rank operators, respectively.

**II. Problem Statement**

A. **Linear Inverse Problem**

We consider a physical field \( \mathbf{f} \in \mathbb{R}^N \) which can be described as

\[
\mathbf{f} = \mathbf{\Phi}\alpha
\]

where \( \alpha \in \mathbb{R}^n \) is a vector of parameters to be estimated, \( n \ll N \), and \( \mathbf{\Phi} \in \mathbb{R}^{N \times n} \) is a known full column-rank matrix, which we call the signal representation matrix and its column vectors compose a basis of the physical field.

From sensor networks, we can obtain isolated observations of the physical field. We denote the number of sensor nodes by \( M(n \leq M \ll N) \). One sensor observation can be described as

\[
y_i = \mathbf{h}_i^T \mathbf{f} + \nu_i = \varphi_{s_i}^T \alpha + \nu_i, \quad i \in \{1, 2, \ldots, M\}
\]

where \( \nu_i \) is the measurement noise, \( \mathbf{h}_i = \mathbf{e}_{s_i} \), \( s_i \) corresponds to the \( i \)-th sensing location, and \( \varphi_{s_i} \) represents the observation model of the \( i \)-th sensor node, which we call the observation vector. Clearly, \( \varphi_{s_i}^T \) is the \( s_i \)-th row of \( \mathbf{\Phi} \). Arraying all the sensor observations as a vector, we can obtain the following observation model

\[
y = \mathbf{H}\mathbf{f} + \nu = \mathbf{\Phi}\alpha + \nu
\]

where \( \mathbf{H} \in \mathbb{R}^{M \times N} \) is the sampling kernel, and the observation matrix, \( \mathbf{\Phi} = \mathbf{H}\mathbf{\Phi} = [\varphi_{s_1}, \varphi_{s_2}, \ldots, \varphi_{s_M}]^T \), is a pruned matrix from the rows of \( \mathbf{\Phi} \) indexed by \( \{s_1, s_2, \ldots, s_M\} \).

We assume that the sensor noises \( \nu_1, \nu_2, \ldots, \nu_M \) are zero-mean i.i.d. Gaussian random variables with variance \( \sigma^2 \). From
Consequently, the two error indicators are equivalent due to
For more detail about WCEV, do refer to [1]. Since
(2), we can obtain the following
where \( \Phi = \Phi^T \Phi \), which we call the dual observation matrix, and \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n \) stand for the eigenvalues of \( \Psi \). With some simple operations, we can obtain the variance of \( \alpha \)

\[
\Sigma = E \left[ (\hat{\alpha} - \alpha) (\hat{\alpha} - \alpha)^T \right] = E \left[ \Phi^+ \nu \nu^T (\Phi^+)^T \right] = \sigma^2 \Phi^+ (\Phi^+)^T = \sigma^2 \Psi^{-1}
\]

Then, we introduce the following worst case error variance
(WCEV) of the MVUE \( \hat{\alpha} \)

\[
WCEV(\hat{\alpha}) = \max_{\|x\|_2 = 1} x^T \Sigma x = \lambda_{\text{max}}(\Sigma) = \frac{\sigma^2}{\lambda_n}
\]

For more detail about WCEV, do refer to [1]. Since \( \Phi^+ (\Phi^+)^T = \Psi^{-1} \), it is easy to find from [4] and [5] that

\[
\text{MSE}(\hat{\alpha}) = \sigma^2 \| \Phi^+ \|_F^2 \quad \text{WCEV}(\hat{\alpha}) = \sigma^2 \| \Phi^+ \|_2^2
\]

Consequently, the two error indicators are equivalent due to the equivalence of the two matrix norms. Specifically,

\[
WCEV(\hat{\alpha}) \leq \text{MSE}(\hat{\alpha}) \leq \sqrt{n} WCEV(\hat{\alpha})
\]

It is clear in [4] and [5] that both the MSE and the WCEV depend on the eigenvalues of the dual observation matrix \( \Psi \), which fully depends on the sensing locations.

B. Sensor Placement Problem

We denote the set of selected sensing locations by \( S = \{s_1, s_2, \ldots, s_M\} \), and the set of potential sensing locations by \( \mathcal{N} = \{1, 2, \ldots, N\} \), which correspond to the row index of \( \Phi^+ \) and \( \Phi \), respectively. Then, we formulate the following sensor placement problem.

Problem 1: Given a matrix \( \tilde{\Phi} \in \mathbb{R}^{N \times n} \), select \( M \) rows of \( \tilde{\Phi} \) indexed by \( s_1, s_2, \ldots, s_M \) to construct the observation matrix \( \Phi \in \mathbb{R}^{M \times n} \), such that the error of the estimated parameters \( \hat{\alpha} \) in [3] is small enough and the number of rows of \( \Phi \), i.e., \( M \), is minimized.

We aim to find the minimum number of sensing locations with which the error of \( \hat{\alpha} \) is less than a predefined threshold. In this paper, we use the WCEV as the error indicator, and equation [6] shows that a small WCEV can guarantee a small MSE. Then, this sensor placement problem can be formulated by the following cardinality minimization problem

\[
\hat{S} = \arg \min_{S \subseteq \mathcal{N}} |S| \quad \text{subject to} \quad \lambda_n \geq \gamma
\]

where \( | \cdot | \) returns the cardinality of a set, and \( \gamma \) corresponds to the maximum accepted WCEV.

C. The state of the art

The sensor placement problem [7] is a combinatorial optimization problem which is typical NP-hard [27]. Here, we briefly review three current related methods.

1) Convex relaxation [17]: When the number of sensor nodes is fixed, the sensor placement problem can be formulated as the optimization problem

\[
\begin{align*}
\text{maximize} & \quad \log \det \left( \sum_{i=1}^{N} w_i \varphi_i \varphi_i^T \right) \\
\text{subject to} & \quad 1^T w = M \\
& \quad w_i \in \{0, 1\}, \quad i \in \mathcal{N}
\end{align*}
\]

with variable \( w \in \mathbb{R}^N \). Here \( w_i = 1 \) means \( i \in S \) and \( w_i = 0 \) means \( i \notin S \). Performing a convex relaxation, i.e., replacing the nonconvex Boolean constraints \( w_i \in \{0, 1\} \) by \( w_i \in [0, 1] \), we can obtain the following convex optimization problem:

\[
\begin{align*}
\text{maximize} & \quad \log \det \left( \sum_{i=1}^{N} w_i \varphi_i \varphi_i^T \right) \\
\text{subject to} & \quad 1^T w = M \\
& \quad w_i \in [0, 1], \quad i \in \mathcal{N}
\end{align*}
\]

with variable \( w \). This problem can be solved by the interior-point methods [28]. Rearranging the entries of the solution of the relaxed problem [9], i.e., \( w^* \in [0, 1]^N \), in descending order yields the sequence \( \{w^*_1, w^*_2, \ldots, w^*_M\} \). Then, the set of the sensing locations is given by \( \hat{S} = \{s_1, s_2, \ldots, s_M\} \), i.e., the indices of the \( M \) largest elements of \( w^* \).

2) SparSenSe [2]: To minimize the number of required sensor nodes, the following convex optimization, called sparse-aware sensor selection (SparSenSe), is formulated.

\[
\begin{align*}
\text{minimize} & \quad \|w\|_1 \\
\text{subject to} & \quad \left[ \sum_{i=1}^{N} w_i \varphi_i \varphi_i^T e_j e_j^T \right] \geq 0, \quad j = 1, \ldots, n \\
& \quad \|x\|_1 \leq \gamma, \quad x_j \geq 0, \quad j = 1, \ldots, n \\
& \quad w_i \in [0, 1], \quad i \in \mathcal{N}
\end{align*}
\]

This is a linear matrix inequalities problem and can be solved using the Matlab LMIs toolbox [29]. With the solution \( w^* \), the sensing locations can be found like the convex relaxation method.

3) FrameSense [3]: According to the frame theory, if the number of sensor nodes is fixed, one frame \( \Phi \), whose rows are extracted from the equal-norm frame \( \tilde{\Phi} \), achieves the minimum MSE when it achieves the minimum frame potential [3]. Here, the ensemble of the rows of \( \Phi \) or \( \tilde{\Phi} \) is viewed as one frame. For the basic concept of the frame theory, do refer to [31]. The frame potential is considered to be a proxy for the MSE of \( \alpha \), and the sensor placement problem can be solved by minimizing the following frame potential

\[
FP(\Phi) = \sum_{i,j \in \hat{S}} (\varphi_i^T \varphi_j)^2
\]

One greedy “worst-out” algorithm, called the FrameSense, can provide near-optimal solution in the sense of the minimum
frame potential. At each iteration, it removes the row of \( \Phi \) that maximally increase the frame potential. The row index is exactly the index of the row of \( (\Phi \Phi^T)^2 \) which is with the largest \( l_1 \)-norm. Here, \( A^2 \) denotes a matrix whose entries are square of the corresponding entries of \( A \). For more details about the FrameSense, do refer to [3].

Remark 1: For the convex relaxation method and the SparSense, the minimum number of required sensor nodes can be determined by increasing the sensor number from \( n \) until the constraint in (7) is satisfied. For the FrameSense, when removing each row of \( \Phi_s \), the constraint should be checked until it is unsatisfied.

III. MAXIMAL PROJECTION ON MINIMUM EIGENSPACE

We cannot directly solve the cardinality optimization problem (7) due to the huge computational cost of exhaustively searching \( 2^N \) potential sensor placements. One simple strategy to reduce the number of searched sensor placement is to individually determine the sensing locations. With such a strategy, the minimum number of required sensor nodes, \( M \), can be easily found by judging whether the constraints in (7) are satisfied after each sensing location is determined. Therefore, the number of searched sensor placement can be reduced to \( MN - M(M - 1)/2 \).

Admittedly, the influence of each sensor node is correlated with other sensing locations. One sensor reading is informative for a given sensor placement but may be meaningless for others. We do not know the contribution of each sensor node for the final sensor placement when individually finding the sensing locations; therefore, we cannot find the optimal solution. However, it is possible to find an effective sensor placement with proper sensor number by individually determining each sensing location.

We introduce a new matrix \( \Phi_k \in \mathbb{R}^{k \times n} \) to denote the first \( k(1 \leq k \leq M) \) rows of \( \Phi \), which corresponds to the first \( k \) sensing locations. Accordingly, we introduce \( \Psi_k = \Phi_k^T \Phi_k \) which has the nonincreasing eigenvalue sequence \( \{\lambda_i(k)\}_{i=1}^n \).

For a given sensor placement corresponding to \( \Phi_{k-1}(1 \leq k < M) \), we need to formulate some guidelines to determine the next sensing location \( s_k \). However, we do not know how \( \varphi_{s_k} \) affects \( \lambda_n(M) \) since \( \{\varphi_{s_j}\}_{j=k+1}^M \) and even the required sensor number \( M \) is unknown.

A. Minimum Nonzero Eigenvalue Pursuit

**Question 1:** When determining \( k \)-th sensing location, can we find a variable which is positively correlated with \( \lambda_n(M) \) and meanwhile can be directly obtained from \( \Phi_{k-1} \) and \( \varphi_{s_k} \)?

To help answer this question, we introduce the following rank-one modification theorem.

**Theorem 1:** Suppose \( B = A + c c^T \) where \( A \in \mathbb{R}^{n \times n} \) is symmetric, and \( c \in \mathbb{R}^n \) is a non-zero vector. Then,

\[
\lambda_1(B) \geq \lambda_1(A) \geq \lambda_2(B) \geq \lambda_2(A) \geq ... \geq \lambda_n(B) \geq \lambda_n(A)
\]

**Proof:** See [32] and Theorem 8.1.8 in [33].

It is obvious that

\[
\Psi_k = [\Phi_{k-1}^T \varphi_{s_k} | \Phi_{k-1}^T \varphi_{s_k}]^T = \Psi_{k-1} + \varphi_{s_k} \varphi_{s_k}^T
\]

For all \( k < n \), \( \Psi_k \) is singular and \( \lambda_n(k) = 0 \). Therefore, the number of sensor nodes must not be less than \( n \). We assume that \( \varphi_{s_k} \) is independent with \( \{\varphi_{s_1}, ..., \varphi_{s_{k-1}}\} \), which can be easily satisfied by properly selecting \( k \)-th sensing location. Considering Theorem 1, we have

\[
0 = \lambda_{k-1}^{(k-1)} \leq \lambda_k(k) \leq \lambda_{k-1}^{(k-1)} \quad \text{for all } k \leq n \quad (12)
\]

\[
0 \leq \lambda_{n}^{(k)} \leq \lambda_{n}^{(k-1)} \leq ... \leq \lambda_{1}^{(1)} \quad \text{for all } k \geq n \quad (13)
\]

For any \( k \leq n \), (12) shows that the minimum nonzero eigenvalue of \( \Psi_k \) is \( \lambda_n^{(k)} \), which is bounded by \( \lambda_{k-1}^{(k-1)} \), i.e.,

\[
\lambda_n(k) \leq \lambda_{n}^{(k)} \leq ... \leq \lambda_{1}^{(1)}
\]

Therefore, for all \( k < n \), if we select sensor location to maximize \( \lambda_n^{(k)} \), we maximize the upper bound of \( \lambda_n(M) \). Actually, \( \lambda_n^{(k)} \) is monotonic increasing with respect to \( \lambda_{k-1}^{(k-1)} \), which will be shown later.

For any \( k \geq n \), (13) shows that \( \lambda_n^{(k)} \) is always not less than \( \lambda_n^{(k-1)} \), i.e.,

\[
\lambda_n(M) \geq \lambda_{n}^{(M-1)} \geq ... \geq \lambda_n^{(n)}
\]

Thus, if we select the sensing location properly for all \( k \geq n \) to maximize \( \lambda_n^{(k)} \), we maximize the lower bound of \( \lambda_n \).

Consequently, we can view the eigenvalues \( \lambda_n^{(k)} \), \( \lambda_{k+1}^{(k+1)} \), ..., \( \lambda_n^{(n)} \), \( \lambda_{n+1}^{(n+1)} \), ..., \( \lambda_n^{(M-1)} \) as the bridges between \( \varphi_{s_k} \) and \( \lambda_n^{(M)} \).

To maximize \( \lambda_n^{(M)} \), we can select \( \varphi_{s_k} \) to maximize \( \lambda_n^{(k)} \) for \( k < n \) (\( \lambda_n^{(k)} \) for \( k \geq n \)), i.e., to maximize the minimum nonzero eigenvalue of \( \Psi_k \). The greedy algorithm which we call minimum nonzero eigenvalue pursuit (MNEP) is summarized in Algorithm 1.

**Algorithm 1:** Minimum nonzero eigenvalue pursuit

**Input:** \( \Phi = [\varphi_1, \varphi_2, ..., \varphi_N]^T \in \mathbb{R}^{N \times n} \)

**Output:** \( \Phi \in \mathbb{R}^{M \times n}, S \)

1. **Initialization:** \( N = \{1, 2, ..., N\}, S = \emptyset \)

2. **Determine the first \( n - 1 \) sensing locations:**
   (a) \( \Phi_0 = [\cdot], k = 1 \)
   (b) \( \tilde{s}_k = \arg \max_{i \in N \setminus S} \lambda_k(\Phi_{k-1}^T \Phi_{k-1} + \varphi_i \varphi_i^T) \)
   (c) Update: \( S = S \cup \{\tilde{s}_k\}, \Phi_k = [\Phi_{k-1}^T \varphi_{s_k}]^T \)
   (d) Set \( k = k + 1 \) and repeat step (b-c) until \( k = n \)

3. **Determine the remaining sensing locations:**
   (a) \( \tilde{s}_k = \arg \max_{i \in N \setminus S} \lambda_n(\Phi_{k-1}^T \Phi_{k-1} + \varphi_i \varphi_i^T) \)
   (b) Update: \( S = S \cup \{\tilde{s}_k\}, \Phi_k = [\Phi_{k-1}^T \varphi_{s_k}]^T \).
   (c) If \( \lambda_n(k) \geq \gamma \) return \( S \) and \( \Phi = \Phi_k \), else set \( k = k + 1 \) and repeat step (a-b).

B. MAXIMAL PROJECTION ON MINIMUM EIGENSPACE

When determining the \( k \)-th sensing location, we need to traverse \( N - k + 1 \) rows of \( \Phi \) and evaluate the minimum nonzero eigenvalue of \( \Psi_k \) (\( \lambda_n^{(k)} \) when \( k < n \) or \( \lambda_n^{(k)} \) when \( k \geq n \)) for each case. Solving an eigenvalue problem for each case is computationally expensive.
Question 2: Can we find a simpler alternative criteria for each row of \( \Phi \) to access its contribution for the minimum nonzero eigenvalue of \( \Psi_k \)?

To address this question, firstly we introduce the following theorem and show the relation between the eigenvalues of \( \Psi_k \) and the rows of \( \Phi_k \).

**Theorem 2:** For any matrix \( \Phi_k \in \mathbb{R}^{k \times n} \), the symmetric matrix \( \Psi_k = \Phi_k^T \Phi_k \) has the nonincreasing eigenvalue sequence \( \{\lambda_i(k)\}_{i=1}^n \), and

\[
\lambda_i(k) = \|\Phi_k u_i(k)\|_2^2
\]

where \( u_i(k) \) is the normalized eigenvector associated with \( \lambda_i(k) \).

**Proof:** The spectrum decomposition of \( \Psi_k \) is

\[
\Psi_k = \Phi_k^T \Phi_k = U_k \Lambda_k U_k^T
\]

where \( U_k = [u_1(k), u_2(k), \ldots, u_n(k)] \) is an orthonormal matrix, and \( \Lambda_k \) is a diagonal matrix whose diagonal entry \( \lambda_i(k) \) is \( \lambda_i(k) \). Then, we can obtain

\[
\Lambda_k = U_k^T \Phi_k^T \Phi_k U_k
\]

from which (14) can be directly found.

We denote

\[
\xi_i(k) = \Phi_k u_i(k)
\]

where the \( j \)-th element of \( \xi_i(k) \), \( \xi_{ij}(k) = \Phi_{ij} u_i(k) \) and \( |\xi_i(k)| \) represents the magnitude of the projection of \( \varphi_{s_j} \) onto \( u_i(k) \). Then, we introduce the following set of vectors to represent all the rows of \( \Phi_k \)

\[
P_k = \{\varphi_{s_1}, \varphi_{s_2}, \ldots, \varphi_{s_k}\}
\]

Considering (14), we can obtain \( \lambda_i(k) = \|\xi_i(k)\|^2 \). It means that \( \lambda_i(k) \) is the square summation of the projections of all the vectors in \( P_k \) onto the eigenspace \( U_k^{(k)} = \operatorname{span}(u_i(k)) \).

Next, we introduce the Courant-Fischer Minimax Theorem, and show one requirement for the rows of \( \Phi = \Phi_M \) to meet the requirement of the estimation accuracy.

**Theorem 3 (Courant-Fischer Minimax Theorem):** If \( A \in \mathbb{R}^{n \times n} \) is symmetric, then for \( i = 1 : n \),

\[
\lambda_i(A) = \max_{\dim(U) = i} \min_{x \notin U} x^T Ax / \|x\|^2
\]

**Proof:** See the proof of Theorem 8.1.2 in [33].

Since \( x^T \Psi_k x = x^T \Phi_k^T \Phi_k x = \|\Phi_k x\|^2 \), from Theorem 3 we have

\[
\lambda_n(k) = \min_{\|x\|_2 = 1} x^T \Psi_k x = \min_{\|x\|_2 = 1} \|\Phi_k x\|^2
\]

Then, from (14) and (17), we can obtain

\[
u_n(k) = \arg \min_{\|x\|_2 = 1} \|\Phi_k x\|^2
\]

Considering (14) and (18), we conclude that the minimum eigenvalue of \( \Psi_k \) (i.e., \( \lambda_n(k) \)) equals the square summation of the projections of all the vectors in \( P_k \) onto the eigenspace \( U_n^{(k)} \), and that we cannot find any other subspace of \( \mathbb{R}^n \) that the square summation of the projections of all the vectors in \( P_k \) onto it is less than \( \lambda_n(k) \). Therefore, to meet the minimum requirement on \( \lambda_n \) (i.e., \( \lambda_n \geq \gamma \)), we need to guarantee that the square summation of the projections of all the vectors in \( P_M \) onto any subspace of \( \mathbb{R}^n \) is greater than \( \gamma \).

For simplicity, we introduce a new concept, the minimum eigenspace, as follows.

**Definition 1:** For any symmetric matrix \( A \in \mathbb{R}^{n \times n} \) with the nonincreasing eigenvalue sequence \( \{\lambda_i(A)\}_{i=1}^n \), the minimum eigenspace of \( A \) is the eigenspace associated with all the minimum eigenvalues of \( A \), i.e.,

\[
\bigcup_{k=n}^1 (A) = \operatorname{span}(u_k, u_{k+1}, \ldots, u_n)
\]

where \( u_i \) is the eigenvector associated with \( \lambda_i(A) \), and \( \lambda_{k-1}(A) > \lambda_k(A) = \lambda_n(A) \).

When \( k \geq n \), \( \lambda_{k-1}(A) \) is a simple and the minimum eigenvalue of \( \Psi_{k-1} \). Thus, the minimum eigenspace of \( \Psi_{k-1} \) is \( U_{k,n}^{(k-1)} \) and \( U_{n}^{(k-1)} \) when \( k < n \) and \( k \geq n \), respectively.

For a given sensor placement associated with \( \Phi_{k-1} \), the square summation of the projections of all the vectors in \( P_{k-1} \) onto \( U_{k,n}^{(k-1)} \) is minimum when \( k \geq n \). Consequently, to determine the \( k \)-th sensor location and maximize the minimum nonzero eigenvalue of \( \Psi_k \) (i.e., \( \lambda_n(k) \) for \( k < n \) and \( \lambda_n(k) \) for \( k \geq n \)), one intuitive idea is to select \( \varphi_{s_k} \) as the one that has a maximum projection onto the minimum eigenspace of \( \Psi_{k-1} \).

We introduce a new vector

\[
z = U_{k-1}^T \varphi_{s_k}
\]

It is clear that the square of the projection of \( \varphi_{s_k} \) onto the minimum eigenspace of \( \Psi_{k-1} \) is

\[
\zeta_k = \begin{cases} 
\sum_{i=k}^n z_i^2 & \text{if } k < n \\
\sum_{i=k}^n z_i^2 & \text{if } k \geq n
\end{cases}
\]

Then, we provide the following theorem with a corollary from which we show that the square of the projection onto the minimum eigenspace of \( \Psi_{k-1} \) is the criteria that we want to answer Question 2.

**Theorem 4:** For any matrix \( \Phi_k = [\varphi_{s_1}, \varphi_{s_2}, \ldots, \varphi_{s_k}]^T \in \mathbb{R}^{k \times n} \), the symmetric matrix \( \Phi_k = \Phi_k^T \Phi_k \) has a nonincreasing eigenvalue sequence \( \{\lambda_i(k)\}_{i=1}^n \). If \( k < n \),

\[
\lambda_n(k) = \frac{\zeta_k}{1 + \sum_{i=1}^{k-1} \sum_{j=0}^{i} \zeta_i^2 / \lambda_{i+1} \lambda_k / \lambda_i}
\]

else if \( k \geq n \),

\[
\lambda_n(k) = \lambda_n(k-1) + \frac{\zeta_k}{1 + \sum_{i=1}^{n-1} \sum_{j=0}^{i} \zeta_i^2 / \lambda_{i+1} \lambda_{n} / \lambda_i}
\]

**Proof:** See Appendix A.

**Corollary 5:** The minimum nonzero eigenvalue of \( \Psi_k \), i.e., \( \lambda_n(k) \) for \( k < n \) and \( \lambda_n(k) \) for \( k \geq n \), is strictly monotonically increasing with respect to \( \zeta_k \), the square of the projection of \( \varphi_{s_k} \) onto the minimum eigenspace of \( \Psi_{k-1} \).
Proof: See Appendix B.

Corollary 1 implies that for each row of \( \Phi \), the square of the projection onto the minimum eigenspace of \( \Psi_{k-1} \) can be a criteria to access its contribution for the minimum nonzero eigenvalue of \( \Psi_k \). Therefore, \( c_k \) is positively correlated with \( \lambda_n(\Psi_k) \), i.e., negatively correlated with the WCEV.

Let \( P_{k-1} \) be a projection matrix which can project any vectors in \( \mathbb{R}^n \) onto the minimum eigenspace of \( \Psi_{k-1} \). When \( k < n \), the minimum eigenspace of \( \Psi_{k-1} \) is the null space of \( \text{span}\{\varphi_{s_1}, \varphi_{s_2}, \ldots, \varphi_{s_{k-1}}\} \). Then, we can find that

\[
P_{k-1} = I_n - R_{k-1} R_{k-1}^T
\]

where \( R_{k-1} = \text{orth}(\Phi_{k-1})^T \) whose column vectors are obtained from the Gram-Schmidt Orthonormalization of all the column vectors of \( \Phi_{k-1} \) (i.e., \{\varphi_{s_i}\}_{i=1}^{k-1} \). When \( k \geq n \), it is clear that

\[
P_{k-1} = u_n^{(k-1)}(u_n^{(k-1)})^T
\]

Then we can obtain that

\[
\zeta_k = \|P_{k-1}\varphi_{s_k}\|_2^2
\]

The computational cost to find the eigenvalue of a \( n \times n \) matrix is much more expensive than projecting one vector in \( \mathbb{R}^n \) onto a known subspace. Therefore, we maximize the projection of \( \varphi_{s_k} \) onto the minimum eigenspace of \( \Psi_{k-1} \) instead of the minimum nonzero eigenvector of \( \Psi_k \), and provide the following greedy sensor placement algorithm, which we call maximal projection on minimum eigenspace (MPME).

Algorithm 2: maximal projection on minimum eigenspace

Input: \( \Phi = [\varphi_1, \varphi_2, \ldots, \varphi_N]^T \in \mathbb{R}^{N \times n} \)

Output: \( \Phi \in \mathbb{R}^{M \times n} \), \( S \)

1. Initialization: \( N = \{1, 2, \ldots, N\} \), \( S = \emptyset \).

2. Determine the first \( n-1 \) sensing locations:
   (a) Set \( \Phi_0 = [1] \), \( P_0 = I_n \), and \( k = 1 \).
   (b) \( \delta_k = \arg \max_{i \in N \setminus S} \|P_{k-1}\varphi_i\|_2^2 \).
   (c) Update: \( S = S \cup \{\delta_k\} \), \( \Phi_k = [\Phi_{k-1} \varphi_{\delta_k}]^T \), \( R_k = \text{orth}(\Phi_k^T) \), \( P_k = I_n - R_k R_k^T \).
   (d) Set \( k = k + 1 \) and repeat step (b-c) until \( k = n \).

3. Determine the remaining sensing locations:
   (a) \( \delta_k = \arg \max_{i \in N \setminus S} \|P_{k-1}\varphi_i\|_2^2 \).
   (b) Update: \( S = S \cup \{\delta_k\} \), \( \Phi_k = [\Phi_{k-1} \varphi_{\delta_k}]^T \), \( \Phi_k^T \Phi_k = U_k \Lambda_k U_k^T \), \( P_k = u_k u_k^T \).
   (c) If \( \lambda_n^{(k)} \geq \gamma \) return \( S \) and \( \Phi = \Phi_k \), else set \( k = k + 1 \) and repeat step (a-b).

In the MNEP and MPME algorithms, the minimum number of required sensor nodes is determined by judging whether \( \lambda_n^{(k)} \geq \gamma \) is satisfied after each sensing location is determined when \( k \geq n \).

It is easily found from (21a) that \( \lambda_n^{(k)} \) is monotonic increasing with respect to \( \lambda_{k-1}^{(k-1)} \) for \( k < n \). Similarly, equation (22) shows that \( \lambda_n^{(k)} \) is monotonic increasing with respect to \( \lambda_n^{(k-1)} \) for \( k \geq n \). These are the reasons why in Section III-A we maximize the minimum nonzero eigenvalue to maximize \( \lambda_n^{(M)} (= \lambda_n) \) when individually finding each sensing location.

Although the starting point of maximizing the projection onto the minimum eigenspace of \( \Psi_{k-1} \) is to maximize the minimum nonzero eigenvalue of \( \Psi_k \), the MPME algorithm may provide a better result than the MNEP in terms of a larger \( \lambda_n^{(k)} \) when \( k \geq n \). Equations (21a) and (22) show that \( \lambda_n^{(k)} \) is monotonic decreasing with respect to \( z_i^2 \) for all \( i < k < n \), and \( \lambda_n^{(k)} \) is monotonic decreasing with respect to \( z_i^2 \) for all \( i < n \). Therefore, the MNEP algorithm prefers to select the row with large \( \zeta_k \) and small \( z_i^2 \), which may not be with the largest \( \zeta_k \) but with a balance between a large \( \zeta_k \) and small \( z_i^2 \). It is clear that \( \sum_{i=1}^{n} z_i^2 = ||\varphi_{s_k}||_2^2 \) and \( \sum_{i=1}^{k} ||\varphi_{s_i}||_2^2 = \sum_{i=1}^{k} \lambda_i^{(k)} \); therefore, small \( z_i^2 \) means small increment of \( \lambda_i^{(k)} \) from \( \lambda_i \) for all \( i < k \) when \( k < n \) and all \( i < n \) for \( k \geq n \). Additionally, equations (21a) and (22) show that both \( \lambda_n^{(k)} \) and \( \lambda_n^{(k)} \) are monotonic increasing with respect to \( \lambda_n^{(k-1)} \).

Therefore, compared with the MPME algorithm, selecting the row with small \( z_i^2 \) may lead to a relatively smaller \( \lambda_n^{(k+1)} \) and \( \lambda_n^{(k)} \), where \( k+1 > k \), which will be shown in the following examples.

IV. EFFECTIVENESS OF THE MPME ALGORITHM

Firstly, we provide a simple example and compare the MPME with the MNEP, the current convex relaxation [1], the SparSensE [2], and the FrameSense [3].

Example 1: \( \Phi \in \mathbb{R}^{100 \times 20} \) is a random matrix with components \( \varphi_{ij} \sim \mathcal{N}(0, 1) \), and the variance of the sensor noise \( \sigma^2 = 1 \).

To easily compare with the state of the art, in this simulation, we do not consider the stopping criteria for the proposed two algorithms but show the mean WCEV index and the mean MSE index for the number of sensor nodes increasing from 20 to 40. The mean WCEV index and the mean MSE index of 100 Monte-Carlo simulation run results are given in Fig. 1 and Fig. 2, respectively.

For \( i \)-th (1 \( \leq i \leq 100 \)) simulation run, we find the observation matrix \( \Psi_k^{(i)} \) (20 \( \leq k \leq 40 \)) from the random signal representation matrix \( \Phi^{(i)} \), and obtain the following MSE index and WCEV index from (4) and (5), respectively.

\[
\text{MSE}_{k}^{(i)} = \text{tr}((\Psi_{k}^{(i)})^{-1})
\]

\[
\text{WCEV}_{k}^{(i)} = \lambda_{\text{max}}((\Psi_{k}^{(i)})^{-1})
\]

where \( \Psi_{k}^{(i)} = (\Phi^{(i)})^{T}_{1:k} \Phi^{(i)}_{1:k} \). Then, the mean MSE index and the mean WCEV index of 100 Monte-Carlo simulation run results are given by

\[
\overline{\text{MSE}}_{k} = \frac{1}{100} \sum_{i=1}^{100} \text{MSE}_{k}^{(i)}
\]

\[
\overline{\text{WCEV}}_{k} = \frac{1}{100} \sum_{i=1}^{100} \text{WCEV}_{k}^{(i)}
\]

It is shown in Fig. 1 and Fig. 2 that the solution of MPME is better than all the other solutions in terms of the WCEV index and the MSE index. If the same number of sensor nodes
Fig. 1: The mean WCEV index of $\hat{\alpha}$ estimated from 20 to 40 sensor observations based on the five sensor placement algorithms.

Fig. 2: The mean MSE index of $\hat{\alpha}$ estimated from 20 to 40 sensor observations based on the five sensor placement algorithms.

are used, the MPME can provide the most accurate result of the inverse problem as compared with the other methods.

If we set the WCEV index threshold equals 0.3 (i.e., $\gamma = 10/3$), Fig. 1 shows that the minimum number of required sensor nodes $M_{\text{MNEP}} = M_{\text{MPME}} = 23$, $M_{\text{convex relaxation}} = M_{\text{SparSenSe}} = 28$, and $M_{\text{FrameSense}} = 37$. If we set the MSE index threshold equals 1.5, Fig. 2 shows that the minimum number of required sensor nodes $M_{\text{MPME}} = 23$, $M_{\text{MNEP}} = 25$, $M_{\text{convex relaxation}} = M_{\text{SparSenSe}} = 26$, and $M_{\text{FrameSense}} = 37$. Therefore, to meet the required accuracy, compared with other methods, the MPME algorithm requires the minimum number of sensor nodes.

Fig. 1 and Fig. 2 show that the MPME outperforms the MNEP, which we have analyzed in Section II-B. Additionally, we find that for all the five sensor placement algorithms, the improvement of the WCEV index and the MSE index are increasingly insignificant with the increase in the number of the sensor nodes. It means that the influence of the additional sensor observation declines and its location is not so critical as the previously determined sensing locations. It is clearly shown in Fig. 1 and Fig. 2 that the MPME is much better than the convex relaxation method, the SparSenSe, and the FrameSense in finding the critical sensing locations, which will be analyzed as follows.

A. Compared with the convex relaxation and the SparSenSe

Fig. 2 shows that the mean MSE indices of both the convex relaxation method and the SparSenSe are much larger than that of the MPME when the number of sensor nodes is near the dimension of the vector to be estimated (i.e., 20 in this example). Comparing Fig. 2 with Fig. 1, we can find that when using the convex relaxation method or the SparSenSe, the WCEV index (i.e., the maximum eigenvalue of $\Psi_k^{-1}$, see equation (24)) contributes the main part of the MSE index (i.e., the trace of $\Psi_k^{-1}$, see equation (23)), especially when the number of sensor nodes is small. It means that the maximum eigenvalue of $\Psi_k^{-1}$ is overwhelmingly larger than the others when $k \geq n$ (i.e., $k$ is slightly larger than $n$). In other words, the minimum eigenvalue of $\Psi_k$ is much smaller than the other eigenvalues, which implies that $\Psi_k$ is ill-conditioned, and hence the estimated vector $\hat{\alpha}$ is inaccurate.

The mean condition number of Monte-Carlo simulation result is shown in Fig. 3 For $i$-th ($1 \leq i \leq 100$) simulation run, the condition number of $\Psi_k^{(i)}$ is denoted by $\kappa(\Psi_k^{(i)})$. The mean condition number is given by

$$\kappa_k = \frac{1}{100} \sum_{i=1}^{100} \kappa(\Psi_k^{(i)})$$

Fig. 3 shows that for $k \leq 23$, $\Psi_k$ obtained from the convex relaxation method or the SparSenSe is ill-conditioned, whereas $\Psi_k$ obtained from the MNEP or the MPME is well-conditioned. Accordingly, as shown in Fig. 1 and Fig. 2, the mean MSE index and the mean WCEV index of the convex relaxation or the SparSenSe are much larger than those of the MPME, respectively.

It is clear that the convex relaxation method and the SparSenSe is not suitable for the case that the available sensor nodes are very limit. Both the two methods use the convex relaxation technique. In practice, the solution of the convex
optimization problem (9) or (10), $w^* \in [0, 1]^N$, is mapped into $\{0, 1\}^N$ to find the sensing locations. The largest $M$ elements of $w^*$ are mapped to 1 and other elements are mapped to 0. In such a mapping, the singularity of $\Psi$ is not considered. The number of sensor nodes is nearer the dimension of the estimated vector $\alpha$, the higher the probability of $\Psi$ being ill-conditioned.

The SparSenSe has a little improvement from the convex relaxation method in [1] by adding the sparsity-promoting penalty term. However, even in other works [11], [16]–[19], once the nonconvex Boolean constraints is relaxed to be a convex constraint, the action of mapping $w^* \in [0, 1]^N$ to a suboptimal solution of the original nonconvex optimization problem in $\{0, 1\}^N$ is necessary. Such an operation cannot guarantee that $\Psi$ is well-conditioned, which has been shown in Fig. [1] Fig. 3. This drawback can be overcome by the so called local optimization [1], but it is computationally expensive and we will discuss later.

Unlike the convex relaxation method and the SparSenSe, the proposed MPME selects the $k$-th sensor location with the maximum projection of $\varphi_{s_k}$ onto the minimum eigenspace of $\Psi_k$, which can guarantee a large minimum nonzero eigenvalue of $\Psi_k$ and accordingly a well-conditioned $\Psi$.

**B. Compared with the FrameSense**

It is apparently shown in Fig. 1 to Fig. 3 that the FrameSense provides the worst result for Example 1 in which $\Phi$ is not an equal-norm frame, i.e., the norm of the rows of $\Phi$ is not equal. For this case, the minimization of the frame potential in [11] will select the rows of $\Phi$ with small norm to construct the observation matrix $\Phi$. The FrameSense prefers to drop the rows with large norm [3]. It is easily found that

$$
\sum_{i=1}^{M} ||\varphi_{s_i}||^2 = \text{tr}(\Psi) = \sum_{i=1}^{n} \lambda_i
$$

Compared this equation with the MSE in [3], we conclude that a small norm of the rows of the observation matrix $\Phi$ leads to a large MSE of the estimated vector $\hat{\alpha}$. From another perspective, we find that the smaller the norm of $\varphi_{s_k}$ for $i = 1, ..., M$, the smaller the signal-noise-ratio (SNR) is. Therefore, the FrameSense is only suitable for the case that $\Phi$ corresponds to an equal-norm frame.

Actually, even if $\Phi$ corresponds to a equal-norm frame, the proposed MPME still outperforms the FrameSense, which will be illustrated by the following example.

**Example 2:** $\Phi \in \mathbb{R}^{100 \times 20}$ is a random matrix with normalized rows, whose $i$th row $\varphi_i^T = \frac{\phi_i}{\|\phi_i\|_2}$, and $\phi_i \in \mathbb{R}^{20}$ is a random vector with the components $\phi_{ij} \sim \mathcal{N}(0, 1)$.

The mean WCEV index and the mean MSE index of 100 Monte-Carlo run results are shown in Fig. 4 and Fig. 5 respectively. The two figures show that the FrameSense outperforms the convex relaxation method and the SparSenSe when the number of sensor nodes is small. It is shown in Fig. 5 that the FrameSense, the MPME, the convex relaxation method and the SparSenSe provide almost the same mean MSE indices when the number of sensor nodes is large enough, which indicates the effectiveness of the FrameSense in pursuing the minimum MSE. However, like the convex relaxation method and the SparSenSe, in Fig. 4 and Fig. 5 both the mean WCEV index and the mean MSE index of the FrameSense are much larger than those of the MPME when the number of sensor nodes is near the dimension of $\alpha$ (e.g., $20 \leq k \leq 22$), which indicates that $\Psi_k$ obtained from the FrameSense is ill-conditioned for $k \approx n$. If the signal representation matrix $\Phi$ corresponds to an equal-norm frame, the minimum frame potential of the observation matrix implies the minimum MSE, but the “worst-out” strategy in the FrameSense algorithm cannot find the optimal solution in the sense of the minimum frame potential, and even cannot guarantee that $\Psi$ is well-conditioned when the sensor number is limited. Therefore, the same as the convex relaxation method and the SparSenSe, the FrameSense is not suitable for the case that the available sensor nodes are very limited.

For the Example 2, again the proposed MPME outperforms the other methods in terms of both the WCEV index and the MSE index. As shown in Fig. 4 and Fig. 5 to meet the required
estimation accuracy (i.e., $\gamma = 1/5$ or the MSE index threshold equals 35), the proposed MPME requires minimum number of sensor nodes.

### C. Local optimization

**Definition 2 (Local optimization):** For a given set of sensing locations $S$, exchange one-at-a-time all the sensing location in $S$ with each available candidate location in $\mathcal{N}/S$ to relocate the sensor nodes at new position that can further reduce one criteria of interest (e.g., MSE or WCEV) until there is no further decrease.

The local optimization is a way to improve any given sensor placement, which can even be randomly generated. It can improve the solution of the convex relaxation method, the SparSenSe, and the FrameSense such that $\Psi$ is well-conditioned when $M \geq n$. This technique is similar with Fedorov’s exchange algorithm [34], and Wynn’s algorithm [35]. It has also been discussed in [1], [7]. After the local optimization, replacing any one selected sensing location by any unselected location cannot improve the solution. The result of local optimization is called 2-opt.

We apply the local optimization technique for the solutions of Example 1 obtained from the five methods mentioned before. The mean WCEV indices and the mean MSE indices of the improved sensor placement, together with those obtained from the MPME, are shown in Fig. 6 and Fig. 7 respectively.

---

**Fig. 6:** The mean WCEV index of $\alpha$ estimated from 20 to 40 sensor observations based on six different sensor placements.

**Fig. 7:** The mean MSE index of $\alpha$ estimated from 20 to 40 sensor observations based on six different sensor placements.

---

It is shown in Fig. 6 and Fig. 7 that the solution of the MPME with local optimization provides the best performance in terms of both the WCEV index and the MSE index. Comparing the Fig. 6 and Fig. 7 with the Fig. 1 and Fig. 2 we can see that the solutions of the convex relaxation method, the FrameSense, and the SparSenSe are remarkably improved by the local optimization, especially when the number of sensor nodes $M \geq 20$. On the contrary, the improvement of the solution of the MPME is more limited, which implies that the MPME can provide good result without the local optimization.

Again we set the WCEV index threshold as 0.3 (i.e., $\gamma = 10/3$) and the MSE index threshold as 1.5. Fig. 6 shows that with the local optimization, the solutions of the convex relaxation, the SparSenSe, and the FrameSense require 23, 23, and 24 sensor nodes, respectively. Whereas the solution of the MPME without local optimization only needs 23 sensor nodes, and with the local optimization, the required sensor nodes reduces to 22.

In this section, we compare the computation cost of the MPME with the other sensor placement methods mentioned before.

The computational effort of the convex relaxation method is $O(i_c N^3)$ [1]. The convex optimization problem is solved using the interior-point method and $i_c$ is the iteration number typically. The iteration number is of a few tens [1].

The main computational effort of the SparSenSe is to solve the LMI problem (10). The computational cost of solving one LMI problem is $O(\tau N^3)$ [29] where $\tau$ is the total row size of the LMI system, and $N$ is the total number of scalar decision variables. For the LMI problem (10), $\tau = (n+1)^2 + N$. Hence, the computational effort of the SparSenSe is $O\left(N^4 + N^3(n+1)^2\right)$. If $n \ll N$, the computational effort is $O(N^4)$.

When using the FrameSense, $N - M$ rows is removed from $\Phi$. It costs $O((N - k + 1)N)$ to determine the $k$-th removed row. Therefore, the total cost of the FrameSense is $O(N^3)$.

Finding the eigenvalues of $\Psi_k$ costs $O(n^3)$ operations. When we find the $k$-th sensor location in the MNEP, the main computational cost is to solve the minimum nonzero eigenvalue maximization problem in which $N - k + 1$ eigenvalue problems will be solved. The computation cost is $O(N n^3)$. Therefore, to determine all the $M$ sensor locations using the MNEP, the total computational effort is $O(N M n^3)$.

In the MPME, to determine the $k$-th sensor location, the main computational cost is attributed to the optimization problem

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**V. COMPUTATIONAL COST OF MPME ALGORITHM**

In this section, we compare the computation cost of the MPME with the other sensor placement methods mentioned before.

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In the MPME, to determine the $k$-th sensor location, the main computational cost is attributed to the optimization problem...
\[ \hat{s}_k = \arg \max_{i \in N \setminus S} \| \mathbf{P}_{k-1} \mathbf{v}_i \|_2 \] which costs \( O((N - k + 1)n^2) \).

Hence, finding all the \( M \) sensor locations costs \( O(NMn^2) \).

In using the local optimization technique, for a given set of sensing location \( S \), the process of all the locations in \( S \) are exchanged with all the other locations out of \( S \) one time to check whether the objective criteria is reduced, is called one iteration. In each iteration, the computational effort is \( O(NMn^3) \). We denote the iteration number of the local optimization by \( i_{l} \). We summarize the computational efforts of all the mentioned sensor placement methods in Table I.

**TABLE I: The computational effort of the different sensor placement methods**

<table>
<thead>
<tr>
<th>Method</th>
<th>Convex Relaxation</th>
<th>SparSenSe</th>
<th>FrameSense</th>
<th>MNEP</th>
<th>MPME</th>
<th>Local Optimization</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( O(i_{l}N^3) )</td>
<td>( O(N^3) )</td>
<td>( O(N^3) )</td>
<td>( O(NMn^3) )</td>
<td>( O(NMn^3) )</td>
<td>( O(i_{l}Nn^2) )</td>
</tr>
</tbody>
</table>

**Fig. 8:** The mean computational time of the five sensor placement algorithms for Example 4.

**Example 7** is run in a 2-GHz labtop using Matlab, and the mean computational time of 100 Monte-Carlo simulation runs is shown in Fig. 8. It is clear that the MPME is computationally most efficient amongst the five sensor placement algorithms.

The local optimization technique is used to improve the WCEV index and the MSE index of the sensor placement solutions obtained from the five methods in Table I. For different solutions, the computational efforts for each iteration of the local optimization are the same. The mean iteration numbers of 100 Monte-Carlo simulation runs are shown in Fig. 9 and Fig. 10.

**Fig. 9:** The iteration number of the local optimization of the WCEV index applied for the solutions obtained from five different methods.

**Fig. 10:** The iteration number of the local optimization of the MSE index applied for the solutions obtained from five different methods.

Sensor placement for linear inverse problem is an interesting but challenging combinatorial problem. The optimal solution can be solved by the exhaustive search and the branch-and-bound method \([9], [10]\), but the two types of methods are both impractical due to the extremely expensive computational cost, especially for some large scale problems. Therefore, in the last decade many work focused on finding a effective suboptimal solution via computationally efficient algorithms. To the best of our knowledge, the proposed MPME is one of the most computationally efficient sensor placement algorithms.

Different with many popular methods, the proposed MPME can guarantee that the observation matrix is well-conditioned nearer the indices of the nonzero elements of the solution of the nonconvex sensor placement problem \([8]\).

**VI. CONCLUSION**

We perform Monte-Carlo simulations to compare the MPME with the convex relaxation method \([1]\), the SparSenSe...
where for any matrix $A$, the dimension of a linear space spanned by all the column vectors of $A$. Therefore, $\lambda_i^{(k-1)}$ is an eigenvalue of $\tilde{\Lambda}_k$ with multiplicity $\mu_i - 1$.

**Lemma 9:** If $\lambda_n^{(k)}$ is a simple eigenvalue of $\Lambda_{k-1}$ and $z_n \neq 0$, $\lambda_n^{(k)} \neq \lambda_i^{(k)}$ for all $i$ satisfying $z_i \neq 0$.

**Proof:** Denote the eigenvector of $\Lambda_k$ associated with $\lambda_n^{(k)}$ by $v$, then we have

$$(\Lambda_k - \lambda_n^{(k)}I)v + zz^Tv = 0 \quad (28)$$

If $\lambda_n^{(k)} = \lambda_n^{(k-1)}$, from the last row of (28), considering $z_n \neq 0$, we can obtain $z^Tv = 0$. Substituting it into (28) yields $\Lambda_k - \lambda_n^{(k)}v = \lambda_n^{(k)}v$. Since $\lambda_n^{(k)} = \lambda_n^{(k-1)}$ is a simple eigenvalue of $\Lambda_{k-1}$, we have $v \in \text{span}(e_n) \neq 0$ and so $z^Tv = \tilde{v}_nz_n \neq 0$, which is a contradiction. Therefore, $\lambda_n^{(k)} \neq \lambda_n^{(k-1)}$.

Denote the multiplicity of $\lambda_n^{(k-1)}$ with respect to $\Lambda_k$ by $\mu_n - 1$. If $\sum_{i=n-\mu_n+1}^{n-1} z_i^2 \neq 0$, considering Lemma 8, $\lambda_n^{(k-1)}$ is an eigenvalue of $\Lambda_k$ with multiplicity $\mu_n - 1 - 1$. Then, if $\lambda_n^{(k)} = \lambda_n^{(k-1)}$, we can obtain

$$\lambda_n^{(k)} < \lambda_n^{(k-1)} - 1 = \lambda_n^{(k-1)}$$

which obviously is a contradiction. Hence, $\lambda_n^{(k)} \neq \lambda_n^{(k-1)}$.

Consequently, if $\sum_{i=n-\mu_n+1}^{n-1} z_i^2 = 0$, we have $\lambda_n^{(k)} < \lambda_n^{(k-1)} \leq \lambda_j^{(k-1)}$, $j < n - 1$ else $\lambda_n^{(k)} < \lambda_n^{(k)} \leq \lambda_j^{(k-1)}$, $j < n - \mu_n - 1$ which implies this lemma.

**Lemma 10:** If $\lambda_n^{(k-1)}$ is a multiple eigenvalue of $\Lambda_{k-1}$ with multiplicity $\mu_n$ and $\sum_n^{n-n-\mu_n+1} z_i^2 \neq 0$, then $\lambda_n^{(k)} = \lambda_n^{(k-1)}$ for all $i$ satisfying $z_i \neq 0$.

**Proof:** It follows Theorem 1 and Lemma 8 that $\lambda_n^{(k)} > \lambda_n^{(k-1)}$. Similarly with Lemma 8, we can proof that if $z_i \neq 0$, $\lambda_n^{(k)} > \lambda_n^{(k-1)}$ for all $i \leq n - \mu_n$.

Next, leveraging the five Lemmas, we proof Theorem 4.

Firstly, consider the case $k < n$. When, $k < n$, $\lambda_n^{(k)} = 0$ is a multiple eigenvalue of $\Psi_{k-1}$ with multiplicity $n - k + 1$. From Lemma 8 we can directly obtain (21b).

If $\sum_{i=k}^{n} z_i^2 = 0$, we can obtain $\zeta_k = 0$, and (21a) can be rewritten as $\lambda_n^{(k)} = 0$. According to Lemma 6, we can find that $\lambda_n^{(k)} = \lambda_n^{(k-1)}$ for all $k \leq i \leq n$.

If $\sum_{i=k}^{n} z_i^2 \neq 0$, according to Lemma 7 we can obtain

$$(\Lambda_k - \lambda_n^{(k)}I)v_k + z(\tilde{z}^Tv_k) = 0 \quad (29)$$

Lemma 10 can guarantee that $\Lambda_k - \lambda_n^{(k)}I$ is nonsingular, and therefore from (29), we find that $\tilde{z}^Tv_k \neq 0$. Then, left multiplying $\tilde{z}^T(\Lambda_k - \lambda_n^{(k)}I)^{-1}$ to both sides of (29) yields

$$\tilde{z}^T \tilde{v}_k (1 + \tilde{z}^T(\Lambda_k - \lambda_n^{(k)}I)^{-1} \tilde{z}) = 0$$

Hence, $1 + \tilde{z}^T(\Lambda_k - \lambda_n^{(k)}I)^{-1} \tilde{z} = 0$
which implies (21a).

Then, we consider the case 2: \( k \geq n \). When \( k \geq n \), \( \lambda_n^{(k-1)} \) is a simple eigenvalue of \( A_{k-1} \).

If \( z_n = 0 \), we can find that \( \zeta_k = 0 \). Considering Lemma 9 (22) obviously holds.

If \( z_n \neq 0 \), from Lemma 9 we know that \( \tilde{A}_{k-1} - \lambda_n^{(k)} I \) is nonsingular. Similarly with case 1, we can obtain that \( \tilde{z}^T v_n \neq 0 \), and that

\[
1 + \tilde{z}^T (\tilde{A}_{k} - \lambda_n^{(k)} I)^{-1} \tilde{z} = 0
\]

from which we can obtain (22).

B. Proof of Corollary 5

From (21a) and (22), we can obtain

\[
\zeta_k = \begin{cases} 
\lambda_n^{(k-1)} + \sum_{i=1, z_i \neq 0}^{k-1} \lambda_i^{(k)} - \lambda_n^{(k)} & \text{if } k < n \\
\lambda_n^{(k-1)} + \sum_{i=1, z_i \neq 0}^{n-1} \frac{(\lambda_i^{(k-1)} - \lambda_n^{(k-1)}) z_i^2}{\lambda_i^{(k-1)} - \lambda_n^{(k-1)}} & \text{if } k \geq n
\end{cases}
\]

Taking the derivative of \( \zeta_k \) with respect to the minimum nonzero eigenvalue of \( \Psi_k \), we can obtain that

\[
\frac{d\zeta_k}{d\lambda_k^{(k)}} = 1 + \sum_{i=1, z_i \neq 0}^{k-1} \frac{\lambda_i^{(k-1)} - \lambda_n^{(k-1)}}{(\lambda_i^{(k-1)} - \lambda_n^{(k-1)})^2}, \quad k < n \tag{30}
\]

and that

\[
\frac{d\zeta_k}{d\lambda_n^{(k)}} = 1 + \sum_{i=1, z_i \neq 0}^{n-1} \frac{(\lambda_i^{(k-1)} - \lambda_n^{(k-1)}) z_i^2}{(\lambda_i^{(k-1)} - \lambda_n^{(k-1)})^2}, \quad k \geq n \tag{31}
\]

Since \( \lambda_i^{(k-1)} \geq 0 \) and \( \lambda_n^{(k-1)} \geq \lambda_i^{(k-1)} \), it is obvious from (30) and (31) that \( d\zeta_k/d\lambda_k^{(k)} \geq 1 \) for \( k < n \) and \( d\zeta_k/d\lambda_n^{(k)} \geq 1 \) for \( k \geq n \). Therefore, \( 0 < \lambda_n^{(k)} / d\zeta_k \leq 1 \) for \( k < n \) and \( 0 < d\lambda_n^{(k)} / d\zeta_k \leq 1 \) for \( k \geq n \), which implies Corollary 5.

REFERENCES


