Note on pre-Courant algebroid structures for parabolic geometries

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Abstract

This note aims to demonstrate that every parabolic geometry has a naturally defined per-Courant algebroïd structure. This structure is a Courant algebroïd if and only if the the curvature κ of the Cartan connection vanishes. In all other cases, if the parabolic geometry is regular, there does not exist a natural universal expression for a Courant bracket.

This note assumes familiarity with both parabolic geometry, and Courant algebroïds. See [ČS00] and [ČG02] for a good introduction to the first case, and [KS05] and [Vai05] for the second. Some of the basic definitions will be recalled here:

Definition 0.1 (Parabolic Geometry)**.** Let G be a semi-simple Lie group with Lie algebra $\mathfrak{g}$, and P a parabolic subgroup with Lie algebra $\mathfrak{p}$. A parabolic geometry on a manifold M is given by a principal P bundle $\mathcal{P}$, an inclusion $\mathcal{P} \subset \mathcal{G}$, and a principal connection $\overrightarrow{\omega}$ on $\mathcal{G}$. This connection is required to satisfy the condition that $\overrightarrow{\omega}|_{\mathcal{P}}$ is a linear isomorphism $T\mathcal{P} \to \mathfrak{g}$.

Let $\mathcal{A} = \mathcal{P} \times_P \mathfrak{g} = \mathcal{G} \times_G \mathfrak{g}$ and denote by $\overrightarrow{\nabla}$ the affine connection on $\mathcal{A}$ coming from $\overrightarrow{\omega}$. By construction, $\mathcal{A}$ also inherits an algebraïc bracket $\{,\}$ and the Killing form B. It moreover has a well defined subbundle $\mathcal{A}_{(0)} = \mathcal{P} \times_P \mathfrak{p}$, and the properties of $\overrightarrow{\nabla}$ give an equivalence $\mathcal{A}/\mathcal{A}_{(0)} \cong T$, thus a projection $\pi : \mathcal{A} \to T$. The Killing form B then defines an inclusion $T^* \subset \mathcal{A}$, with $(T^*)^{\perp} = \mathcal{A}_{(0)}$. This implies that for v a one form, x any section of $\mathcal{A}$:

$$B(v, x) = v \llcorner \pi(x).$$

Let x, y be sections of $\mathcal{A}$. By the above inclusion, we may see $\overrightarrow{\nabla} x$ as section of $T^* \otimes \mathcal{A} \subset \mathcal{A} \otimes \mathcal{A}$, and thus we can directly write expressions like $\overrightarrow{\nabla}_x y$ (which is equal to $\overrightarrow{\nabla}_{\pi(x))} y$ in more traditional notation. The curvature of $\overrightarrow{\nabla}$ is κ; by inclusion, we can see this as a section of $\wedge^2 \mathcal{A} \otimes \mathcal{A}$. The parabolic structure gives a filtration on $\mathcal{A}$ and hence a concept of minimum homogeneity for sections of any associated bundle. There is also a Lie algebra co-differential $\partial^* : \wedge^2 \mathfrak{p}^{\perp} \otimes \mathfrak{g} \to \mathfrak{p}^{\perp} \otimes \mathfrak{g}$.

Definition 0.2. A parabolic geometry is *regular* if $hom(\kappa) \geq 1$, and is normal if $\partial^* \kappa = 0$.

Drawing on the definition of [KS05]:

Definition 0.3 (Courant algebroïd)**.** A Courant algebroïd is vector bundle $E \to M$, with a pseudo-Riemannian metric B on it, a projection $\pi : E \to TM$ called an anchor, and an inclusion $T^* \subset E$. It has a differential, $\mathbb{R}$ -linear bracket $[,] : \Gamma(E) \otimes \Gamma(E) \to \Gamma(E)$. This is required to obey the following properties, for sections x, y, z of E:

1. The Jacobi identity $\mathcal{J}(x, y, z) = [x, [y, z]] - [[x, y], z] - [y, [x, z]] = 0$.
2. $\pi(x) \cdot B(y, y) = 2B(x, [y, y])$.
3. $\pi(x) \cdot B(y, z) = B([x, y], z) + B([x, z], y)$.

Note that property 2 implies that $[,]$ is not a skew bracket. A pre-Courant algebroïd, as defined in [Vai05], is a structure that obeys all the pervious conditions, except for property 1. Instead, it is required to simply to have a linear Jacobian $\mathcal{J} \in \Gamma(\wedge^3 \mathcal{A} \otimes \mathcal{A})$.

The point of this note is:

Theorem 0.4. *Let $(M, \mathcal{A}, \overrightarrow{\nabla})$ be a parabolic geometry. Then $\mathcal{A}$ is a pre-Courant algebroïd for a natural choice of bracket $[,]$. If $\overrightarrow{\nabla}$ is flat, then it is a Courant algebroïd. If $\overrightarrow{\nabla}$ is non-flat, then there is no natural universal choice of $[,]$ that makes it into a Courant algebroïd.*

By 'natural', we mean a bracket constructed from the natural structures on $\mathcal{A}$: the algebraïc bracket $\{,\}$, the Killing form B, the connection $\overrightarrow{\nabla}$, its curvature κ, and the iterated derivatives of κ. By 'universal' we mean a single expression that will work for all parabolic geometries (or for all normal parabolic geometries). It may still be the case that for certain geometries, or for certain specific geometric objects, there would exist a natural choice of $[,]$ with vanishing Jacobian.

First, it is easy to see that $\mathcal{A}$ has all the required algebraïc properties of a Courant algebroïd: a projection to T, an inclusion of T^*, and a metric coming from the Killing form B.

Paper [ČG02] defines a differential bracket $<,>$ defined on $\mathcal{A}$ as:

$$< x, y >= \overrightarrow{\nabla}_x y - \overrightarrow{\nabla}_y x - \{x, y\} - \kappa(x, y).$$

Because of its original definition (defined as the bracket of right-invariant vector fields on $\mathcal{P}$), it must obey the Jacobi identity. This allows us to construct the (non-skew) bracket:

$$[x, y] =< x, y > +B(y, \kappa(x, -)) - B(x, \kappa(y, -)) + B(\overrightarrow{\nabla} x, y).$$

The last term is the contraction of the second component of $\overrightarrow{\nabla} x$ with y; it is thus always a section of T^*. Most of the remaining properties of the pre-Courant algebroïd are not hard to show (paper [Vai05] demonstrates them directly). For instance:

$$[x, x] = B(\overrightarrow{\nabla} x, x) = \frac{1}{2} dB(x, x), \qquad (1)$$

is immediate, (implying that $[x, y] = -[y, x] + d(B(x, y))$, while

$$\begin{aligned}
B([x, y], z) + B(y, [x, z]) &= B(\overrightarrow{\nabla}_x y - \overrightarrow{\nabla}_y x - \{x, y\} - \kappa(x, y) + B(\overrightarrow{\nabla} x, y), z) \\
&\quad B(y, \overrightarrow{\nabla}_x z - \overrightarrow{\nabla}_z x - \{x, z\} - \kappa(x, z) + B(\overrightarrow{\nabla} x, z)) \\
&\quad B(B(y, \kappa(x, -), z) - B(B(x, \kappa(y, -), z) \\
&\quad +B(B(z, \kappa(x, -), y) - B(B(x, \kappa(y, -), z) \\
&= B(\overrightarrow{\nabla}_x y, z) - B(\overrightarrow{\nabla}_y x, z) + B(\overrightarrow{\nabla}_z x, y) \\
&\quad +B(\overrightarrow{\nabla}_x z, y) - B(\overrightarrow{\nabla}_z x, y) + B(\overrightarrow{\nabla}_y x, y) \\
&\quad -B(\kappa(x, y), z) - B(\kappa(x, z), y) + B(y, \kappa(x, z)) + B(z, \kappa(x, y)) \\
&\quad -B(x, \kappa(y, z)) - B(x, \kappa(z, y)) \\
&= \pi(x) \cdot B(y, z) + 0
\end{aligned}$$

gives

$$\pi(x) \cdot B(y, z) = B([x, y], z) + B(y, [x, z]). \qquad (2)$$

Finally:

Proposition 0.5. *If $\mathcal{J} = [x, [y, z]] - [[x, y], z] - [y, [x, z]]$ is the Jacobian, then it is a section of $\wedge^3 \mathcal{A} \otimes \mathcal{A}$.*

Proof. First, we need to note that for any function $f \in C^\infty(M)$, $[df, y] = 0$. This follows from the fact that $[df, y]$ is clearly a one-form (the only ambiguous term is $\overrightarrow{\nabla}_y df - \{y, df\}$, which is a one-form as the negative homogeneity components of $\overrightarrow{\nabla}_y$ and $\{y, -\}$ are the same) and from:

$$\begin{aligned}
[df, y](Z) &= (-\overrightarrow{\nabla}_y df + \{y, df\} + B(\overrightarrow{\nabla} df, y) - B(df, \kappa(y, -)))(Z) \\
&= -(\nabla df)(y, Z) - B(\{\mathsf{P}(Y), df\}, Z) + B(\{A + v, df\}, Z) + \\
&\quad (\nabla df)(Z, y) + B(\{Z, df\}, y) + B(\{\mathsf{P}(Z), df\}, y) - B(df, \kappa(y, Z)) \\
&= d(df)(Z, y) - df(\nabla_Z Y - \nabla_Y Z - [Z, Y]) - B(df, \kappa(y, Z)) \\
&\quad + B(df, \{\mathsf{P}(Y), Z\} - \{\mathsf{P}(Z), y\} - \{A + v, Z\} + \{y, Z\}) \\
&= B(df, (\{\mathsf{P}(Y), Z\} - \{\mathsf{P}(Z), Y\} + \{Y, Z\} + \nabla_Y Z - \nabla_Z Y - [Y, Z])_-) \\
&\quad - B(df, \kappa(y, Z)) \\
&= B(df, \kappa(Y, Z)_-) - B(df, \kappa(y, Z)) \\
&= 0,
\end{aligned}$$

where Z is any section of T, and we have used a preferred connection ∇ (see [ČG02]) to give a splitting of $\overrightarrow{\nabla}_X = X + \nabla_X + \mathsf{P}(X)$ and $y = Y + A + v$.

This allows us to calculate:

$$\begin{aligned}
\mathcal{J}(fx, y, z) &= [fx, [y, z]] - [[fx, y], z] - [y, [fx, z]] \\
&= f[x, [y, z]] - ([y, z] \cdot f)x + B(x, [y, z])df \\
&\quad -[f[x, y], z] - [-(y \cdot f)x, z] - [B(x, y)df, z] \\
&\quad -[y, f[x, z]] + [y, (z \cdot f)x] - [y, B(z, x)df] \\
&= f\mathcal{J}(x, y, z) - ([y, z] \cdot f)x + B(x, [y, z])df \\
&\quad +(z \cdot f)[x, y] - B([x, y], z)df + (y \cdot f)[x, z] - (z \cdot y \cdot f)x + B(x, z)d(y \cdot f) \\
&\quad -B(x, y)[df, z] + (z \cdot B(x, y))df - B(df, z)dB(x, y) \\
&\quad -(y \cdot f)[x, z] + (z \cdot f)[y, x] + (y \cdot z \cdot f)x - B(z, x)[y, df] - (y \cdot B(z, x))df.
\end{aligned}$$

Re-arranging, and using the equations (1) and (2) extensively gives:

$$\begin{aligned}
&= f\mathcal{J}(x, y, z) + B(x, z)(d(y \cdot f) - [y, df]) - B(x, y)[df, z] \\
&= f\mathcal{J}(x, y, z) + B(x, z)[df, y] - B(x, y)[df, z] \\
&= f\mathcal{J}(x, y, z).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\mathcal{J}(x, y, z) &= [x, [y, z]] - [[x, y], z] - [y, [x, z]] \\
&= -[x, [z, y]] + [z, [x, y]] + [[x, z], y] \\
&\quad +[x, dB(y, z)] - dB(z, [x, y]) - dB(y, [x, z]) \\
&= -\mathcal{J}(x, z, y) + [x, dB(y, z)] - d(x \cdot B(y, z)) \\
&= -\mathcal{J}(x, z, y) - [dB(y, z), x] \\
&= -\mathcal{J}(x, z, y).
\end{aligned}$$

Similarly

$$\begin{aligned}
\mathcal{J}(x, y, z) &= [x, [y, z]] - [[x, y], z] - [y, [x, z]] \\
&= -[y, [x, z]] + ([[y, x], z] + [dB(y, x), z]) + [x, [y, z]] \\
&= -\mathcal{J}(y, x, z).
\end{aligned}$$

So $\mathcal{J}$ is totally skew, and $C^\infty(M)$-linear in the first entry, hence in every entry. □

Given the previous result, to actually calculate $\mathcal{J}$ at a point, it suffices to choose a particular frame at that point. Let $\{e_j\}$ be a local frame of $\mathcal{A}$ around $p \in M$ chosen so that $(\overrightarrow{\nabla} e_j)_p = 0$. Then it is immediately evident that:

$$[e_2, e_3]_p = (\{e_2, e_3\} + \kappa(e_2, e_3) + B(e_3, \kappa(e_2)) - B(e_2, \kappa(e_3)))_p.$$

Now consider $[e_1, [e_2, e_3]]_p$. The terms in that expression will be either second derivatives of the e_j, or linear terms. This gives us, at p:

$$\begin{aligned} [e_1, [e_2, e_3]] &= < e_1, < e_2, e_3 >> + B(e_3, (\overrightarrow{\nabla}_{e_1} \kappa)(e_2)) - B(e_2, (\overrightarrow{\nabla}_{e_1} \kappa)(e_3)) \\ &\quad - \{e_1, B(e_3, \kappa(e_2)) - B(e_2, \kappa(e_3))\} + B(B(e_3, \kappa(e_2)) - B(e_2, \kappa(e_3)), \kappa(e_1)) \\ &\quad + B(\overrightarrow{\nabla}_{e_1}(\overrightarrow{\nabla} e_2), e_3) \end{aligned}$$

$$\begin{aligned} [e_2, [e_1, e_3]] &= < e_2, < e_1, e_3 >> + B(e_3, (\overrightarrow{\nabla}_{e_2} \kappa)(e_1)) - B(e_1, (\overrightarrow{\nabla}_{e_2} \kappa)(e_3)) \\ &\quad - \{e_2, B(e_3, \kappa(e_1)) - B(e_1, \kappa(e_3))\} + B(B(e_3, \kappa(e_1)) - B(e_1, \kappa(e_3)), \kappa(e_2)) \\ &\quad + B(\overrightarrow{\nabla}_{e_2}(\overrightarrow{\nabla} e_1), e_3) \end{aligned}$$

$$\begin{aligned} [[e_1, e_2], e_3] &= -[e_3, [e_1, e_2]] + dB(e_3, [e_1, e_2]) \\ &= - < e_3, < e_1, e_2 >> - B(e_2, (\overrightarrow{\nabla}_{e_3} \kappa)(e_1)) + B(e_1, (\overrightarrow{\nabla}_{e_3} \kappa)(e_2)) \\ &\quad + \{e_3, B(e_2, \kappa(e_1)) - B(e_1, \kappa(e_2))\} - B(B(e_2, \kappa(e_1)) - B(e_1, \kappa(e_2)), \kappa(e_3)) \\ &\quad - B(\overrightarrow{\nabla}_{e_3}(\overrightarrow{\nabla} e_1), e_2) \\ &\quad + B(e_3, \overrightarrow{\nabla}(\overrightarrow{\nabla}_{e_1} e_2 - \overrightarrow{\nabla}_{e_2} e_1) - (\overrightarrow{\nabla} \kappa)(e_1, e_2) + B(e_2, (\overrightarrow{\nabla} \kappa)(e_1)) - B(e_1, (\overrightarrow{\nabla} \kappa)(e_2)) \\ &\quad + B(e_3, B(\overrightarrow{\nabla}(\overrightarrow{\nabla} e_1), e_2)) \end{aligned}$$

Now $<,>$ obeys the Jacobi identity. We get further simplifications of the type $(\overrightarrow{\nabla}_{e_1} \kappa)(e_2) = (\overrightarrow{\nabla}_{e_2} \kappa)(e_1) - (\overrightarrow{\nabla} \kappa)(e_1, e_2)$ by the Bianci identity on $\overrightarrow{\nabla}$. Remembering the identity $B(e_3, v) = v(e_3)$ for any one-form v gives simplifications in the $[[e_1, e_2], e_3]$ term. Together, these give the relation:

$$\begin{aligned} \mathcal{J}(e_1, e_2, e_3) &= -\{e_1, B(e_3, \kappa(e_2)) - B(e_2, \kappa(e_3))\} + B(e_3, \kappa(e_2, \kappa(e_1))) - B(e_2, \kappa(e_3, \kappa(e_1))) \\ &\quad + \text{cyclic terms}. \end{aligned}$$

This demonstrates that if $\kappa = 0$, then we are in the presence of a Courant algebroïd. However, if this is not the case:

Proposition 0.6. *If $\kappa \neq 0$ at a point then $\mathcal{J} \neq 0$ at that point, and $hom(\mathcal{J}) = hom(\kappa)$. Furthermore, if κ is regular, there does not exist a natural universal Courant bracket with vanishing Jacobian.*

Proof. Assume $\kappa \neq 0$ at p. Let $h = hom(\kappa)$. Since B and $\{,\}$ preserve homogeneity,

$$hom(\mathcal{J}) \geq h.$$

At p, let us project κ onto its lowest homogeneity component κ_H (if κ is normal, this is just the lowest homogeneity harmonic curvature [Čap06]). This κ_H can further be decomposed into sections $\kappa_{a,b,c}$ of $T^*_a \wedge T^*_b \otimes \mathcal{A}_c$ for integers a, b and c, with $a + b + c = h$. Pick a, b and c such that,

$\kappa_{a,b,c} \neq 0$ at p.

Chose a Weyl structure ∇ to give a splitting $\mathcal{A} = \sum_{i=-k}^{k} \mathcal{A}_i$. The Killing form B ensures that $\mathcal{A}_i \perp \mathcal{A}_j$ whenever $j \neq -i$. Define $\mathcal{A}_{(j)} = \sum_{i=j}^{k} \mathcal{A}_i$ (this does not depend on the choice of ∇).

Let E be the grading section in $\mathcal{A}_0$, X_{-a} a section of $\mathcal{A}_{-a} = T_{-a}$ and Y_{-c} a section of $\mathcal{A}_{-c}$. Call $\mathcal{J}_h$ the homogeneity h component of $\mathcal{J}$. Now $\kappa(E) = 0$, so $\mathcal{J}_h(E, Y_{-c}, X_{-a})$ reduces to

$$\Big(\{E, B(Y_{-c}, \kappa_H(X_{-a}))\} - \{E, B(X_{-a}, \kappa_H(Y_{-c}))\}\Big)/\mathcal{A}_{(b+1)}$$
$$=$$
$$\{E, (Id - m)(\kappa_H)(X_{-a}, -, Y_{-c})\}/\mathcal{A}_{(b+1)},$$

where m is the operator interchanging the first and last component of $\otimes^3 \mathcal{A}$. Basic representation theory implies that $Id - m$ is injective on $\wedge^2 \mathcal{A} \otimes \mathcal{A}$. Moreover $Id - m$ preserves homogeneity, so there must exist X_{-a}, Y_{-c} such that $(Id - m)(\kappa_H)(X_{-a}, -, Y_{-c})/(\mathcal{A}_{(b+1)})$ is a non-zero section of $\mathcal{A}_b$ around p. The bracket with E does not change this, as E acts by multiplication by b on these sections, and $b > 0$ since κ is a curvature.

This demonstrates that $\mathcal{J} \neq 0$ whenever $\kappa \neq 0$ and further, that $hom(\mathcal{J}) = h$.

Now assume that our geometry is regular, i.e. that $h = hom(\kappa) \geq 1$. By [Vai05], if we want to construct another pre-courant bracket on $\mathcal{A}$, we may only do this by twisting $[,]$ with a section of K of $\wedge^3 \mathcal{A}_{(0)} = \wedge^3(\ker \pi)$.

If K is naturally defined, that means that it is constructed from the natural geometric objects: B, $\{,\}$, $\overrightarrow{\nabla}$ and κ. Now $\overrightarrow{\nabla}$ preserves the first two objects, and since the action of $\mathcal{A}$ preserves B and $\{,\}$, the only sections $\wedge^3 \mathcal{A}$ that we can construct from the first three elements must also be preserved by the action of $\mathcal{A}$. Since $\mathfrak{g}$ is semi-simple, the only candidates are $\mathbb{R}$-linear sums the Lie brackets of the simple pieces of $\mathcal{A}$. None of these are sections of $\wedge^3 \mathcal{A}_{(0)}$ (since one of the requirements for being a parabolic geometry is that $\mathcal{A}_{(0)}$ contain no simple ideals of $\mathcal{A}$). Consequently, any such natural K must contain the curvature in its definition. Since all the other natural objects have homogeneity zero,

$$hom(K) \geq h.$$

Define $\mathcal{J}_K$ to be the Jacobian of the twisted bracket $[,]_K = [,] + K$. Then, for the e_j's as before:

$$[e_1, [e_2, e_3]_K]_K \quad = \quad [e_1, [e_2, e_3]] + K(e_1, [e_2, e_3]) + K(e_1, K(e_2, e_3)) + [e_1, K(e_2, e_3)].$$

We now want to look at the terms of homogeneity h in this expression. Since both K and κ have homogeneity $h > 0$, we may ignore any terms where more than one of these appear. Because $\pi K = 0$, the relevant terms in $[e_1, K(e_2, e_3)]$ are $\overrightarrow{\nabla}_{e_1} K(e_2, e_3) - \{e_1, K(e_2, e_3)\} = (D_{e_1} K)(e_2, e_3) - K(\{e_1, e_2\}, e_3) - K(e_2, \{e_1, e_3\})$, with the D operator defined as [ČG02]

$$D_x y = \overrightarrow{\nabla}_x y - \{x, y\}.$$

However, the D operator is of strictly positive homogeneity, so we are left with $-K(\{e_1, e_2\}, e_3) + K(e_2, \{e_3, e_1\})$ in homogeneity h. The homogeneity h components from $K(e_1, [e_2, e_3])$ are

$$K(e_1, \overrightarrow{\nabla}_{e_2} e_3 - \overrightarrow{\nabla}_{e_3} e_2 - \{e_2, e_3\}) = K(e_1, D_{e_2} e_3 - D_{e_3} e_2 + \{e_2, e_3\})$$

Thus in homogenity $\leq h$,

$$[e_1, [e_2, e_3]_K]_K \quad = \quad [e_1, [e_2, e_3]] + K(e_1, \{e_2, e_3\}) + K(e_3, \{e_1, e_2\}) + K(e_2, \{e_3, e_1\}).$$

Now

$$\mathcal{J}_K = [e_1, [e_2, e_3]_K]_K - [e_2, [e_1, e_3]_K]_K + [e_3, [e_1, e_2]_K]_K - dB(e_3, K(e_1, e_2)),$$

so in homogeneity h Using the fact that D is of strictly positive homogeneity, and summing the terms in the Jacobian, we get, in homogeneity h:

$$(\mathcal{J}_K - \mathcal{J})_h(e_1, e_2, e_3) = K(e_1, \{e_2, e_3\}) + K(e_2, \{e_3, e_1\}) + K(e_3, \{e_1, e_2\}) - (dB(e_3, K(e_1, e_2)))_h.$$

At p, $dB(e_3, K(e_1, e_2)) = B(e_3, (\overrightarrow{\nabla} K)(e_1, e_2)) = B(e_3, (DK + \{-, K\})(e_1, e_2)) = \{-, BK\}(e_1, e_2, e_3)$, where BK is K seen as a section of $\wedge^3 \mathcal{A}_{(0)}$. Thus

$$\begin{aligned} B(\mathcal{J}_K - \mathcal{J}, z)_h(e_1, e_2, e_3, z) &= 3(BK(e_1, \{e_2, e_3\}, z) + BK(e_2, \{e_3, e_1\}, z) + BK(e_3, \{e_1, e_2\}, z)) \\ &\quad + BK(\{z, e_3\}, e_1, e_2) + BK(\{z, e_1\}, e_2, e_3) + BK(\{z, e_2\}, e_3, e_1). \end{aligned}$$

Notice that if two of the entries in this expression are one-forms, the whole expression must vanish. So, to show that there is no naturally defined universal Courant bracket, it suffices to find a (normal) parabolic geometry, with a $\mathcal{J}_h$ that does not vanish on two one-forms. There is such an example at the end of paper [Arm07], dealing with the $|2|$-graded geometry of free n-distributions. There, I constructed a splitting ∇ of $\mathcal{A}$ and frames $\{X_j\}$ of T_{-1} and $\{Y_{jk} = \{X_j, X_k\}\}$ of T_{-2} such that the curvature was reduced to

$$\kappa = \kappa_H = Y_{12}^* \wedge X_1^* \otimes Y_{34}.$$

Then

$$\begin{aligned} \mathcal{J}(Y_{12}, Y_{13}, Y_{34}^*, X_3^*) &= B(\{Y_{13}, B(Y_{34}^*, \kappa(Y_{12}))\}, X_3^*) \\ &= B(\{Y_{13}, B(Y_{34}^*, Y_(34))X_1^*\}, X_3^*) \\ &= B(\{Y_{13}, X_1^*\}, X_3^*) = B(X_3, X_3^*) = 1. \end{aligned}$$

□

References

[Arm07] Stuart Armstrong. Free 3-distributions: holonomy, fefferman constructions and dual distributions. *arXiv*, 2007.

[Čap06] Andreas Čap. Two constructions with parabolic geometries. *Rend. Circ. Mat. Palermo (2)*, 79:11–37, 2006.

[ČG02] Andreas Čap and Rod Gover. Tractor calculi for parabolic geometries. *Trans. Amer. Math. Soc.*, 354(4):1511–1548, 2002.

[ČS00] Andreas Čap and Hermann Schichl. Parabolic geometries and canonical cartan connections. *Hokkaido Math. J.*, 29(3):453–505, 2000.

[KS05] Yvette Kosmann-Schwarzbach. Quasi, twisted, and all that...in poisson geometry and lie algebroid theory. *Progr. Math.*, 232:363–389, 2005.

[Vai05] Izu Vaisman. Transitive courant algebroids. *Int. J. Math. Math. Sci.*, 11:1737–1758, 2005.