Generalized Cayley-Hamilton-Newton identities

A. Isaev

Bogoliubov Laboratory of Theoretical Physics, JINR, 141980 Dubna, Moscow region, Russia

O. Ogievetsky* and P. Pyatov†

Center of Theoretical Physics, Luminy, 13288 Marseille, France

Abstract

The q-generalizations of the two fundamental statements of matrix algebra – the Cayley-Hamilton theorem and the Newton relations – to the cases of quantum matrix algebras of an "RTT-" and of a "Reflection equation" types have been obtained in [2]–[6]. We construct a family of matrix identities which we call Cayley-Hamilton-Newton identities and which underlie the characteristic identity as well as the Newton relations for the RTT- and Reflection equation algebras, in the sense that both the characteristic identity and the Newton relations are direct consequences of the Cayley-Hamilton-Newton identities.

1 Introduction

Let V be a vector space and $\hat{R} \in \mathrm{Aut}(V \otimes V)$ an $\hat{R}$-matrix of Hecke type, that is, $\hat{R}$ satisfies the Yang-Baxter equation and Hecke condition, respectively,

$$\hat{R}_1\hat{R}_2\hat{R}_1 = \hat{R}_2\hat{R}_1\hat{R}_2 \, , \tag{1.1}$$

$$\hat{R}^2 = I + (q - q^{-1})\hat{R} \, . \tag{1.2}$$

We use here the matrix notations of [1] (e.g., $\hat{R}_1 = \hat{R} \otimes I$, $\hat{R}_2 = I \otimes \hat{R}$ in (1.1) etc.), I is an identity operator and $q \neq 0$ is a numeric parameter.

In this note we deal with quantum matrix algebras of two types: an RTT-algebra and a Reflection equation (RE) algebra. They are associative unital algebras generated, respectively, by elements of "q-matrices" $T = ||T^i_j||_{i,j=1,\ldots,\dim V}$ and $L = ||L^i_j||_{i,j=1,\ldots,\dim V}$ subject to relations

$$\hat{R}\, T_1 T_2 = T_1 T_2\, \hat{R} \, , \tag{1.3}$$

$$\hat{R}\, L_1 \hat{R}\, L_1 = L_1 \hat{R}\, L_1 \hat{R} \, . \tag{1.4}$$

*On leave of absence from P. N. Lebedev Physical Institute, Theoretical Department, Leninsky pr. 53, 117924 Moscow, Russia

†On leave of absence from Bogoliubov Laboratory of Theoretical Physics, JINR, 141980 Dubna, Moscow region, Russia

For both these algebras, q-versions of the Newton identities and the Cayley-Hamilton theorem have been recently established (see [2]–[6]). The proofs of these two statements given for the q-matrix T in [5] and [6] turn out to be very similar ideologically and technically, which indicates that there should exist a more wide set of identities containing the Newton and the characteristic identities as particular cases. The main object of the present note is a construction of such generalized Cayley-Hamilton-Newton (CHN) identities.

We prove a q-version of the CHN identities for the RTT-algebra case. The CHN identities for the RE algebra are presented also. In case when both the RTT- and RE algebras originate from a quasitriangular Hopf algebra, the CHN identities for the q-matrix L can be derived from those for the q-matrix T by a procedure described in [6]. An independent proof of the CHN identities for the RE algebra will be given elsewhere.

Note that taking $\hat{R} = P$, the permutation matrix, one obtains – from any of the q-versions of the CHN theorem – a set of identities for usual matrices with commuting entries. It is worth mentioning that the CHN identities appear to be a new result even for the classical matrix algebra.

2 Notation

We shall begin with a brief reminder on the $\hat{R}$-matrix technique (a more complete treatment can be found, e.g., in [7, 4]).

Assume that q is not a root of unity, that is $k_q \equiv (q^k - q^{-k})/(q - q^{-1}) \neq 0$ for any $k = 2, 3, \dots$.

Given a Hecke $\hat{R}$-matrix, one can construct two series of projectors, $A^{(k)}$ and $S^{(k)}$, called q-antisymmetrizers and q-symmetrizers, respectively. They are defined inductively as

$$A^{(1)} := I \ , \qquad A^{(k)} := \frac{1}{k_q}\, A^{(k-1)} \left(q^{k-1} - (k{-}1)_q \hat{R}_{k-1}\right) A^{(k-1)} \ , \tag{2.1}$$

$$S^{(1)} := I \ , \qquad S^{(k)} := \frac{1}{k_q}\, S^{(k-1)} \left(q^{1-k} + (k{-}1)_q \hat{R}_{k-1}\right) S^{(k-1)} \ . \tag{2.2}$$

Further, assume that the q-antisymmetrizers fulfil the conditions

$$\mathrm{rank}\, A^{(n)} = 1 \ , \quad A^{(n+1)} = 0 \tag{2.3}$$

for some n. In this case the corresponding $\hat{R}$-matrix is called *even* and the number n is called the *height* of the $\hat{R}$-matrix.

For an $\hat{R}$-matrix of finite height n one introduces the following two matrices

$$\mathcal{D}_r := \frac{n_q}{q^n} \mathrm{Tr}_{\,(2\dots n)} A^{(n)} \ , \qquad \mathcal{D}_\ell := \frac{n_q}{q^n} \mathrm{Tr}_{\,(1\dots n-1)} A^{(n)} \ , \tag{2.4}$$

Here and below we use notation $\mathrm{Tr}_{\,(i_1 \dots i_k)}$ to denote the operation of taking traces in the spaces on places $(i_1 \dots i_k)$.

3 Cayley–Hamilton–Newton identities

Let us consider three sequences of elements in the RTT-algebra:

$$s_k(T) := \quad \mathrm{Tr}_{(1\dots k)}(\hat R_1 \hat R_2 \dots \hat R_{k-1} T_1 T_2 \dots T_k)\ , \tag{3.1}$$
$$\sigma_k(T) := q^k\, \mathrm{Tr}_{(1\dots k)}(A^{(k)} T_1 T_2 \dots T_k)\ , \tag{3.2}$$
$$\tau_k(T) := q^{-k}\, \mathrm{Tr}_{(1\dots k)}(S^{(k)} T_1 T_2 \dots T_k)\ , \quad k = 1, 2, \dots\ . \tag{3.3}$$

We also put $s_0(T) = \sigma_0(T) = \tau_0(T) = 1$.

To clarify the meaning of these elements, consider the classical limit $\hat R = P$. Denote $\{x_a\}$ the spectrum of the semisimple part of an operator $X \in \mathrm{Aut}(V)$. Then the elements $s_k(X)$, $\sigma_k(X)$, $\tau_k(X)$ are symmetric polynomials in x_a. Namely, $s_k(X) = \mathrm{Tr}\, X^k = \sum_a x_a^k$ are *power sums*, $\sigma_k(X) = \sum_{a_1 < \dots < a_k} x_{a_1} \dots x_{a_k}$ are *elementary symmetric functions*, and $\tau_k(X) = \sum_{a_1 \le \dots \le a_k} x_{a_1} \dots x_{a_k}$ are *complete symmetric functions*. We keep this notation for the elements $s_k(T)$, $\sigma_k(T)$, $\tau_k(T)$ of the RTT-algebra also.

The q-version of power sums $s_k(T)$ has been introduced by J.M. Maillet, who established their important property — the commutativity [8]. Just as in the classical case, the elementary and complete symmetric functions admit an expression in terms of the power sums (see Corollary 2 below) and, hence, the commutativity property extends to any pair of elements of the sets $\{s_k(T)\}$, $\{\sigma_k(T)\}$, $\{\tau_k(T)\}$.

If $\hat R$ is an even $\hat R$-matrix of height n, then one has $\sigma_k(T) = 0$ for $k > n$ and the last nontrivial element $\sigma_n(T)$ is proportional to a quantum determinant of T, $det_q T$

$$\sigma_n(T) = q^n det_q T\ . \tag{3.4}$$

Finally, we need an appropriate generalization of the matrix multiplication in the RTT-algebra. Inspired by the definition of the quantum power sums (3.1), one can introduce two versions of a k-th power of the q-matrix T [6]:

$$T^{\underline{k}} \ := \ \mathrm{Tr}_{(1\dots k-1)} \left(\hat R_1 \hat R_2 \dots \hat R_{k-1} T_1 T_2 \dots T_k \right)\ , \tag{3.5}$$
$$T^{\overline{k}} \ := \ \mathrm{Tr}_{(2\dots k)} \left(\hat R_1 \hat R_2 \dots \hat R_{k-1} T_1 T_2 \dots T_k \right)\ . \tag{3.6}$$

We use the superscripts $\underline{k}$ and $\overline{k}$ here for denoting different types of the k-th power of matrix T. This should not make a confusion with the usual matrix power (one has $T^{\underline{k}} = T^{\overline{k}} = T^k$ in the classical limit only).

In the same manner one can introduce a pair of versions of k-*wedge* (k-*symmetric*) powers of the q-matrix T, $T^{\underline{\wedge k}}$ and $T^{\overline{\wedge k}}$ ($T^{\underline{s k}}$ and $T^{\overline{s k}}$), removing the last or the first trace in the definition of the elementary (complete) symmetric functions, respectively,

$$T^{\underline{\wedge k}} := \mathrm{Tr}_{(1\dots k-1)} \left(A^{(k)} T_1 \dots T_k \right)\ , \qquad T^{\overline{\wedge k}} := \mathrm{Tr}_{(2\dots k)} \left(A^{(k)} T_1 \dots T_k \right)\ , \tag{3.7}$$
$$T^{\underline{s k}} := \mathrm{Tr}_{(1\dots k-1)} \left(S^{(k)} T_1 \dots T_k \right)\ , \qquad T^{\overline{s k}} := \mathrm{Tr}_{(2\dots k)} \left(S^{(k)} T_1 \dots T_k \right)\ . \tag{3.8}$$

With these definitions we can formulate the main result.

Theorem. (Cayley-Hamilton-Newton identities for the RTT-algebra).
Let $\hat{R}$ be Hecke R-matrix. For any j, the following identities hold

$$j_q\, T^{\underline{\wedge j}} \;=\; \sum_{k=0}^{j-1}(-1)^{j-k+1}\sigma_k(T)\, T^{\underline{j-k}}\ , \qquad (3.9)$$

$$j_q\, T^{\overline{\wedge j}} \;=\; \sum_{k=0}^{j-1}(-1)^{j-k+1} T^{\overline{j-k}}\,\sigma_k(T)\ , \qquad (3.10)$$

$$j_q\, T^{\underline{Sj}} \;=\; \sum_{k=0}^{j-1}\tau_k(T)\, T^{\underline{j-k}}\ , \qquad (3.11)$$

$$j_q\, T^{\overline{Sj}} \;=\; \sum_{k=0}^{j-1} T^{\overline{j-k}}\,\tau_k(T)\ . \qquad (3.12)$$

Proof. We shall give the details of the proof of the eq.(3.9). The relations (3.10)–(3.12) can be proved analogously.

For $k = 1, \ldots, j-1$ we have

$$\begin{aligned} \sigma_k(T)T^{\underline{j-k}} &= q^k \mathrm{Tr}_{\,(1\ldots k)}(A^{(k)}T_1\ldots T_k)\mathrm{Tr}_{\,(k+1\ldots j-1)}(R_{k+1}\ldots R_{j-1}T_{k+1}\ldots T_j) \\ &= q^k \mathrm{Tr}_{\,(1\ldots j-1)}(A^{(k)}T_1\ldots T_k\hat{R}_{k+1}\ldots\hat{R}_{j-1}T_{k+1}\ldots T_j) \\ &= q^k \mathrm{Tr}_{\,(1\ldots j-1)}(A^{(k)}\hat{R}_{k+1}\ldots\hat{R}_{j-1}T_1\ldots T_j) \end{aligned} \qquad (3.13)$$

We use (2.1) in the form $q^k A^{(k)} = (k+1)_q A^{(k+1)} + k_q A^{(k)}\hat{R}_k A^{(k)}$ to rewrite (3.13) as

$$\begin{aligned} &(k+1)_q \mathrm{Tr}_{\,(1\ldots j-1)}(A^{(k+1)}\hat{R}_{k+1}\ldots\hat{R}_{j-1}T_1\ldots T_j) \\ &+ k_q \mathrm{Tr}_{\,(1\ldots j-1)}(A^{(k)}\hat{R}_k A^{(k)}\hat{R}_{k+1}\ldots\hat{R}_{j-1}T_1\ldots T_j)\ . \end{aligned}$$

In the last term, the right antisymmetrizer $A^{(k)}$ commutes with the expression $R_{k+1}\ldots R_{j-1}T_1\ldots T_j$, so one can move $A^{(k)}$ through this expression to the right. Next, we can move $A^{(k)}$ to the left using the cyclic property of the trace. Finally, $(A^{(k)})^2 = A^{(k)}$ and we obtain

$$\begin{aligned} \sigma_k(T)T^{\underline{j-k}} &= (k+1)_q \mathrm{Tr}_{\,(1\ldots j-1)}(A^{(k+1)}\hat{R}_{k+1}\ldots\hat{R}_{j-1}T_1\ldots T_j) \\ &\quad + k_q \mathrm{Tr}_{\,(1\ldots j-1)}(A^{(k)}\hat{R}_k\ldots\hat{R}_{j-1}T_1\ldots T_j)\ . \end{aligned}$$

We have also $\sigma_0(T)T^{\underline{j}} = T^{\underline{j}}$. Taking the alternative sum, we obtain the relation (3.9). ■

Corollary 1. (**Newton identities for the RTT-algebra [5]**.)
Let $\hat{R}$ be Hecke R-matrix. The following iterative relations hold for the elements of the sets $\{s_k(T)\}$, $\{\sigma_k(T)\}$ and $\{\tau_k(T)\}$

$$q^{-j} j_q\,\sigma_j(T) \;=\; \sum_{k=1}^{j-1}(-1)^{k-1}\sigma_{j-k}(T)\,s_k(T) + (-1)^{j-1}s_j(T)\ , \qquad (3.14)$$

$$q^{j} j_q\,\tau_j(T) \;=\; \sum_{k=1}^{j-1}\tau_{j-k}(T)\,s_k(T) + s_j(T)\ , \qquad (3.15)$$

$$0 \;=\; \sum_{k=0}^{j}(-1)^k q^{2(j-k)}\tau_{j-k}(T)\,\sigma_k(T)\ , \quad \forall j = 1, 2, \ldots\ . \qquad (3.16)$$

Proof. To obtain the eqs. (3.14) and (3.15) one just takes the last trace (in the space with number j) in (3.9) and (3.11), correspondingly. The eq. (3.16) then follows from (3.14) and (3.15). ■

Corollary 2. (**Cayley-Hamilton theorem for the RTT-algebra [6]**).
Let $\hat{R}$ be even Hecke $\hat{R}$-matrix of rank n. The q-matrix T satisfies identities

$$\sum_{k=1}^{n} \sigma_{n-k}(T)(-T)^{\underline{k}} + \sigma_n(T)\,\mathcal{D}_\ell \;=\; 0\;, \tag{3.17}$$

$$\sum_{k=1}^{n} (-T)^{\overline{k}}\sigma_{n-k}(T) + \sigma_n(T)\,\mathcal{D}_r \;=\; 0\;. \tag{3.18}$$

Proof. Let $j = n$ in (3.9). We also have $A^{(n)}T_1 \dots T_n = A^{(n)} det_q T$. Then, the eq. (3.17) follows by an application of (2.4) and (3.4). The eq. (3.18) is similarly derived from (3.10). ■

Corollary 3. (**Inverse CHN theorem for the RTT-algebra**).
The formulas inverse to the eqs. (3.9)–(3.12) are

$$T^{\underline{j}} \;=\; \sum_{k=1}^{j} (-1)^{k+1} q^{2(j-k)} k_q\, \tau_{j-k}(T)\, T^{\underline{\wedge k}}\;, \tag{3.19}$$

$$T^{\overline{j}} \;=\; \sum_{k=1}^{j} (-1)^{k+1} q^{2(j-k)} k_q\, T^{\overline{\wedge k}} \tau_{j-k}(T)\;, \tag{3.20}$$

$$T^{\underline{j}} \;=\; \sum_{k=1}^{j} (-1)^{j-k} q^{-2(j-k)} k_q\, \sigma_{j-k}(T)\, T^{\underline{S k}}\;, \tag{3.21}$$

$$T^{\overline{j}} \;=\; \sum_{k=1}^{j} (-1)^{j-k} q^{-2(j-k)} k_q\, T^{\overline{S k}} \sigma_{j-k}(T)\;. \tag{3.22}$$

Proof. Consider two lower triangular matrices:

$$H \;:=\; \{H^j_k = q^{2(j-k)}\tau_{j-k}(T) \text{ if } j \geq k\,; \quad H^j_k = 0 \;\text{ otherwise}\}\,,$$
$$E \;:=\; \{E^j_k = (-1)^{j-k}\sigma_{j-k}(T) \text{ if } j \geq k\,;\; E^j_k = 0 \;\text{ otherwise}\}\,.$$

By the eq. (3.16) one has $HE = I$.

With this notation one rewrites (3.9) as $(-1)^{j+1} j_q\, T^{\underline{\wedge j}} = \sum_{k=1}^{j} E^j_k T^{\underline{k}}$. Then $T^{\underline{j}} = \sum_{k=1}^{j} (-)^{k+1} k_q H^j_k T^{\underline{\wedge k}}$, which is equivalent to (3.19).

The relations (3.20)–(3.22) are proved similarly. ■

We conclude by formulating the CHN theorem for the RE algebra.

Theorem. (Cayley-Hamilton-Newton identities for the RE algebra).
Let $\hat{R}$ be Hecke R-matrix and the q-matrix L generate the RE algebra (1.4). Then the following identities hold

$$j_q \, L^{\wedge j} = \sum_{k=0}^{j-1} (-1)^{j-k+1} \sigma_k(L) \, L^{j-k} \ , \qquad j_q \, L^{\mathcal{S} j} = \sum_{k=0}^{j-1} \tau_k(L) \, L^{j-k} \ . \tag{3.23}$$

Here the notation is as follows:

$$L^{\wedge k} := \mathrm{Trq}_{\,(2\dots k)}(A^{(k)} L_{\overline{1}} \dots L_{\overline{k}}) \ , \qquad L^{\mathcal{S} k} := \mathrm{Trq}_{\,(2\dots k)}(S^{(k)} L_{\overline{1}} \dots L_{\overline{k}})$$

are the k-wedge and the k-symmetric powers of the q-matrix L, respectively; L^k is the usual matrix power; $\sigma_k(L) := q^k \mathrm{Tr_q} \, L^{\wedge k}$ and $\tau_k(L) := q^{-k} \mathrm{Tr_q} \, L^{\mathcal{S} k}$ are the elementary and complete symmetric functions on the spectrum of L, respectively; $\mathrm{Tr_q} \, X := \mathrm{Tr} \, (\mathcal{D}_r X)$ *is a q-trace operation, and $L_{\overline{k}}$ is defined inductively by*

$$L_{\overline{1}} := L_1 \ , \qquad L_{\overline{k}} := \hat{R}_{k-1} L_{\overline{k-1}} \hat{R}^{-1}_{k-1} \ .$$

Acknowledgements: We are grateful to D. Gurevich and P. Saponov for discussions. This work is supported in part by the grants for promotion of french–russian scientific cooperation: the CNRS grant PICS No. 608 and the RFBR grant No. 98-01-2033. The work of P.P. and A.I. is also partly supported by the RFBR grant No. 97-01-01041.

References

[1] Faddeev L. D., Reshetikhin N. Yu., and Takhtajan L. A.: Algebra i Analiz *1* no. 1 (1989) 178. English translation in: Leningrad Math. J. *1* (1990) 193.

[2] Nazarov M. and Tarasov V.: Publications RIMS *30* (1994) 459.

[3] Pyatov P.N. and Saponov P.A.: J. Phys. A: Math. Gen., *28* (1995) 4415.

[4] Gurevich D.I., Pyatov P.N., Saponov P.A.: Lett. in Math. Phys. *41* (1997) 255.

[5] Pyatov P., Saponov P.: "Newton relations for quantum matrix algebras of RTT-type", Preprint IHEP 96-76 (1996).

[6] Isaev A., Ogievetsky O., Pyatov P. and Saponov P.: "Characteristic polynomials for Quantum Matrices", Preprint CPT-97/P3471 (1997). To appear in Proc. of the Intern. Conf. in memory of V.I. Ogievetsky (Dubna, Russia, 1997): Springer-Verlag, 1998.

[7] Gurevich D.I.: Algebra i Analiz *2* (1990). English translation in: Leningrad Math. J. *2* (1991) 801.

[8] Maillet J.M.: Phys. Lett. *B245* (1990) 480.