Open Gromov–Witten invariants and SYZ under local conifold transitions

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Abstract

For a local non-toric Calabi–Yau manifold which arises as a smoothing of a toric Gorenstein singularity, this paper derives the open Gromov–Witten invariants of a generic fiber of the special Lagrangian fibration constructed by Gross and thereby constructs its Strominger-Yau-Zaslow (SYZ) mirror. Moreover, it proves that the SYZ mirrors and disk potentials vary smoothly under conifold transitions, giving a global picture of SYZ mirror symmetry.

1. Introduction

Let $N$ be a lattice and $P$ be a lattice polytope in $N$. It is an interesting classical problem in combinatorial geometry to construct Minkowski decompositions of $P$.

On the other hand, consider the family of polynomials $\sum_{v \in P \cap N} c_v z^v$ for $c_v \in \mathbb{C}$ with $c_v \neq 0$ when $v$ is a vertex of $P$. All these polynomials have $P$ as their Newton polytopes. This establishes a relation between geometry and algebra, and the geometric problem of finding Minkowski decompositions is transformed to the algebraic problem of finding polynomial factorizations.

One purpose of this paper is to show that the beautiful algebro-geometric correspondence between Minkowski decompositions and polynomial factorizations can be realized via Strominger-Yau-Zaslow (SYZ) mirror symmetry.

A key step leading to the miracle is brought by a result of Altmann [3]. First of all, a lattice polytope corresponds to a toric Gorenstein singularity $X$. Altmann showed that Minkowski decompositions of the lattice polytope correspond to (partial) smoothings of $X$. From a string-theoretic point of view, different smoothings of $X$ belong to different sectors of the same stringy Kähler moduli. Under mirror symmetry, this should correspond to a complex family of local Calabi–Yau manifolds, and the various sectors (coming from Minkowski decompositions) correspond to various (conifold) limits of the complex family.

We show that such a complex family can be realized via SYZ construction by using special Lagrangian fibrations on smoothings of toric Gorenstein singularities constructed by Gross [19]. This paper follows the framework of SYZ given in [10] which uses open Gromov–Witten invariants rather than their tropical analogs. All the relevant open Gromov–Witten invariants are computed explicitly (Theorem 4.7), which gives a more direct understanding to symplectic geometry.

From the computations, we obtain an explicit expression of the SYZ mirror.

Theorem 1.1 (see Theorem 4.8 for the detailed statement). For a smoothing of a toric Gorenstein singularity coming from a Minkowski decomposition of the corresponding polytope,
its SYZ mirror is
\[ uv = \prod_{i=0}^{p} \left( 1 + \sum_{l=1}^{k_i} z^{u_i^l} \right), \]
where \( u_i^l \) are vertices of simplices appearing in the Minkowski decomposition.

See Sections 3 and 4 for more details on Minkowski decomposition and SYZ construction. Thus the SYZ mirror gives a factorization of a polynomial corresponding to a Minkowski decomposition of \( P \). The following diagram summarizes the dualities that we have and their relations.

Polytope decompositions \[\text{Newton} \rightarrow \text{Polyomial factorizations}\]
\[\text{Altmann} \leftrightarrow \text{Smoothings of toric Gorenstein singularity} \]

Another motivation for this paper comes from global SYZ mirror symmetry. Currently, most studies in SYZ mirror symmetry focus on the large complex structure limit, and other limit points of the moduli are less understood. Open Gromov–Witten invariants and SYZ for toric Calabi–Yau manifolds were studied in \([8, 10, 12]\), which are around the large complex structure limit. This paper studies SYZ for conifold transitions of toric Calabi–Yau manifolds, which are other limit points of the moduli. This gives a more global understanding of SYZ mirror symmetry.

More concretely, we prove the following theorem.

**Theorem 1.2** (see Theorem 5.2 for the complete statement). Let \( Y \) be a toric Calabi–Yau manifold associated to a triangulation of a lattice polytope \( P \), and \( X_t \) be the conifold transition of \( Y \) induced by a Minkowski decomposition of \( P \).

Then their SYZ mirrors \( \check{X} \) and \( \check{Y}_q \) are connected by an analytic continuation: there exists an invertible change of coordinates \( q(\check{q}) \) and a specialization of parameters \( q = \check{q} \) such that
\[ \check{X} = \check{Y}_{q(\check{q})}|_{q=\check{q}}. \]

The above theorem is proved by combining Theorem 1.1 and the toric open mirror theorem given by Chan–Cho–Lau–Tseng \([8]\), which gives explicit expressions of the disk potential and SYZ mirror of a toric Calabi–Yau manifold in terms of the mirror map. This is similar to the strategy for proving Ruan’s crepant resolution conjecture via mirror symmetry \([17, 21]\).

Note that the smoothings \( X_t \) are no longer toric. Open Gromov–Witten invariants for non-toric cases are usually difficult to compute and only known for certain isolated cases (such as the real Lagrangian in the quintic \([25]\)). This paper gives a class of non-toric manifolds that the open Gromov–Witten invariants can be explicitly computed.

In this paper, we do not need Kuranishi’s obstruction theory to define the invariants: there is no non-constant holomorphic sphere (Lemma 4.2), and hence no sphere bubbling can occur. (And no disc-bubbling occurs since we consider discs with the minimal Maslov index.) This makes the theory of open Gromov–Witten invariants easier to deal with. On the other hand, the local Calabi–Yau manifold here is non-toric, and it involves new geometric arguments to compute the invariants.

One can also construct the mirror via the Gross–Siebert program \([20]\) using tropical geometry of the base of the fibration, which will give the same answer. In this case, the Gross–Siebert
program is rather easy to run since all the walls are parallel and so they do not interact with each other. Nevertheless, since the correspondence between tropical and symplectic geometry in the open sector is still conjectural, this paper takes the more geometric approach based on symplectic geometry.

There are several interesting related works to the author’s knowledge. The work by Castano-Bernard and Mateessi [6] gave a detailed treatment of Lagrangian fibrations and affine geometry of the base under conifold transitions, and studied mirror symmetry in terms of tropical geometry along the lines of Gross–Siebert. For $A_n$-type surface singularities and the three-dimensional local conifold singularity $\{xy = zw\}$, SYZ construction by tropical wall-crossing was studied by Chan–Pomerleano–Ueda [7, 13, 14]. Auroux [4, 5] studied wall-crossing of open Gromov–Witten invariants and the SYZ mirror of the complement of an anticanonical divisor and gave several beautiful and illustrative examples. Abouzaid–Auroux–Katzarkov [1] gave a beautiful treatment of SYZ for blowups of toric varieties which is useful for studying mirror symmetry for hypersurfaces in toric varieties. The relation between open Gromov–Witten invariants of the Hirzebruch surface $F_2$ and its conifold transition was studied by Fukaya–Oh–Ohta–Ono [18] with an emphasis on non-displaceable Lagrangian tori (which are not fibers).

A related story in the Fano setting was studied by Akhtar–Coates–Galkin–Kasprzyk [2], where they considered Minkowski polynomials and mutations in relation with mirror symmetry for Fano manifolds. This paper works with local Calabi–Yau manifolds instead, and stresses more on open Gromov–Witten invariants and SYZ constructions.

2. **Smoothing of a toric Gorenstein singularity by Minkowski decomposition**

Let $N$ be a lattice, $M$ be the dual lattice and $\nu \in M$ be a primitive vector. Let $P$ be a lattice polytope in the affine hyperplane $\{v \in N_\mathbb{R} : \nu(v) = 1\}$. Then $\sigma = \text{Cone}(P) \subset N_\mathbb{R}$ is a Gorenstein cone. We denote by $m$ the number of corners of the polytope $P$, and by $\tilde{m}$ the number of lattice points contained in the (closed) polytope $P$. The corresponding toric variety $X = X_\sigma$ using $\sigma$ as the fan is a toric Gorenstein singularity. We assume that the singularity is an isolated point.

We choose a lattice point $v_0$ in the (closed) lattice polytope $P$ to translate it to a polytope in the hyperplane $\nu_\mathbb{R}^+ \subset N_\mathbb{R}$, and by abuse of notation we still denote this by $P$.

By Altmann [3], from a Minkowski decomposition

$$P = R_0 + R_1 + \cdots + R_p,$$

where $R_i$ are convex subsets in $\nu_\mathbb{R}^+$ and $p \in \mathbb{N}$, one obtains a (partial) smoothing of $X$ as follows. Let $\tilde{N} := (\nu_\mathbb{R}^+) \oplus \mathbb{Z}^{p+1}$. Define $\tilde{\sigma}$ to be the cone over the convex hull of

$$\bigcup_{i=0}^{p} (R_i \times \{e_i\}),$$

where $\{e_i\}_{i=0}^p$ is the standard basis of $\mathbb{Z}^{p+1}$. The total space of the family is $X = X_{\tilde{\sigma}}$, and $X_t$ is defined as fibers of

$$[-t_0, -t_1, \ldots, -t_p] = [0, t_0 - t_1, \ldots, t_0 - t_p] : X = X_{\tilde{\sigma}} \rightarrow \mathbb{C}^{p+1}/\mathbb{C}\langle (1, \ldots, 1) \rangle,$$

with $X_0 = X$. Here $t_0, \ldots, t_p$ are functions corresponding to $(0, \tilde{e}_i) \in \tilde{M}$, where $\tilde{M}$ is the dual lattice to $\tilde{N}$ and $\{e_i\}_{i=0}^p$ is dual to the standard basis $\{e_i\}_{i=0}^p$ of $\mathbb{Z}^{p+1}$.

The total space $X = X_{\tilde{\sigma}}$ of the family is a toric variety, but a generic member $X_t$ is not toric. Each member $X_t$ is equipped with a Kähler structure $\omega_t$ induced from $X$ and a holomorphic volume form $\Omega_t := (u_1, \ldots, u_p) \hat{\Omega}|_{X_t}$, where

$$\hat{\Omega} = d \log z_0 \wedge \cdots \wedge d \log z_{n-1} \wedge dt_0 \wedge \cdots \wedge dt_p$$

is the volume form on $X$. 
From now on, we assume that \( \mathcal{X}_t \) is smooth for generic \( t \). In particular, we assume that every summand \( R_i \) is a unimodular \( k_i \)-simplex for some \( k_i = 1, \ldots, n - 1 \). (Here a \( k \)-simplex with one of its vertices being 0 is said to be unimodular if its vertices generate all the lattice points in the \( k \)-plane containing the simplex.) The vertices of \( R_i \) are \( 0, u_1^i, \ldots, u_k^i \) for some lattice points \( u_j^i \in \mathbb{Z}^k \).

3. Gross fibration and its variants

Let us fix the smoothing parameter \( c = [c_0, c_1, \ldots, c_p] \in \mathbb{C}^{p+1}/\mathbb{C}\langle(1, \ldots, 1)\rangle \) from now on, where, \( c_0 = 0 \), \( c_1, \ldots, c_p \in \mathbb{C} \) are taken to be distinct constants, and consider \( \mathcal{X}_c \) which is supposed to be smooth. By relabeling \( c_i \)'s if necessary, then we assume \( 0 = |c_0| \leq |c_1| \leq \cdots \leq |c_p| \). The functions \( t_i \) restricted to \( \mathcal{X}_c \) equal \( t - c_i \), where \( t := t_0 \).

Gross [19] constructed a proper special Lagrangian fibration on \( \mathcal{X}_c \), which is

\[
\pi^K := (\pi_0, |t - K|^2 - K^2) : \mathcal{X}_c \to B,
\]

where

\[
B = \frac{M_\mathbb{R}}{\mathbb{R}(\nu)} \times \mathbb{R}_{\geq -K^2},
\]

\( K \in \mathbb{R}_{>0}, \pi_0 \) is the moment map of the action \( T^1\mathcal{Z} \) on \( (\mathcal{X}_c, \omega_c, \Omega_c) \). Here \( \omega_c \) is the symplectic form restricted from \( \omega \) to \( \mathcal{X}_c \), and

\[
\Omega_c = t_i \cdots t_p \Omega \big|_{\mathcal{X}_c} = d \log z_0 \wedge \cdots \wedge d \log z_{n-1} \wedge d \log (t - K) \quad (3.1)
\]

and

\[
\hat{\Omega} = d \log z_0 \wedge \cdots \wedge d \log z_{n-1} \wedge d \log (t_0 - K) \wedge dt_1 \wedge \cdots \wedge dt_p \quad (3.2)
\]

on \( \mathcal{X}_c \), which are different from the ones we described at the end of Section 2. The form \( \Omega_c \) is nowhere zero and has a simple pole on the boundary divisor \( D_0 = \{ t = K \} \subset \mathcal{X}_c \). As \( K \) tends to \( \infty \), \( \pi^K \) tends to the fibration

\[
\pi^\infty := (\pi_0, \Re t) : \mathcal{X}_c \to B,
\]

which is no longer proper, and we will not use \( \pi^\infty \) in this paper. (On the side of toric resolutions, the fibration \( \pi^\infty \) and the related open Gromov–Witten invariants in dimension 3 have been well studied by the theory of topological vertices [23].)

By Gross [19, Proposition 3.3], away from the boundary \( \partial B \), the discriminant loci have codimension 2, and they lie in the hyperplanes

\[
H_i = \frac{M_\mathbb{R}}{\mathbb{R}(\nu)} \times \{|c_i - K|^2 - K^2\}.
\]

The level sets of \( |t - K|^2 - K^2 \) give a fibration of the complex \( t \)-plane over \( \mathbb{R}_{\geq 0} \) by circles. For a singular fiber of \( \pi^K \), its image under \( t \) is a circle centered at \( K \) hitting one of the points \( c_i \); see Figure 1.

The boundary divisor \( \pi^{-1}(\partial B) = \{ t = K \} \) is denoted by \( D_0 \), and the discriminant loci supported in \( H_i \) are denoted by \( \Gamma_i \). For a generic choice of \( K \), the values \( |c_i - K| \) are pairwise distinct for \( i = 0, \ldots, p \) and so all the hyperplanes \( H_i \) and \( \partial B \) are disjoint. In the rest of this paper, we assume such a choice of \( K \).

We can deform the Lagrangian fibration \( \pi^K \) and make all walls \( H_i \) collide into one: deform the metric to \( d(\cdot, \cdot) \) on the complex \( t \)-plane from the standard norm \( | \cdot | \), so that all the points \( c_i \) are of the same distance to \( K \) for \( i = 0, \ldots, p \). Then the Lagrangian fibration \( d(\cdot, K)^2 - K^2 \) on the \( t \)-plane pulls back to a Lagrangian fibration \( \pi \) on \( \mathcal{X}_t \). Since all \( c_i \) lie in the same fiber, the discriminant locus away from the boundary \( \partial B \) lies in a single hyperplane \( H \). The fibers are no
The image of a fiber of the Lagrangian fibration $\pi^K$ is a circle centered at $K$, and that of a singular fiber hits one of the $c_i$. The function $t$ defines a fibration over $\pi^{-1}_{0}\{a\}$, where the fiber at each point other than one of the points $c_i$ can be identified as the torus $T_{\perp}^{\nu}$. At the points $c_i$ the $T_{\perp}^{\nu}$-orbit can degenerate to a smaller torus.

longer special with respect to the holomorphic volume form $\Omega_c$, but this does not hurt since we are only concerned with symplectic geometry. Interesting singular Lagrangian fibers arise for the fibration $\pi$. We will study the disc potential of a Clifford torus fiber $T$ (see Definition 4.15 and Corollary 4.16) of $\pi$ and its behavior under conifold transitions (Theorem 5.2).

For the purpose of SYZ construction along the lines of [10], we consider the following compactification of $X_c$. Add the ray generated by $-e_0 - \cdots - e_p$ to the fan $\hat{\sigma}$ and corresponding cones to produce a complete fan $\bar{\sigma}$. Then consider the corresponding toric variety $\bar{X}_c = X_{\bar{\sigma}}$ and fibers $\bar{X}_c \subset \bar{X}$ of $(t_0 - t_1, \ldots, t_0 - t_p)$ as before, which are preserved under the action of $T_{\perp}^{\nu}$. The Gross fibration has the same definition $\bar{\pi}^K = (\pi_0, |t - K|^2 - K^2)$ on $\bar{X}_c$, which has boundary divisor $D_0 = \{t = K\}$ and $D_\infty = \{t = \infty\}$ (the pole divisor of $t$). The meromorphic volume form $\Omega_c$ defined by the same expression (3.1) has simple poles along $D_0$ and $D_\infty$.

4. SYZ of smoothings of toric Gorenstein singularities

Here we follow the construction of [10] to obtain the SYZ mirror from disc countings. Adapted to this case, the procedures are as follows.

(1) Define the semi-flat mirror to be

$$\hat{X}_0 = \{\text{flat } U(1) \text{ connections } \nabla \text{ on a fiber of } \hat{\pi}^K \text{ over } r \in B_0\},$$

where $B_0 = B - \Gamma$, $\Gamma$ is the discriminant locus. We have the bundle map $\hat{\pi} : \hat{X}_0 \to B_0$. We have semi-flat complex coordinates on $\hat{X}_0$ which are monodromy invariant.

(2) Let $H \subset B_0$ be the wall (see Definition 4.3). Define the generating functions of open Gromov–Witten invariants (see Definition 4.4)

$$u(r, \nabla) = \sum_{\beta, D_0=1} n_\beta \exp \left( - \int_{\beta} \omega \right) \text{Hol}_\nabla(\partial \beta)$$

(4.1)
and
\[ v(r, \nabla) = \sum_{\beta:D_{0}\neq1} n_{\beta} \exp \left( -\int_{\beta} \omega \right) \text{Hol}_{\nu}(\partial\beta), \] (4.2)
on the semi-flat mirror \( \hat{\pi}^{-1}(B_{0} - H) \) associated to the boundary divisors \( D_{0} \) and \( D_{\infty} \). They serve as quantum-corrected complex coordinates.

(3) Take \( R \) to be the ring of functions on \( \hat{\pi}^{-1}(B_{0} - H) \) generated by \( u, v \) and \( z^{v} \) for \( v \in \nu^{\perp} \). Then the SYZ mirror is defined to be Spec(\( R \)).

Readers are referred to [10, Section 2] for more details.

Since \( \pi^{K} \) is a special Lagrangian fibration with respect to \( \Omega_{c} \) which has simple poles along \( D_{0} \) (and \( \bar{\pi}^{K} \) is a special Lagrangian with respect to \( \Omega_{c} \) which has simple poles along \( D_{0} \) and \( D_{\infty} \)), by Auroux [4] we have the following formula for the Maslov index of a disc class.

**Lemma 4.1** [4, Lemma 3.1]. The Maslov index of \( \beta \in \pi_{2}(\mathcal{X}_{c}, F_{r}) \), where \( F_{r} \) is a fiber of \( \pi^{K} \) over \( r \in B \), is \( \mu(\beta) = 2\beta \cdot D_{0} \). Similarly, the Maslov index of \( \beta \in \pi_{2}(\bar{\mathcal{X}}_{c}, F_{r}) \), where \( F_{r} \) is a fiber of \( \bar{\pi}^{K} \), is \( \mu(\beta) = 2\beta \cdot (D_{0} + D_{\infty}) \).

In the following lemma, we see that there is no holomorphic sphere in \( \mathcal{X}_{c} \) (or \( \bar{\mathcal{X}}_{c} \)). This simplifies the moduli theory for open Gromov–Witten invariants a lot because no sphere bubbling occurs.

**Lemma 4.2.** There is no non-constant holomorphic map from \( \mathbb{P}^{1} \) to \( \mathcal{X}_{c} \) (or \( \bar{\mathcal{X}}_{c} \)).

**Proof.** Suppose that \( u: \mathbb{P}^{1} \to \mathcal{X}_{c} \) (or \( u: \mathbb{P}^{1} \to \bar{\mathcal{X}}_{c} \)) is a holomorphic map. Then \( t \circ u \) is a holomorphic function on \( \mathbb{P}^{1} \) which can only be constant. Thus the image of \( u \) lies in a fiber \( t^{-1}\{a\} \) for some \( a \in \mathbb{C} \). But \( t^{-1}\{a\} \) is just a product of \( (\mathbb{C}^{\times})^{k} \) with \( \{ (z_{1}, \ldots, z_{n-k}) \in \mathbb{C}^{n-k} : z_{1} \cdots z_{n-k} = 0 \} \), and so \( H_{2}(t^{-1}\{a\}) = 0 \). Hence \( u \) can only be a constant. \( \square \)

For each \( i \), we may identify \( H_{i} \) with \( M_{R}/\mathbb{R}(\nu) \). Then the discriminant locus \( \Gamma_{i} \subset H_{i} \) can be identified as the normal fan of the simplex \( R_{i} \). Thus \( H_{i} - \Gamma_{i} \) consists of \( k_{i} + 1 \) components which will be referred to as chambers in \( H_{i} \). The chambers are one-to-one corresponding to the vertices of the standard simplex \( R_{i} \). Each chamber is (up to translation) the dual cone of the cone \( (R_{i})_{\nu} \) obtained by localizing the simplex \( R_{i} \) to its vertex \( v \) corresponding to the chamber.

From now on, we assume \( |c_{i+1} - K| \geq |c_{i} - K| \) for all \( i = 0, \ldots, p - 1 \), and \( K > 0 \) is chosen generically such that all the hyperplanes \( H_{i} \) for \( i = 1, \ldots, p \) are distinct.

To obtain a local trivialization of the torus bundle \( (\pi^{K})^{-1}(B_{0}) \to B_{0} \), we fix a chamber \( C_{i} \) in \( H_{i} \) for each \( i \). Without loss of generality, we fix \( C_{i} \) to be the chamber corresponding to the vertex \( 0 \) of the simplex \( R_{i} \). The vectors normal to the facets of the chamber \( C_{i} \) are \( u_{1}, \ldots, u_{k_{i}} \in \nu^{\perp} \). Define the open subset \( U = B_{0} - \bigcup_{i=0}^{p} (H_{i} - C_{i}) \subset B_{0} \), which is contractible. Thus \( (\pi^{K})^{-1}(U) \) over \( U \) is a trivial torus bundle, and every fiber can be identified as the fiber at a chosen point \( p \in U \). We always stick with this local trivialization in the rest of the paper.

Now we consider open Gromov–Witten invariants bounded by a fiber of \( \pi^{K} \) above points in \( U \subset B \). The invariants are well defined when the fiber has minimal Maslov index 2. Recall the following definition of a wall from [10].
Definition 4.3 (Wall). The wall $H$ of the Lagrangian fibration $\pi^K : \mathcal{X}_t \to B$ (or $\bar{\pi}^K$) is the set
\[ \{ r \in B_0 : \text{the fiber } F_r \text{ bounds non-constant holomorphic discs with Maslov index } 0 \}. \]

Away from the wall, the one-pointed genus-zero open Gromov–Witten invariants are well defined, and we can use them to construct the SYZ mirror.

Definition 4.4 (Open Gromov–Witten invariants). Let $r \in B_0 - H$ be away from the wall $H$, and $\beta \in \pi_2(\bar{\mathcal{X}}_t, F_r)$ be a disc class bounded by the fiber $F_r$. We have the moduli space $M_1(\beta)$ of stable discs with one boundary marked point representing $\beta$. The open Gromov–Witten invariant associated to $\beta$ is
\[ n_\beta = \int_{M_1(\beta)} \text{ev}^*[\text{pt}], \]
where $\text{ev} : M_1(\beta) \to F_r$ is the evaluation map at the boundary marked point.

The open Gromov–Witten invariant $n_\beta$ is non-zero only when the Maslov index $\mu(\beta)$ is 2. Moreover, by Lemma 4.2, away from the wall there is no disc bubbling and sphere bubbling in the moduli. Thus $M_1(\beta)$ consists of (classes of) holomorphic maps $(\Delta, \partial \Delta) \to (X, F_r)$. No virtual perturbation theory is needed in this situation.

For the Lagrangian fibration $\pi^K$, the wall is a union of disjoint hyperplanes as stated in the following proposition.

Proposition 4.5. The wall of the Lagrangian fibration $\pi^K : \mathcal{X}_t \to B$ (or $\bar{\pi}^K$) is the union of hyperplanes $H_i$ for $i = 0, \ldots, p$, which are disjoint from each other.

Proof. Suppose that the fiber $F_r$ bounds a non-constant holomorphic disc $u : (\Delta, \partial \Delta) \to (\mathcal{X}_c, L)$ with Maslov index. By the Maslov index formula (Lemma 4.1), the disc does not hit the divisor $D_0$ (nor $D_\infty$). This means that the function $(t - K) \circ u$ is never zero (and never equal to infinity). Since $|t - K| \circ u$ is constant on $\partial \Delta$, by the maximal principle it follows that $t$ is constant on the disc. Thus the disc $u$ lies in the fiber of $t$. A fiber $t^{-1}\{a\}$ for $a \neq 0, c_1, \ldots, c_p$ is isomorphic to $(\mathbb{C}^\times)^{n-1}$, and $F_r \cap t^{-1}\{a\}$ is a moment-map fiber of $(\mathbb{C}^\times)^{n-1}$, which cannot bound any non-constant holomorphic disc. Hence we have $t = 0, c_1, \ldots, c_p$. This implies that the base points $r$ lies in one of the hyperplanes $H_i$. \hfill \Box

We need to identify all the disc classes in order to compute their open Gromov–Witten invariants. The next definition gives a label to every basic disc class.

Definition 4.6 (Disc classes). Let $F_r$ be a fiber of the Lagrangian fibration $\bar{\pi}^K$ contained in the trivialization $(\bar{\pi}^K)^{-1}(U)$. The disc class of Maslov index 2 emanating from the boundary divisor $D_0$ (or $D_\infty$) is denoted by $\beta_0$ (or $\beta_\infty$, respectively). Moreover, each discriminant locus $\Gamma_i \subset H_i$ gives rise to $k_i$ disc classes of Maslov index 0, which are in one-to-one correspondence with the normal vectors $u_{i1}, \ldots, u_{ik_i} \in \nu^\perp$ to the facets of the chamber, and they are denoted by $\beta_j^i$ for $j = 1, \ldots, k_i$.

The disc class $\beta_0$ gives a local coordinate function $z_0$ on the semi-flat mirror $\bar{\pi}^{-1}(U) \subset \bar{\mathcal{X}}_0$ (which is not monodromy invariant):
\[ z_0 := \exp \left( - \int_{\beta} \omega \right) \text{Hol}_\nu(\partial \beta). \]
All stable disc classes bounded by \( F_r \) of Maslov index 2 must be of the form

\[
\beta_0 + \sum_{i=0}^{p} \sum_{j=1}^{k_i} n_j^i \beta_j^i \quad \text{or} \quad \beta_\infty + \sum_{i=0}^{p} \sum_{j=1}^{k_i} n_j^i \beta_j^i,
\]

where \( n_j^i \in \mathbb{Z}_{\geq 0} \). In the following theorem, we classify all the stable disc classes of Maslov index 2 and compute their open Gromov–Witten invariants. In the toric case, an analogous result is obtained by Cho–Oh [15]. Since here the symplectic manifold \( \mathcal{X}_r \) (or \( \mathcal{X}_c \)) is non-toric and the Lagrangian fibration \( \pi^K \) (or \( \pi^K \)) has interior discriminant loci, we need a non-trivial argument here.

**Theorem 4.7.** Let \( r \) be a regular value between the walls \( H_l \) and \( H_{l+1} \) of the Lagrangian fibration \( \pi^K \), and let

\[
\beta_0 + \sum_{i=0}^{p} \sum_{j=1}^{k_i} n_j^i \beta_j^i
\]

be a disc class which has Maslov index 2 bounded by \( F_r \). Then the moduli space \( \mathcal{M}_1(\beta) \) is non-empty only when \( n_j^i = 0 \) for all \( i, j \) except that for each \( 0 \leq i \leq l \), there could be at most one \( j = 1, \ldots, k_i \), with \( n_j^i = 1 \). In such a case the open Gromov–Witten invariant is \( n_\beta = 1 \).

For the Lagrangian fibration \( \pi^K \), we have exactly the same statement for the disc class

\[
\beta_0 + \sum_{i=0}^{p} \sum_{j=1}^{k_i} n_j^i \beta_j^i
\]

emanating from the boundary divisor \( D_\infty \), the moduli space \( \mathcal{M}_1(\beta) \) is non-empty only when \( n_j^i = 0 \) for all \( i, j \) except that for each \( l + 1 \leq i \leq p \), there could be at most one \( j = 1, \ldots, k_i \), with \( n_j^i = 1 \). In such a case the open Gromov–Witten invariant is \( n_\beta = 1 \).

**Proof.** First consider the disc class \( \beta_0 + \sum_{i=0}^{p} \sum_{j=1}^{k_i} n_j^i \beta_j^i \). The analysis for the disc class \( \beta_\infty + \sum_{i=0}^{p} \sum_{j=1}^{k_i} n_j^i \beta_j^i \) is similar.

Since the disc class has zero intersection with \( D_\infty \), it follows from the maximal principle on \((t - K) \circ u\) that a disc \( u \) bounded by \( F_r \) must have \(|t - K|^2 \leq K^2 + r_n\) (where \( r_n \) is the last coordinate entry of \( r \)), and hence never hits \((\pi^K)^{-1}(H_i)\) for \( i > l \). Thus the moduli space \( \mathcal{M}_1(\beta) \) is non-empty only when \( n_j^i = 0 \) for all \( i > l \).

Since there is no holomorphic sphere in \( \mathcal{X}_c \) (Lemma 4.2), no sphere bubbling occurs in the moduli. Also there is no disc bubbling because the fiber under consideration is not lying over the walls so that the minimal Maslov index of non-constant holomorphic discs is 2. Thus elements in \( \mathcal{M}_1(\beta) \) are holomorphic maps \( u : (\Delta, \partial \Delta) \to (\mathcal{X}_c, F_r) \). Using local coordinates \((\zeta_1, \ldots, \zeta_{n-1}, t)\) of \( \mathcal{X}_c \) (where \( \zeta_1, \ldots, \zeta_{n-1} \) corresponds to a basis of \( M/\mathbb{Z}(\ell) \)), the map \( u \) can be written as \((\zeta_1(z), \ldots, \zeta_{n-1}(z), t(z))\) for \( z \in \Delta \). Moreover, \( F_r \) is defined by the equations

\[
|\zeta_1|^2 = r_1, \ldots, |\zeta_{n-1}|^2 = r_{n-1}, \quad |t - K|^2 = K^2 + r_n.
\]

Consider the holomorphic function \( t \circ u \) on the disc \( \Delta \). Since \( u \) represents \( \beta_0 + \sum_{i=0}^{p} \sum_{j=1}^{k_i} n_j^i \beta_j^i \), the winding number of \( t \circ u \) around \( K \) is 1. Since \( r \) lies between the walls \( H_l \) and \( H_{l+1} \), the winding numbers of \( t \circ u \) around \( c_0 = 0 \) and \( c_1, \ldots, c_l \) are also 1. Thus \( t \circ u \) attains each value \( c_0, \ldots, c_l \) exactly once. On the other hand, \( u \) hits the critical fibers over \( \Gamma_i \), \( \sum_{j=1}^{k_i} n_j^i \) times (counting multiplicity), and so \( t \circ u \) attains the value \( c_i \) at least \( \sum_{j=1}^{k_i} n_j^i \) times. This forces \( n_j^i \) to be either 0 or 1, and among the integers \( n_1^1, \ldots, n_k^i \), at most one of them is 1 for each \( i \).
Suppose that this is the case. Then we equate \( M_1(\beta) \) and \( M_1(\beta_0) \) as follows. Let \( u : (\Delta, \partial \Delta) \to (X_1, F_r) \) represent the class \( \beta \). For \( i = 0 \), extend the vertices \( u_1^0, \ldots, u_{k_0}^0 \) of the polytope \( R_0 \) to a basis of \( \mathcal{M}^\perp \) and let \( \zeta_1, \ldots, \zeta_{n-1} \) be functions corresponding to the dual basis of \( M/\mathbb{Z}[\nu] \). The holomorphic map \( u \) can be written as \( (\zeta_1(z), \ldots, \zeta_{n-1}(z), t(z)) \) in terms of these coordinates. If \( n_j^0 = 1 \) for a certain \( j = 1, \ldots, k_0 \) (otherwise we do nothing), then there exists \( z_0 \in \Delta^0 \) such that \( \zeta_j(z_0) = 0 \) and \( t(z_0) = 0 \). Define

\[
\tilde{u}(z) = \left( \zeta_1(z), \ldots, \left( \frac{z - z_0}{1 + z_0 z} \right)^{-1} \zeta_j(z), \ldots, \zeta_{n-1}(z), t(z) \right).
\]

Then \( \tilde{u} \) does not hit any singular fibers at \( \Gamma_0 \) and hence belongs to the moduli space \( M_1(\beta_0 + \sum_{i=1}^p \sum_{j=1}^{k_i} n_{i,j}^1 \beta_j^1) \). Conversely, let \( u : (\Delta, \partial \Delta) \to (X_1, F_r) \) represent the class \( \beta_0 + \sum_{i=1}^p \sum_{j=1}^{k_i} n_{i,j}^1 \beta_j^1 \). By the above analysis on the winding number of \( t \), there exists exactly one \( z_0 \in \Delta^0 \) such that \( t(z_0) = 0 \). Then define

\[
u(z) = \left( \zeta_1(z), \ldots, \left( \frac{z - z_0}{1 + z_0 z} \right) \zeta_j(z), \ldots, \zeta_{n-1}(z), t(z) \right),
\]

which belongs to \( M_1(\beta) \). This set up an isomorphism

\[
M_1(\beta) \cong M_1 \left( \beta_0 + \sum_{i=1}^p \sum_{j=1}^{k_i} n_{i,j}^1 \beta_j^1 \right).
\]

Doing the same thing for \( i = 1, \ldots, p \) inductively, we obtain an isomorphism between \( M_1(\beta) \) and \( M_1(\beta_0) \).

Now for the class \( \beta_0 \), by the maximum principle any holomorphic map in \( \beta_0 \) has coordinates \( z_1, \ldots, z_{n-1} \) that are constants. The moduli space \( M_1(\beta_0) \) does not have disc bubbling even for a Lagrangian fiber at the walls \( H_i \), and thus it remains to be the same under wall-crossing. This means \( M_1(\beta_0) \) for \( F_r \) between the walls \( H_i \) and \( H_{i+1} \) is the same as that for a fiber below the wall \( H_0 \), which is the same as a toric fiber in \( \mathbb{C} \times (\mathbb{C}^\times)^{n-1} \). It follows that \( n_{\beta_0} = 1 \). Since \( M_1(\beta) \cong M_1(\beta_0) \), we also have \( n_\beta = 1 \).

For \( \beta = \beta_\infty + \sum_{i=0}^p \sum_{j=1}^{k_i} n_{i,j}^1 \beta_j^1 \), we use the winding number of \( t \) around \( \infty \) instead of around \( K \), and use the same argument to deduce the statement.

Now we are ready to compute the SYZ mirror of \( \pi : X_t \to B \).

**Theorem 4.8.** By the construction defined at the beginning of this section, the SYZ mirror of the Lagrangian fibration \( \pi : X_t \to B \) is

\[
\mathcal{X} = \{ ((u, v), z_1, \ldots, z_{n-1}) \in \mathbb{C}^2 \times (\mathbb{C}^\times)^{n-1} : uv = g(z) \},
\]

where

\[
g(z) = \prod_{i=0}^p \left( 1 + \sum_{l=1}^{k_i} z^{u_i^1} \right) = \sum_{v \in P \cap \nu} n_v z^v,
\]

for some explicit positive integers \( n_v \) attached to each \( v \in P \cap \nu^\perp \), and \( n_v = 1 \) for all vertices \( v \) of \( P \). Note that this is independent of the deformation parameter \( t \).
Proof. By Proposition 4.5, the wall $H$ is a union of $(p + 1)$ parallel hyperplanes $H_k$ for $k = 0, \ldots, p$. Hence $B_0 - H$ consists of $(p + 2)$ components denoted by $V_k$ for $k = -1, \ldots, p$, where the hyperplane $H_k$ is a boundary of $V_{k-1}$ and also $V_k$.

Recall that $u|_{\pi^{-1}(V_k)}$ is the generating function of open Gromov–Witten invariants $n_\beta$ of disc classes with $\beta \cdot D_0 = 1$ and $\beta \cdot D_\infty = 0$, bounded by a Lagrangian torus fiber at a point in $V_k$ (see equation (4.1)). Such disc classes must take the form

$$\beta = \beta_0 + \sum_{i=0}^{p} \sum_{j=1}^{k_i} n_j^i \beta_j,$$

which contributes to the monomial

$$z_0 \prod_{i=0}^{p} \prod_{j=1}^{k_i} z^{n_j^i u_j^i}.$$

By Theorem 4.7, $n_\beta$ is either 0 or 1, and it equals 1 exactly when all $n_j^i = 0$ except that for each $i \leq k$ there could be at most one $j$ with $n_j^i = 1$. It means that only the monomials

$$z_0 \prod_{i=0}^{k} u_j^i,$$

are present in the generating function $u|_{\pi^{-1}(V_k)}$, where $j_i \in \{0, \ldots, k_i\}$ and $u_0^i = 0$, and the coefficient of each of these monomials is 1. Hence

$$u|_{\pi^{-1}(V_k)} = z_0 \prod_{i=0}^{k} \left( 1 + \sum_{l=1}^{k_i} z^{u_j^i} \right).$$

Similarly, $v|_{\pi^{-1}(V_k)}$ is the generating function of open Gromov–Witten invariants $n_\beta$ of disc classes with $\beta \cdot D_\infty = 1$ and $\beta \cdot D_0 = 0$ (see equation (4.2)). Such disc classes must take the form

$$\beta = \beta_\infty + \sum_{i=0}^{p} \sum_{j=1}^{k_i} n_j^i \beta_j,$$

which contributes to the monomial

$$z_0^{-1} \prod_{i=0}^{p} \prod_{j=1}^{k_i} z^{n_j^i u_j^i}.$$

By Theorem 4.7, $n_\beta$ is either 0 or 1, and it equals 1 exactly when all $n_j^i = 0$ except that for each $i > k$ there could be at most one $j$ with $n_j^i = 1$. It means that only the monomials

$$z_0^{-1} \prod_{i=k+1}^{p} z^{u_j^i},$$

are present in the generating function $v|_{\pi^{-1}(V_k)}$, where $j_i \in \{0, \ldots, k_i\}$ and $u_0^i = 0$, and the coefficient of each of these monomials is 1. Hence

$$v|_{\pi^{-1}(V_k)} = z_0^{-1} \prod_{i=k+1}^{p} \left( 1 + \sum_{l=1}^{k_i} z^{u_j^i} \right).$$

It follows that we have the relation

$$uv = \prod_{i=0}^{p} \left( 1 + \sum_{l=1}^{k_i} z^{u_j^i} \right),$$
for the functions $u, v, z_1, \ldots, z_{n-1}$ on $B_0 - H$. Thus $\tilde{\mathcal{X}} = \text{Spec}(R)$, where $R$ is the subring of functions generated by $u, v, z_1, \ldots, z_{n-1}$, is the subvariety defined by the above equation.

**Remark 4.9.** The above calculation of open Gromov–Witten invariants matches with the expectation from the Gross–Siebert program [20] (which reconstructs the mirror manifold from tropical geometry instead of directly using symplectic geometry): the wall-crossing function attached to each wall $H_i$ is

$$1 + \sum_{l=1}^{k_i} z^{u_{i,l}},$$

and the coefficient of $z^v$ for $v \in \nu^\perp$ in $\prod_{i=0}^{p} (1 + \sum_{l=1}^{k_i} z^{u_{i,l}})$ equals $\sum_{\beta \circ \beta'} n_{\beta_0 + \beta'}$, where the summation is over all holomorphic disc classes $\beta'$ of Maslov index 0 with $\partial \beta' = v$.

**Remark 4.10.** All the complex geometric information of $\tilde{\mathcal{X}}$ is recorded by the hypersurface $g(z) = 0$ in $(\mathbb{C}^*)^{n-1}$. Components of the hypersurface, which are products of $(k-1)$-dimensional pair-of-pants with $(\mathbb{C}^*)^{n-k}$, are one-to-one corresponding to discriminant loci of $\pi^K$ in the hyperplanes $H_i$. It matches with the philosophy of T-duality that a big torus is dual to a small torus and vice versa.

**Remark 4.11.** We may also consider the dual picture of Minkowski decomposition, namely, decomposition of the dual fan of the polytope. By considering the pair-of-pant factor of each component of the hypersurface $g(z) = 0$ and taking the tropical limit, we obtain a fan (which is standard simplicial of possibly lower dimension). The union of all these fans recovers the dual fan of the original polytope. See Figure 11.

**Remark 4.12.** In the above expression of mirror, only the integer coefficients of $g$ depend on the Minkowski decomposition that we start with. Geometrically it means that different smoothings of $\mathcal{X}$ correspond to local patches of different limit points of the same stringy Kähler moduli.

As a consequence, the open Gromov–Witten invariants of $F_r$ for $r$ above all the walls $H_i$ are non-zero only for $\beta_v$ where $v$ is a lattice point in $P$. This is a non-trivial consequence because a priori all stable disc classes of Maslov index 2,

$$\beta_0 + \sum_{i=0}^{p} \sum_{j=1}^{k_i} n_{i,j} \beta_{i,j},$$

may have non-trivial open Gromov–Witten invariants.

Now consider the Lagrangian fibration $\pi$ (defined in Section 3) and its disc potential. There is only one wall and open Gromov–Witten invariants are well defined away from this wall. We follow the terminologies of [4, Definition 5.1] and make the following analogous definition.

**Definition 4.13** (Clifford and Chekanov tori). A regular fiber $T$ of the Lagrangian fibration $\pi$ is called a Clifford torus if its based point $r = \pi(T)$ is above the wall. It is called a Chekanov torus if its based point is below the wall.

Auroux [4] studied an analogous Lagrangian fibration for $\mathbb{P}^2$, where there is a wall in the base. A fiber above the wall is called a Clifford torus, which is Hamiltonian isotopic to a
regular moment-map fiber; a fiber below the wall is called a Chekanov torus, which is not a moment-map fiber up to Hamiltonian isotopy.

A Chekanov torus bounds only one class of holomorphic discs, namely $\beta_0$. Thus the disc potential (Definition 4.15) for Chekanov tori is simply $z_0$, which is not quite interesting. On the other hand, the disc potential for Clifford tori is an interesting object to study.

There exists a Lagrangian isotopy between the fiber $F_r$ of $\pi^K$ at $r = (b, a) \in B$ for $a \gg 0$ and a product torus fiber $T$ of $\pi$, such that each member in the isotopy never bounds holomorphic discs of Maslov index 0. Thus their open Gromov–Witten invariants are equal:

$$n_T^\beta = n_T^\beta,$$

where $\beta \in \pi_2(X, T)$ is identified as a disc class in $\pi_2(X, F_r)$ by this isotopy. As a consequence, we have the following corollary.

**Corollary 4.14.** Let $T$ be a product torus fiber of $\pi$ and $\beta \in \pi_2(X_c, T)$. Write

$$\beta = \beta_0 + \sum_{i=0}^{p} \sum_{j=1}^{k_i} n_i^j \beta_i^j. $$

Then $n_T^\beta = 1$ when $n_i^j$ is either 0 or 1 and for every $i$ there is at most one $j$ such that $n_i^j = 1$. Otherwise $n_T^\beta = 0$.

We also consider the disc potential $W^T$ for a product torus fiber $T$. In Section 5, we will see that $W^T$ can be obtained from the toric disc potential of the conifold transition of $X_t$.

**Definition 4.15 (Disc potential).** Let $T$ be a product torus fiber of $\pi$. The disc potential is defined as

$$W^T = \sum_{\beta \in \pi_2(X_c, T)} n_\beta \exp \left(-\int_{\beta} \omega\right) \text{Hol}_\varphi(\partial \beta).$$

(4.3)

By Corollary 4.14, we have the following corollary.

**Corollary 4.16.** Let $T$ be a product torus fiber of $\pi$ and $\beta \in \pi_2(X_c, T)$. Then the disc potential for $T$ is

$$W^T = z_0 \prod_{i=0}^{p} \left(1 + \sum_{l=1}^{k_i} z_u^l\right) = z_0 \sum_{v \in P \cap N} n_v z_v,$$

where $n_v$ are the same integers as in Theorem 4.8.

**Remark 4.17.** If we consider the singular toric variety $X_0$, then there are $m$ basic disc classes corresponding to vertices of the polytope. This gives the Laurent polynomial

$$z_0 \sum_{v: \text{vertex of } P} z_u,$$

which is NOT equal to $W^T$ in general.

5. Local conifold transitions

Instead of taking Minkowski decompositions of the polytope $P$ to obtain smoothings of toric Calabi–Yau Gorenstein singularities, one can instead consider triangulations of $P$ giving rise
Another smoothing $X_t$ gives a conifold limit point in the moduli, and different smoothings give different limit points.

to a fan $\Sigma$ which corresponds to a toric resolution $Y = Y_\Sigma$, and construct the mirror of $Y$ via SYZ. This has been done in [10]. The procedure of degenerating $X_t$ to a toric Gorenstein canonical singularity $X_0$ and taking toric resolution $Y$ is called conifold transition. On both sides, the geometry of Lagrangian fibrations has been explained beautifully in [19].

String theorists expect that quantum geometry undergoes a smooth deformation under conifold transition from $Y$ to $X_t$, even though there are singularities developed in the procedure of varying from $Y$ to $X_t$. (See Figure 2 showing a picture of the Kähler moduli.) Since we have computed all the open Gromov–Witten invariants of the smoothing $X_t$, and from [8] we also have good understanding of open Gromov–Witten invariants of the toric resolution $Y$, we can compare their disc potentials (which capture the quantum geometry relative to a torus fiber) and also their SYZ mirrors. The phenomenon is the same as that described by Ruan’s crepant resolution conjecture [26] and also its counterpart in the open sector [9]. The main difference is that now we consider crepant resolution of an isolated Gorenstein toric singularity which is not of orbifold type, and its smoothing is no longer a toric manifold.

The aim of this section is to deduce Theorem 5.2 which gives a positive response to the string theorists’ expectation. In order to do so, we will use the explicit expression of the disc potential $W^{X_t}$ given in Corollary 4.16. We also need an explicit expression of the disc potential $W^Y(r, \nabla) = \sum_{\beta \in \pi_2(Y, F_r)} n_\beta \exp \left( -\int_\beta \omega \right) \text{Hol}_\nabla(\partial \beta)$, where $F_r$ is the moment-map fiber at a regular value $r$ and $\nabla$ is any flat $U(1)$-connection on $F_r$. The disc potential $W^Y$ can be expressed in terms of the Kähler parameters $q$ of $Y$, and it has an explicit expression given by the toric open mirror theorem [8, Theorem 1.5] (which is [10, Conjecture 5.1]), which states that $W^Y_q$ has a simple combinatorial expression (called the Hori–Vafa mirror) by a change of coordinates $q(\tilde{q})$ called the mirror map. An explicit expression of the mirror map can be found in [11, Definition 3.2]. (The paper [11] is about toric semi-Fano toric manifolds rather than toric Calabi–Yau manifolds. Nevertheless, the mirror maps have the same combinatorial expressions. The mirror map expressed in [8] is for orbifolds and looks more complicated. Since we will not use the explicit expression of the mirror map $q(\tilde{q})$ in deducing Theorem 5.2, we do not include it here.)

**Theorem 5.1 (Toric open mirror theorem [8]).** Let $Y$ be a toric Calabi–Yau manifold arising as a toric resolution of an isolated Gorenstein canonical singularity given by a lattice polytope $P$ in the $(n-1)$-dimensional affine hyperplane $\{ \nu = 1 \}$. Fix a choice of lattice points $\{ v_i \}_{i=1}^n$ in $P$ such that they form a basis of $N$. For any $v \in (P \cap N) - \{ v_i : i = 1, \ldots, n \}$, $v$ can be expressed as a linear combination $\sum_{i=1}^n a_i v_i$. Then $\beta_v - \sum_{i=1}^n a_i \beta_i$ gives a curve class in $H_2(Y, \mathbb{Z})$, where $\beta_v$ and $\beta_i$ are the basic disc classes corresponding to $v$ and $v_i$, respectively.
The corresponding Kähler parameters and mirror complex parameters are denoted by \( q_v \) and \( \tilde{q}_v \), respectively. The mirror map, which expresses \( q_v \)'s in terms of \( \tilde{q}_v \)'s, is denoted by \( q(\tilde{q}) \).

Then the disc potential \( W^Y \) for its regular moment-map fibers can be expressed as

\[
W^Y(q(\tilde{q})) = z_0 \left( z^{v_1} + \cdots + z^{v_n} + \sum_{v \in (P \cap N) - \{v_i\}} \tilde{q}_v z^v \right),
\]

up to an (explicit) affine change of coordinates on \((\mathbb{C}^\times)^n\). Moreover, the SYZ mirror \( \tilde{Y}_q \) of \( Y \) has the expression

\[
\tilde{Y}_{q(\tilde{q})} = \left\{ (u, v, z) \in \mathbb{C}^2 \times (\mathbb{C}^\times)^{n-1} : uv = z^{v_1} + \cdots + z^{v_n} + \sum_{v \in (P \cap N) - \{v_i\}} \tilde{q}_v z^v \right\}.
\]

Now we are ready to deduce that the disc potential undergoes a smooth change under conifold transitions.

**Theorem 5.2 (Open conifold transition theorem).** Let \( Y \) be a toric resolution of an isolated Gorenstein canonical singularity given by a lattice polytope \( P \), and suppose that there exists a Minkowski decomposition of \( P \) into standard simplices (of possibly smaller dimensions than \( P \)), such that the corresponding smoothing \( X_t \) is smooth. Let \( W^Y : (\mathbb{C}^\times)^n \to \mathbb{C} \) be the disc potential of moment-map fibers of \( Y \), and \( W^X : (\mathbb{C}^\times)^n \to \mathbb{C} \) be the disc potential of product torus fibers of \( \mathbb{P}^n_v \) (Definition 4.15). Let \( q_i \) be the Kähler parameters of \( Y \) for \( i = 1, \ldots, \tilde{m} - n \), where \( \tilde{m} \) is the number of lattice points contained in \( P \). Then there exists an invertible change of coordinates \( q(\tilde{q}) \) and a specialization of parameters \( \tilde{q} = \tilde{q} \in \mathbb{C}^{\tilde{m} - n} \) such that the following properties hold.

1. The disc potential \( W^Y(q) : = W^Y(q(\tilde{q})) \) can be analytic continued to all \( q \in \mathbb{C}^{\tilde{m} - n} \).
2. The disc potential \( W^Y(q) = \tilde{q} \) equals \( W^X \) up to an affine change of coordinates on the domain \((\mathbb{C}^\times)^n\).

The SYZ mirrors \( \tilde{X} \) and \( \tilde{Y}_q \) also have such a relation, that is, the same change of coordinates \( q(\tilde{q}) \) and same specialization of variables \( \tilde{q} = \tilde{q} \) gives

\[
\tilde{X} = \tilde{Y}_{q(\tilde{q})}|_{\tilde{q} = \tilde{q}}.
\]

**Proof.** By Theorem 5.1, the disc potential \( W^Y \) equals the Hori–Vafa mirror under the mirror map \( q(\tilde{q}) \):

\[
W^Y(q(\tilde{q})) = z_0 \left( z^{v_1} + \cdots + z^{v_n} + \sum_{v \in (P \cap N) - \{v_i\}} \tilde{q}_v z^v \right).
\]

We may choose \( v_1 \) to be a vertex of \( P \) and \( v_2, \ldots, v_n \) to be lattice points contained in \( P \). Note that \( v_2 - v_1, \ldots, v_n - v_1 \) form a basis of \( \mathbb{Z}^n \). In particular, we can write

\[
v_1 = a_1(v_2 - v_1) + \cdots + a_{n-1}(v_n - v_{n-1}),
\]

for some integers \( a_1, \ldots, a_{n-1} \). Thus \( z^{v_1} = (z^{v_2-v_1})^{a_1} \cdots (z^{v_n-v_{n-1}})^{a_{n-1}} \).

The disc potential \( W^X \) has the expression

\[
W^X = z_0 \sum_{v \in P \cap N} n_v z^v,
\]

by Corollary 4.16, where \( n_v = 1 \) for vertices \( v \) of \( P \). (In particular, \( n_{v_1} = 1 \).)
Under the change of coordinates
$$\begin{align*}
z_0 &\mapsto \lambda_0 z_0, \\
z^{v_2-v_1} &\mapsto \lambda_1 z^{v_2-v_1}, \\
&\vdots \\
z^{v_{n-1}-v_n} &\mapsto \lambda_{n-1} z^{v_{n-1}-v_n},
\end{align*}$$
we have
$$\begin{align*}
z_0 z^{v_1} &\mapsto (\lambda_0 \lambda_1^{a_1} \cdots \lambda_{n-1}^{a_{n-1}}) z_0 z^{v_1}, \\
z_0 z^{v_2} &\mapsto (\lambda_0 \lambda_1^{a_1} \cdots \lambda_{n-1}^{a_{n-1}}) \lambda_1 z_0 z^{v_2}, \\
&\vdots \\
z_0 z^{v_{n-1}} &\mapsto (\lambda_0 \lambda_1^{a_1} \cdots \lambda_{n-1}^{a_{n-1}}) \lambda_{n-1} z_0 z^{v_{n-1}}.
\end{align*}$$
Thus by setting
$$\begin{align*}
\lambda_1 &= n v_2, \\
\lambda_{n-1} &= n v_n, \\
\lambda_0 &= \frac{1}{n v_2 \cdots n v_{n-1}},
\end{align*}$$
under the change of coordinates $W_Y(q(\check{q}))$ becomes
$$z_0 \left( n v_1 z^{v_1} + n v_2 z^{v_2} + \cdots + n v_n z^{v_n} + \sum_{v \in (P \cap N) - \{v_i\}_{i=1}^n} c_v \check{q}_v z^v \right),$$
for some constants $c_v$. Then the specialization of parameters
$$\check{q}_v = \frac{n v}{c_v},$$
for $v \in (P \cap N) - \{v_i\}_{i=1}^n$ equates the above expression with $W_X\hat{t}$. The SYZ mirrors have similar expressions to $W_Y$ and $W_X\hat{t}$: the SYZ mirror of $Y$ is defined by
$$uv = \left( z^{v_1} + \cdots + z^{v_n} + \sum_{v \in (P \cap N) - \{v_i\}_{i=1}^n} \check{q}_v z^v \right),$$
while the SYZ mirror of $X_t$ is defined by
$$uv = \prod_{i=0}^p \left( 1 + \sum_{l=1, \ldots, k_i} z^{v_i l} \right).$$
Thus the mirror map $q(\check{q})$ and the same specialization of parameters give
$$\hat{X} = \check{Y}_q |_{\check{q} = \check{q}}.$$

**Remark 5.3.** In general, the coefficients of the SYZ mirror $\check{Y}_q$ (or the disc potential $W_Y^q$) are a series in $q$, and it may not make sense to specify a value of $q$ (because the series may not converge at that value). A change of variable $q(\check{q})$ is necessary in order to have an analytic continuation of $\check{Y}_q$ from the large complex structure limit to the conifold limit.

6. **Examples**

6.1. **A-type surface singularities**

Consider the $A_p$ singularity $\mathbb{C}^2/\mathbb{Z}_{p+1}$, which is described by the fan whose rays are generated by $(k, 1)$ for $k = 0, \ldots, p + 1$. Then the polytope is the line segment $P = [0, p + 1]$ and the fan is obtained by cone over $P$. By taking the Minkowski decomposition
$$P = R_0 + \cdots + R_p,$$
where $R_i = [0, 1]$ for all $i = 0, \ldots, p$, one obtains a smoothing $\mathcal{X}_t$ of the $A_p$ singularity; see Figure 3.

The Gross fibration $\pi^K$ has $p + 1$ parallel walls $H_i$ for $i = 0, \ldots, p$ in the base, and each of them contains a singular value of the fibration (see Figure 3(C)). There is an $A_p$ chain of Lagrangian two-spheres hitting the singular fibers, and they do not contribute to computation of open Gromov–Witten invariants (in contrast to the other side of the resolution). We can also deform the fibration and make all the walls collapse to one (described in Section 3), and this is denoted by $\pi$. Then there is one interior singularity left, and the singular fiber is depicted in Figure 4.

From Theorem 4.8, the SYZ mirror is

$$ u v = (1 + z)^{p+1} = \sum_{k=0}^{p+1} \binom{p+1}{k} z^k, $$

for $(u, v, z) \in \mathbb{C}^2 \times \mathbb{C}^\times$, which is again the $A_p$ singularity. Thus we see that the $A_p$ singularity is self-mirror in this sense. Moreover, we get a correspondence between the Minkowski decomposition of an interval and the polynomial factorization $(1 + z)^{p+1} = \sum_{k=0}^{p+1} \binom{p+1}{k} z^k$.

On the other side of the transition, we consider the toric resolution of the $A_n$ singularity. SYZ and relevant open Gromov–Witten invariants were computed in [22]. The moment-map polytope is shown in Figure 5. The SYZ mirror is

$$ u v = (1 + z)(1 + q_1 z)(1 + q_1 q_2 z) \cdots (1 + q_1 \cdots q_p z). $$
It is easy to see that the specialization of Kähler parameters in Theorem 5.2 in this case is
\[ q_1 = \cdots = q_p = 1, \]
and the SYZ mirror of \( Y \) reduces to the SYZ mirror of \( X_t \). (In this case, we do not even need analytic continuation.) This gives the conifold limit point which is also an orbifold limit in this case. We can study SYZ via orbidisc invariants (defined in [16]) of the orbifold \( X \) instead of its smoothing. This was done in [9] and one obtains different flat coordinates around the conifold limit.

6.2. A three-dimensional conifold

Consider the conifold singularity \( X \) defined by \( xy = zw \) for \( x, y, z, w \in \mathbb{C} \), which is described by the three-dimensional fan whose rays are generated by
\[
(0,0,1), (1,0,1), (0,1,1), (1,1,1).
\]
Then the corresponding polytope is the square \( P = [0,1] \times [0,1] \) and the fan is obtained by cone over \( P \). By taking the Minkowski decomposition
\[
P = R_0 + R_1,
\]
where \( R_0 = [1,0] \) and \( R_1 = [0,1] \), one obtains a smoothing \( X_t \) of the conifold singularity; see Figure 6. The manifold \( X_t \) can be identified with \( T^*S^3 \).

The Gross fibration \( \pi^K \) has two parallel walls \( H_0 \) and \( H_1 \) in the base, and the discriminant loci are the boundary of the base, and two lines contained in these two planes, respectively (see Figure 6(C)). There is a Lagrangian three-sphere whose image under \( \pi^K \) is a vertical line segment joining the two planes. Under the identification \( X_t \cong T^*S^3 \), the Lagrangian sphere is the zero section of \( T^*S^3 \to S^3 \). There are no holomorphic two-spheres.

The Lagrangian fibration \( \pi \) is formed by collapsing the two walls into one. See Figure 7 for the topological type of the singular Lagrangian fibers over the discriminant locus (which is a cross consisting of two lines) in the wall.

From Theorem 4.8, the SYZ mirror is
\[
uv = (1 + z_1)(1 + z_2) = 1 + z_1 + z_2 + z_1z_2.
\]
The Minkowski decomposition shown in Figure 6(B) corresponds to the polynomial factorization \( (1 + z_1)(1 + z_2) = 1 + z_1 + z_2 + z_1z_2 \).

On the other side of the transition, we may consider \( \mathcal{O}_{P^1}(-1) \oplus \mathcal{O}_{P^1}(-1) \) which is a toric resolution of \( X \) (see Figure 8). SYZ and relevant open Gromov–Witten invariants have been
A Lagrangian three-sphere comes up in the smoothing, and its image under the Gross fibration is the vertical dotted line in the rightmost figure. (A) The moment-map polytope, (B) the Minkowski decomposition and (C) the Gross fibration and the basic holomorphic discs.

Singular fibers of $\pi$ which are degenerate tori. (A) A singular fiber over a generic point of the discriminant locus. It is a two-torus fibration over the circle with one singular fiber. (B) A singular fiber over the central point of the discriminant locus. It is a two-torus fibration over the circle with two singular fibers.

The dotted line shows the fan picture (projected to the plane) and the solid line shows the polytope picture. The solid line segment in the middle corresponds to the zero section $\mathbb{P}^1$ marked by its Kähler parameter $q$.

computed in [10, Section 5.3.2]. The SYZ mirror is

$$uv = 1 + z_1 + z_2 + qz_1z_2,$$

where $q$ is the Kähler parameter corresponding to the size of the zero section $\mathbb{P}^1 \subset \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. 


Figure 9 (color online). Two possible Minkowski decompositions of the fan polytope of $dP_6$.

Figure 10 (color online). Cone over $dP_6$ and the Gross fibrations over its two different smoothings. The dotted lines are the discriminant loci. The holomorphic disc emanated from the boundary divisor at the bottom is in the class $\beta_0$ which has Maslov index 2, and the discs emanating from singular fibers have Maslov index 0. (A) The moment-map polytope, (B) the Gross fibration of the first smoothing and (C) the Gross fibration of the second smoothing.

Thus the specialization of Kähler parameters in Theorem 5.2 in this case is

$$q = 1.$$  

6.3. Cone over del Pezzo surface of degree 6

Let $N = \mathbb{Z}^3$ and $\nu = (0, 0, 1) \in M$. The polytope $P \in \mathbb{R}^3$ has corners $v_1 = (1, 0), v_2 = (0, 1), v_3 = (1, 1), v_4 = (-1, 0), v_5 = (0, -1)$ and $v_6 = (-1, -1)$, and $\sigma \subset \mathbb{R}^3$ is the cone over $P$. This is [19, Example 3.1]. There are two different Minkowski decompositions as shown in Figure 9 giving rise to two different smoothings of $X = X_\sigma$.

The Gross fibrations and holomorphic discs are shown in Figure 10. In the first smoothing, there are Lagrangian three-spheres whose images are vertical line segments between two consecutive walls (shown as a dotted line in Figure 10(B)). In the second smoothing, there is a Lagrangian cone over the two-torus whose image is a vertical line segment between the two walls (shown as a dotted line in Figure 10(C)).

From Theorem 4.8, the SYZ mirrors are $$uv = g(z_1, z_2),$$ where the first choice of smoothing gives

$$g = g_1^{\text{con}}(z_1, z_2) = (1 + z_1)(1 + z_2)(1 + z_1^{-1}z_2^{-1}) = 2 + z_1 + z_2 + z_1z_2 + z_1^{-1} + z_2^{-1} + z_1^{-1}z_2^{-1},$$
and the second choice of smoothing gives

\[ g = g^\text{com}_{(1)}(z_1, z_2) = (1 + z_1 + z_2^{-1})(1 + z_1^{-1} + z_2) = 3 + z_1 + z_2 + z_1 z_2 + z_1^{-1} + z_2^{-1} + z_1^{-1} z_2^{-1}. \]

These two equations differ only in their constant terms. We see that Minkowski decompositions of the polytope in this example correspond to (integral) factorizations of polynomials.

**Remark 6.1.** The compactification \( \hat{X}_t \) of the first smoothing \( X_t \) (Figure 10(B)) gives the complete flag manifold \( Fl(1, 2, 3) \) (which is Fano). There are three walls in the base of Lagrangian fibration. If one moves the middle and lowest ones to the bottom boundary, and the highest one to the top boundary, then it gives the toric degeneration of \( Fl(1, 2, 3) \) constructed in [24] (see [24, Figure 4]). By using Theorem 4.7, the disc potential for fibers between the middle and highest walls is

\[ W = z_0(1 + z_1)(1 + z_2) + \frac{1}{z_0}(1 + z_1^{-1} z_2^{-1}) = z_0 + \frac{1}{z_0} + z_0 z_1 + z_2 + \frac{1}{z_0} + z_0 z_1 + z_0 z_2 \]

(setting all Kähler parameters to be 1 for simplicity). It coincides with the disc potential computed in [24].

For the second choice of smoothing, the tropical diagrams of the components \( 1 + z_1 + z_2^{-1} = 0 \) and \( 1 + z_2 + z_1^{-1} = 0 \) are shown in Figure 11. We see that they give a decomposition of the dual fan of the polytope \( P \).

On the other side of the transition, we consider toric resolution of the singularity \( X \). There are three choices \( Y_A, Y_B, Y_C \) which differ from each other by flops as shown in Figure 12. In the (complexified) Kähler moduli, they correspond to different chambers around the large volume limit. Their SYZ mirrors are of the same form \( uv = g(z_1, z_2) \), where

\[ \tilde{g}_A^{\text{res}}(z_1, z_2) = z_2 + (1 + \delta_A(q^A))z_1 z_2 + z_1 z_2^2 + q_1^A q_6^A z_1 + q_1^A q_2^A z_1^2 z_2 + q_5^A z_1^2 z_2^2 \]

and \( q_1^A q_6^A = q_3^A q_4^A, q_1^A q_2^A = q_3^A q_5^A, q_2^A q_4^A = q_5^A q_6^A \);

\[ \tilde{g}_B^{\text{res}}(z_1, z_2) = z_2 + (1 + \delta_B(q^B))z_1 z_2 + z_1 z_2^2 + q_6^B + q_1^B q_6^B z_1 + q_1^B q_2^B z_1 z_2 + q_4^B q_1^B q_2^B z_2^2 \]

and \( q_1^B q_6^B = q_3^B, q_1^B q_2^B = q_5^B, q_2^B q_4^B = q_5^B q_6^B \);

\[ \tilde{g}_C^{\text{res}}(z_1, z_2) = z_2 + (1 + \delta_C(q^C))z_1 z_2 + z_1 z_2^2 + q_6^C + q_1^C q_6^C z_1 + q_2^C z_1 z_2 + q_2^C q_4^C z_2^2 \]

and \( q_6^C = q_3^C, q_2^C = q_5^C, q_2^C q_3^C = q_5^C q_6^C \). The Kähler parameters \( q_i \) correspond to the holomorphic spheres as labeled in Figure 12. The Kähler moduli space has dimension 4.

By the change of coordinates \( z_1 \mapsto (1 + \delta(q))^{-1} z_1, \ z_2 \mapsto (1 + \delta(q)) z_2 \) and \( u \mapsto (1 + \delta(q)) u \), all the above expressions are brought to the form

\[ g_t(z_1, z_2) = z_2 + z_1 z_2 + z_1 z_2^2 + t_1 + t_2 z_1 + t_3 z_1^2 z_2 + t_4 z_1^2 z_2^2, \]

for some coordinates \( t_1, t_2, t_3, t_4, z_1, z_2 \).
and so all of them belong to the same mirror family of complex varieties around the same large complex structure limit. The SYZ mirrors after such a change of coordinates are denoted by $g_A^{\text{res}}, g_B^{\text{res}}, g_C^{\text{res}}$, respectively.

By the open mirror theorem for toric Calabi–Yau manifolds [8], $g^{\text{res}}$ can be computed from the mirror maps and expressed as the following:

$$
g_A^{\text{res}}(z_1, z_2) = z_2 + z_1 z_2 + z_1 z_2^2 + g_6^A + q^A g_6^A z_1 + q_2^A z_1^2 z_2 + q_5^A z_1^2 z_2^2,
$$

$$
g_B^{\text{res}}(z_1, z_2) = z_2 + z_1 z_2 + z_1 z_2^2 + g_6^B + q_1^B g_6^B z_1 + q_2^B z_1^2 z_2 + q_4^B z_1^2 z_2^2,
$$

$$
g_C^{\text{res}}(z_1, z_2) = z_2 + z_1 z_2 + z_1 z_2^2 + g_6^C + q_1^C g_6^C z_1 + q_2^C z_1^2 z_2 + q_4^C z_1^2 z_2^2,
$$

where the parameters $q^A$ and $q^B$ (respectively, $B, C$) are related by the mirror maps $q^A_i = q^A_i(q^B)$ (respectively, $B, C$). The complex parameters $q_i$ satisfy the same relations as the Kähler parameters $q$:

$$
\begin{align*}
q_1^A g_6^A &= q_3^A, & q_1^A g_6^B &= q_5^A, & q_1^A g_6^C &= q_5^A; \\
q_2^A g_6^A &= q_5^A, & q_2^B g_6^A &= q_5^A, & q_2^B g_6^B &= q_5^A; \\
q_2^C g_6^A &= q_5^A, & q_2^C g_6^B &= q_5^A, & q_2^C g_6^C &= q_5^A.
\end{align*}
$$

We can see that $g_A^{\text{res}}$ equals $g_B^{\text{res}}$ by the change of variables

$$(q_4^A)^{-1} = q_4^B, \quad q_1^A = q_1^B, \quad q_2^A = q_2^B, \quad q_6^A = q_6^B$$

and $g_B^{\text{res}}$ equals $g_C^{\text{res}}$ by the change of variables

$$(q_4^B)^{-1} = q_4^C, \quad q_3^B = q_3^C, \quad q_4^B = q_4^C, \quad q_5^B = q_5^C.$$

Thus the mirror complex variety undergoes no topological change under the flops, which is a well-known prediction by string theorists.

Going back to smoothings of $\mathcal{X}$, by the change of coordinates $z_1 \mapsto z_1/2$, $z_2 \mapsto 2z_2$ and $u \mapsto 2u$, then $uv = g_1^{\text{res}}$ is equivalent to

$$uv = z_2 + z_1 z_2 + z_1 z_2^2 + \frac{1}{2} z_1 + \frac{1}{2} z_1 z_2 + \frac{1}{2} z_1 z_2^2.$$
Similarly, $uv = g_2^{\text{con}}$ is equivalent to
\[ uv = z_2 + z_1 z_2 + z_1 z_2 + \frac{1}{3} + \frac{1}{3} z_1 + \frac{1}{3} z_1 z_2 + \frac{1}{3} z_1 z_2. \]
Thus we see that the SYZ mirrors of the smoothings correspond to two conifold limit points of the complex moduli:
\[ (t_1, t_2, t_3, t_4) = \left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) \]
and
\[ (t_1, t_2, t_3, t_4) = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right). \]
Then the specialization of variables in Theorem 5.2 for the conifold transitions is
\[
\begin{align*}
\tilde{q}_1^A &= 1, & \tilde{q}_2^A &= \tilde{q}_5^A = \tilde{q}_6^A = \frac{1}{2}; \\
\tilde{q}_2^B &= \tilde{q}_4^B = 1, & \tilde{q}_2^B &= \tilde{q}_6^B = \frac{1}{2}; \\
\tilde{q}_2^C &= \tilde{q}_4^C = 1, & \tilde{q}_2^C &= \tilde{q}_6^C = \frac{1}{2};
\end{align*}
\]
for the first smoothing, and
\[
\begin{align*}
\tilde{q}_1^A &= 1, & \tilde{q}_2^A &= \tilde{q}_5^A = \tilde{q}_6^A = \frac{1}{3}; \\
\tilde{q}_2^B &= \tilde{q}_4^B = 1, & \tilde{q}_2^B &= \tilde{q}_6^B = \frac{1}{3}; \\
\tilde{q}_2^C &= \tilde{q}_4^C = 1, & \tilde{q}_2^C &= \tilde{q}_6^C = \frac{1}{3};
\end{align*}
\]
for the second smoothing.

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