Research Paper

Overlay of Two Simple Polygons with Indeterminate Boundaries

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Abstract
This article outlines procedures that can be used to determine the boundaries and attributes associated with the resultant regions from the topological overlay of two simple polygons with indeterminate boundaries. After reviewing a definition of a simple polygon with indeterminate boundaries, this study first identifies the 10 resultant regions and 1,024 possible topological configurations from the overlay. Procedures for computing the boundaries of the 10 regions are discussed. Methods for determining the attributes associated with the 10 resultant regions are described. The effect of three typical overlay operations – intersection, difference, and union – on resultant regions is also discussed.

1 Introduction

Cognitive and computational issues related to the handling of spatial objects with indeterminate boundaries in a Geographic Information System (GIS) have become very important research topics in the Geographic Information Science (GIScience) literature during recent years (Burrough and Frank 1996). Spatial objects with indeterminate boundaries can be found in many applications in the social and natural sciences. For instance, it is usually a very hard task to clearly delineate the boundaries between urban and rural areas. It is equally difficult to define distinctive boundaries between warm and cold climate regions. In some applications such as the classification of forest and soil types, one often faces the problem of dealing with regions that contain a mix of different types of trees and soils. These regions are essentially polygons with indeterminate boundaries.

Handling spatial objects with indeterminate boundaries in a raster format is relatively simple, but the same cannot be said in a vector environment. There is a whole

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spectrum of research topics that have to be addressed if a comprehensive vector GIS that can be used to handle spatial objects with indeterminate boundaries is to be developed. These topics cover cognitive issues, database models that can support the representation, management, and query of spatial objects with indeterminate boundaries in a digital environment, operations that can be used to facilitate analyses involving spatial objects with indeterminate boundaries, and the visualization and dissemination of analysis results from such a GIS.

Most commercial GIS software packages are designed to handle spatial objects with definitive boundaries (referred to as crisp spatial objects hereafter). Relational algebra provides a solid mathematical foundation for the development of existing GIS software packages, particularly the database management systems behind these GIS software packages. Unfortunately, the GIScience research community has not developed or identified an equally elegant mathematical theory that can be used to develop GIS software packages that can be used to fully support the processing of spatial objects with indeterminate boundaries. It is debatable whether Fuzzy Set Theory (Zadeh 1965) can serve as an adequate foundation for representing and processing spatial objects with indeterminate boundaries in GIS. But, to the best of the authors’ knowledge, no better alternative exists at this time. Therefore, it is worth exploring the potentials of Fuzzy Set Theory in supporting the development of a GIS that can be used to handle spatial objects with indeterminate boundaries.

Among many difficult research issues, the problem of computing the topological overlay of two polygons with indeterminate boundaries has to be solved before a comprehensive GIS can be developed to handle spatial objects with indeterminate boundaries. The reason is that topological overlay is one of the key operations in a vector GIS. Topological overlay is indispensable for supporting many spatial analysis tasks in a GIS. The goal of this study is to formalize operations related to the topological overlay of two simple polygons with indeterminate boundaries, with a focus on the determination of the boundaries of resultant regions and the assignment of attribute values to these regions. Figure 1 shows two example polygons with indeterminate boundaries. Polygon \( T \) represents an area that is classified as soil type \( t \). All points in the central portion (dark area) of \( T \) belong to soil type \( t \) completely, whereas points in the remainder of \( T \) belong

![Figure 1](image-url)

**Figure 1** Two example polygons with indeterminate boundaries
Overlay of Polygons with Indeterminate Boundaries

2 Related Work

Topological overlay of map layers is a classic problem in the analysis of spatial data (Manning 1913, Sherman and Tobler 1957, McHarg 1969, Wang 1993). Because procedures related to the topological overlay of map layers involving only crisp spatial objects are standard knowledge and are readily available in a commercial GIS today, they are not discussed further in this discussion. Procedures related to the topological overlay of map layers concerning polygons with indeterminate boundaries have yet to be developed. The vague nature of some spatial data in general and spatial objects with indeterminate boundaries in particular has received considerable attention in the spatial analysis and GIScience literature (e.g. Freeman 1975; Leung 1982, 1987; Krishnapuram et al. 1993; Burrough and Frank 1996; Cohn and Gotts 1996; Clementini and Di Felice 1996; Wang and Hall 1996; Zhan 1997, 1998; Schneider 2000).

In the database literature, similarity- and possibility-based models have been proposed to handle vagueness in database management. Petry (1996) provided a survey on the topic. The potential of fuzzy set theory in handling uncertainties in a geographic database has long been recognized by researchers (e.g. Robinson 1988, 1990), but so far only a limited amount of work on this topic has been reported in the literature. Wang and Hall (1996) developed methods for representing geographic boundaries using fuzzy set theory. Methods for representing geographic regions with indeterminate boundaries have been proposed by several researchers (Erwig and Schneider 1997; Zhan 1997, 1998). Recent advancements include the development of metric and topological operations on fuzzy spatial objects in a database (Schneider 1999, 2000), as well as cognitive evidence related to the vagueness in linguistic descriptions of topological relations between geographic objects (Zhan 2001).

For spatial objects with indeterminate boundaries, Leung (1982, 1987) developed mathematical models that can be used to characterize some directional and metric spatial relations between spatial objects with indeterminate boundaries two decades ago. Krishnapuram et al. (1993) developed procedures for computing the directional relations between spatial regions with indeterminate boundaries using fuzzy logic. Cohn and
Gotts (1996) presented an ‘Egg-York’ model to represent regions with indeterminate boundaries. In their model, parts of a region with indeterminate boundaries are labeled as egg (the whole region), york (the definite portion of the region), or white (the indeterminate boundaries). Based on this model, a total of 46 topological relations between two regions with indeterminate boundaries are distinguished. Clementini and Di Felice (1996) extended the 9-Intersection model of Egenhofer and Herring (1994) and developed a model to represent topological relations between regions with indeterminate boundaries. They treated the indeterminate boundary as a thick boundary in the 9-Intersection Model, which gives a total of 42 topological relations between two regions with indeterminate boundaries. Zhan (1997, 1998) developed a fuzzy representation of spatial regions with indeterminate boundaries and developed methods for computing spatial relations between spatial regions with indeterminate boundaries based on a-cuts in Fuzzy Set Theory. His methods can be used to quantitatively determine fuzzy degrees of binary topological relations between two regions with indeterminate boundaries.

3 Definition of a Simple Fuzzy Polygon

Based on the observation that a polygon may be perceived to have three parts (an interior, a boundary, and an exterior), a polygon with indeterminate boundaries (A) can be decomposed into three parts: (1) the core, denoted as \(A^\bullet\), (2) the indeterminate boundary, denoted as \(A^d\), and (3) the exterior, denoted as \(A^\circ\) (Zhan 1998). The indeterminate boundary can be further decomposed into the inside edge, denoted as \(A^O\); the outside edge, denoted as \(A^o\); and the region bounded by the inside and outside edges (Zhan 1998).

Let \(\Omega\) be a referential set of a fixed number of attribute values in a domain in question, and that polygon \(A\) be a fuzzy subset defined in a two dimensional space \(R^2\) over \(\Omega\). The membership function of \(A\) can be defined as: \(\mu_A: X \times Y \times \Omega \rightarrow [0, 1]\), where \(X\) and \(Y\) are the sets containing all possible combinations of \(x\) and \(y\) coordinates, and each point \((x, y)\) within the polygon is assigned a membership value for attribute \(\omega\): \(\mu_A(x, y, \omega), \omega \in \Omega\).

Polygon \(A\) as described above is called a fuzzy polygon. Figure 2 presents a typical simple fuzzy region. The core, the indeterminate boundary, the exterior, the inside edge, the outside edge, and the \(\alpha\)-boundary (\(\alpha = 0.5\) in this case; see Definitions 1 and 8 below) of a simple fuzzy region are illustrated in Figure 2. Another important property of a simple fuzzy region is that the membership values of points within its indeterminate boundaries increase from the outside edge to its inside edge monotonically. Zhan (1998) provides a detailed discussion regarding the definition of simple fuzzy polygons. For discussion convenience in the next section, we review and summarize the definition below.

The concept of \(\alpha\)-cut level sets (Klir and Yuan 1995) is used to approximate the indeterminate boundaries of a fuzzy region. The indeterminate boundaries of a fuzzy region can be considered an aggregation of a set of cut level regions. An \(\alpha\)-cut level region, denoted by \(A_\alpha\), is defined below.

**Definition 1.** The \(\alpha\)-cut level region of a fuzzy region \(A\) is defined by Expression (1).

\[
A_\alpha = \{(x, y, \omega) | \mu_A(x, y, \omega) \geq \alpha\} \quad (0 < \alpha < 1)
\]  

(1)

Apparently, \(A_\alpha\) is a region whose boundary is defined by all points (pixels) with membership values equal to \(\alpha\).
Definition 2. A fuzzy region \( A \) is called a simple fuzzy region if it is fully connected and convex. A fuzzy region is said to be fully connected if and only if all of its \( \alpha \)-cut level regions are connected for all \( \alpha > 0 \). A fuzzy region \( A \) is convex if:

\[
\mu_A(\lambda s_1 + (1 - \lambda) s_2) \geq \min[\mu_A(s_1), \mu_A(s_2)], \quad s_1, s_2 \in X \times Y \times \Omega, \quad \forall \lambda \in [0, 1]
\]

where \( \lambda \) is a constant with a value between 0 and 1.

Definition 3. The core \( (A^\bullet) \) of a fuzzy region \( A \) is defined by Expression (3). Clearly, this is the area in which each pixel has a membership value of 1.0.

\[
A^\bullet = \{(x, y, w) \mid \mu_A(x, y, w) = 1\}
\]

Definition 4. The indeterminate boundary of a fuzzy region \( A \) is defined by:

\[
A^\delta = \{(x, y, w) \mid 0 < \mu_A(x, y, w) < 1\}
\]

For a simple fuzzy region, \( A^\delta \) is a belt immediately surrounding the core of the fuzzy region.

Definition 5. The exterior of a fuzzy region \( A \), denoted by \( A^- \), is defined by Expression (5). Apparently, \( A \cap A^- = \emptyset \) (null) and \( A \cup A^- = R^2 \).

\[
A^- = \{(x, y, w) \mid \mu_A(x, y, w) = 0\}
\]

Definition 6. The inside edge of the indeterminate boundary of fuzzy region \( A \) is defined by:
Based on the definitions given above, an region as defined in Section 3. For Polygon associated with each of the 10 regions.

mine the boundaries that define each of these 10 regions as well as the attributes as a result of the overlay. This number gives an exhaustive list of possible topological configurations.

Definition 7. Similarly, the definition of the outside edge of the indeterminate boundary of fuzzy region \(A\) is given by Expression (7).

\[
A^o = \sup(\text{support}(A)) \cup \inf(\text{support}(A))
\]

\[
= \sup\{ (x, y, \omega) | \mu_{\omega}(x, y, \omega) > 0 \}
\]

\[
\cup \inf\{ (x, y, \omega) | \mu_{\omega}(x, y, \omega) > 0 \}
\]

In the definition of an outside edge, \(\sup\{ (x, y, \omega) | \mu_{\omega}(x, y, \omega) > 0 \}\) is the least upper bound of the fuzzy region, and \(\inf\{ (x, y, \omega) | \mu_{\omega}(x, y, \omega) > 0 \}\) is the greatest lower bound of the fuzzy region. Because simple fuzzy regions are connected and convex, the outside edge is the union of the least upper bound and the greatest lower bound of the core.

Definition 8. The \(\alpha\)-boundary of a fuzzy region \(A\), denoted as \(A^\alpha\), is defined by:

\[
A^\alpha = \{(x, y, \omega) | \mu_{\omega}(x, y, \omega) = \alpha\} \quad (0 < \alpha < 1)
\]

Based on the definitions given above, an \(\alpha\)-cut level region of a simple fuzzy region possesses two important properties:

Property 1. All \(\alpha\)-cut level regions are crisp regions without holes.

Property 2. The \(\alpha\)-cut level regions of a fuzzy region are nested, which implies that, for membership values \(1 = \alpha_1 > \alpha_2 > \cdots > \alpha_n > \alpha_{n+1} = 0\), one has \(A_{\alpha_1} \supseteq A_{\alpha_2} \supseteq \cdots \supseteq A_{\alpha_n} \supseteq A_{\alpha_{n+1}}\).

4 Possible Resultant Features in the Overlay of Two Simple Fuzzy Polygons

Let \(A\) and \(B\) be two simple fuzzy polygons with membership values defined over two distinctively different categories of attribute values. Let Polygon \(A\) be the simple fuzzy region as defined in Section 3. For Polygon \(B\), let \(\Psi\) be a referential set of a fixed number of attributes in a given domain, and that Polygon \(B\) be a fuzzy subset defined in a two dimensional space \(R^2\) over \(\Psi\). The membership function of \(B\) can be defined as: \(\mu_B: X \times Y \times \Psi \rightarrow [0, 1]\), where \(X\) and \(Y\) are the sets containing all possible \(x\) and \(y\) coordinates, and each point \((x, y)\) within the polygon is assigned a membership value for attribute \(\theta\):

\[
\mu_B(x, y, \theta), \theta \in \Psi.
\]

The topological overlay of two simple fuzzy polygons \(A\) and \(B\) can be illustrated in Figure 3. Including the exterior (denoted as \(I\)), there are a total of 10 possible resultant regions from the overlay of two simple fuzzy polygons. These 10 regions are denoted as \(I, II, III, IV, V, VI, VII, VIII, IX, X\). These 10 regions can either be empty or non-empty. Therefore, there are a total of \(2^{10} = 1024\) possible topological configurations as a result of the overlay. This number gives an exhaustive list of possible topological configurations in the overlay of two simple fuzzy polygons. The next step is to determine the boundaries that define each of these 10 regions as well as the attributes associated with each of the 10 regions.
4.1 Determination of the Boundaries of the Resulting Regions from the Overlay

There is no need to define the boundaries of Region I because it is always given once the boundaries of other regions are defined. The boundaries of each of the regions II to X can be determined based on the overlay operation of two crisp polygons corresponding to their respective $\alpha$-cut level regions. Therefore, the determination of the boundaries of Regions II to X becomes the overlay operation of two crisp polygons commonly found in an existing GIS.

1. Determination of the boundaries of Region II. The boundaries of Region II are determined by the intersection of the indeterminate boundaries of fuzzy Polygon A and the outside edge of fuzzy Polygon B. This intersection is essentially the intersection between a region and the line segments forming the boundaries of Polygon B.

2. Determination of the boundaries of Region III. The boundaries of Region III are part of the inside edge of Polygon A and part of the outside edge of Polygon B that falls within the core of Polygon A.

3. Determination of the boundaries of Region IV. The boundaries of Region IV are the boundaries of the region resulting from the intersection of the core of Polygon A and the indeterminate boundaries of Polygon B.

4. Determination of the boundaries of Region V. The boundaries of Region V are the boundaries of the resultant region of the intersection of the cores of Polygons A and B.

5. Determination of the boundaries of Region VI. Region VI is the mirror version of Region IV, and its boundaries are the boundaries of the region resulting from the intersection of the core of Polygon B and the indeterminate boundaries of Polygon A.

6. Determination of the boundaries of Region VII. Region VII is the mirror version of Region III, and its boundaries are part of the inside edge of Polygon B and part of the outside edge of Polygon A that falls within the core of Polygon B.

7. Determination of the boundaries of Region VIII. Region VIII is the mirror version of Region II, and its boundaries are determined by the intersection of the indeterminate boundary of Polygon B and the outside edge of Polygon A.
8. Determination of the boundaries of Regions IX and X. The boundaries of Regions IX and X are the boundaries of the resultant regions from the intersection of the indeterminate boundaries of Polygons A and B. The exact locations of the boundaries of Polygons IX and X are different, depending on the exact relative locations of the indeterminate boundaries of Polygons A and B.

4.2 The Intersection, Difference, and Union of Two Simple Fuzzy Polygons

For topological overlay operations involving only crisp polygons, Boolean operators AND (intersection), NOT (difference), and OR (union) are the three basic operators. Complex overlay operations can be defined based on a combination of these three operators. For crisp polygons C and D, the overlay operations and the resultant regions related to Boolean operators AND, OR, and NOT are fairly straightforward (Figure 4).

The geometric aspects of these three operators in the topological overlay of two simple fuzzy polygons are similar to those in the topological overlay of two crisp polygons. As can be derived from Figure 3, when two simple fuzzy polygons A and B intersect (the AND operation) with each other, the resultant regions are Regions I, IV, V, VI, IX, and X. When the NOT operation (difference) is applied to two simple fuzzy polygons A and B, the resultant regions are Regions I, II, and III. When the operation OR (union) is applied to two simple fuzzy polygons A and B, all 10 regions as illustrated in Figure 3 are retained in the resultant features. As an example similar to the one presented in Figure 4, Figure 5 illustrates the resultant regions corresponding to the three typical topological operations (AND, NOT, OR) in the overlay of two fuzzy polygons CF and DF. Both fuzzy polygons CF and DF have a core (the smaller square) and an indeterminate boundary. Shaded areas are the regions that are to be preserved.

Figure 4  Three typical operations of topological overlay of crisp polygons

Figure 5  Three typical operations of topological overlay of simple fuzzy regions
after the overlay. As another example, the resultant regions from the overlay of the two polygons $T$ and $H$ introduced in Section 1 are given in Figure 6. The three typical overlay operations intersection ($T \text{ AND } H$), difference ($T \text{ NOT } H$) and union ($T \text{ OR } H$) lead to different resultant regions (Figure 6).

4.3 Determination of the Attributes of the Resulting Regions from the Overlay

When a resultant region is non-empty, an attribute has to be assigned to that region. The determination of attributes associated with each of the 10 resulting regions is not straightforward. In the case of topological overlay of two map layers containing crisp polygons, four sets of rules can be used to assign attribute values to the resultant regions. These four sets of rules are enumeration rules, dominance rules, contributory rules, and interaction rules (Chrisman 2002). Enumeration rules are used to preserve all possible combinations of attribute values from the input layers and assign them to the resultant regions. Dominance rules pick up one dominant value as the attribute value for each resultant region. Contributory rules combine the attribute values from polygons in the input map layers to form a new attribute value for each resultant region. Interaction rules use certain combinations of attributes of input polygons to derive new attribute values for a resultant region. All these rules may apply to the overlay of two simple fuzzy polygons.

1. **Attributes of Region I.** When Region I is non-empty, the attribute of Region I is either the attribute associated with the exterior of Regions A and B or determined by a set of user-specified criteria.

2. **Attributes of Regions II, III, VII, and VIII.** Because they do not overlap with any portion of the other input polygon, the attributes of Regions II, III, VII, and VIII should either remain the same as those in its corresponding input polygon, A or B.
or be determined by a set of user specified rules. The attribute values of Regions III and VII are special cases and are not fuzzy membership values because they are in the cores of the input polygons and they do not overlap with any other portion of the other input polygon.

3. Attributes of Regions IV, V, VI, IX, and X. These five regions are the result of the overlapping portions of Polygons A and B. The exact attribute associated with each of these regions must be determined by one of the four sets of rules mentioned above, depending on the exact application in question. It should be noted that the attribute value of Region V is a special case and is not a fuzzy membership value because it is in the cores of the input polygons.

Because the determination of the attribute values for any point in Regions I, II, III, VII, and VIII is clear and relatively simple compared to those in Regions IV, V, VI, IX, and X, we will only discuss how the four sets of rules can be extended to determine the attribute values of Regions IV, V, VI, IX, and X in the following paragraphs.

4.3.1 Enumeration rules
The key here is to find all possible combinations of fuzzy membership values for any point in Regions IV, V, VI, IX, and X based on the fuzzy membership function values of the two input simple fuzzy polygons. Follow the definitions of Polygons A and B, and let \( G_A \) be the fuzzy set defining the membership values of all points within Polygon A and \( G_B \) be the fuzzy set containing the membership values of all points within Polygon B. All possible combinations of fuzzy membership values for any point in these resultant regions can be defined by \( G_A \times G_B \).

There are two possible outcomes in the associated attribute values of Regions IV, V, VI, IX, and X, depending on the exact criteria used in the enumeration rules. In the first outcome, each region from Regions IV, V, VI, IX, and X will have two instances. Both instances have the exact boundaries, but fuzzy membership values associated with the points within the boundaries of each instance are inherited from those of their corresponding locations in the input polygon. In the second outcome, every region from Regions IV, V, VI, IX, and X will have one instance and the membership values associated with the points in the instance are defined by a set of user specified criteria. In the second outcome, a set of user specified criteria can be used to assign an attribute value to each point in the resultant region for every combination of fuzzy membership values from the input polygons.

For the example illustrated in Figure 6, example attribute values related to the resultant regions from three topological overlay operations – union, intersection, and difference – are summarized in Table 1. As can be seen from the top portion of Table 1, the membership value \( \mu_R(x, y) \) of a point in a resultant region (\( R \)) can either inherit from its corresponding input polygon or be determined using user-specified criteria when enumeration rules are used to assign the attribute values. We can use the two example polygons \( T \) and \( H \) shown in Figure 1 to illustrate how user-specified rules can be used to assign attribute values. If the two polygons overlap with each other as shown in Figure 6, then one may use the overlay operation to determine whether a location is suitable for growing a specific type of crop. Assuming that soil type \( t \) is suitable for growing the crop and soil type \( b \) is not, then each location can be assigned an attribute value. The attribute value can either be suitable (\( m \)) or not suitable (\( n \)). This user-specified criterion can be stated as: ‘\( m \) if \( \mu_T(x, y) \geq \mu_H(x, y) \); otherwise \( n \).’ The criterion

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### Table 1: Examples of assigning attribute values to the resultant regions given in Figure 6

<table>
<thead>
<tr>
<th>T OR H</th>
<th>T AND H</th>
<th>T NOT H</th>
</tr>
</thead>
<tbody>
<tr>
<td>R ( \mu_0(x, y) )</td>
<td>R ( \mu_0(x, y) )</td>
<td>R ( \mu_0(x, y) )</td>
</tr>
</tbody>
</table>

**Enumeration rules (attributes inherited from input polygons)**

II \( \mu_1(x, y) \)

III \( \mu_2(x, y) \)

IV \( \mu_1(x, y) \) or \( \mu_2(x, y) \)

VI \( \mu_1(x, y) \) or \( \mu_2(x, y) \)

VII \( \mu_1(x, y) \) or \( \mu_2(x, y) \)

VII \( \mu_1(x, y) \) or \( \mu_2(x, y) \)

IX \( \mu_1(x, y) \) or \( \mu_2(x, y) \)

**Enumeration rules (attributes defined by user-specified criteria; m and n are non-fuzzy values)**

II if \( \mu_1(x, y) \geq 0.5 \); otherwise n

III m based on the rule 

IV m if \( \mu_1(x, y) \geq \mu_n(x, y) \); otherwise n

VI m if \( \mu_1(x, y) \geq \mu_n(x, y) \); otherwise n

VI m if \( \mu_1(x, y) \geq \mu_n(x, y) \); otherwise n

I otherwise n

IX m if \( \mu_1(x, y) \geq \mu_n(x, y) \); otherwise n

**Dominance rules (attributes assigned by fuzzy union)**

II \( \mu_1(x, y) \)

III \( \mu_2(x, y) \)

IV \( \mu_1(x, y) \) or \( \mu_2(x, y) \)

VI \( \mu_1(x, y) \) or \( \mu_2(x, y) \)

VII \( \mu_1(x, y) \) or \( \mu_2(x, y) \)

VII \( \mu_1(x, y) \) or \( \mu_2(x, y) \)

IX \( \mu_1(x, y) \) or \( \mu_2(x, y) \)

**Contributory rules (attributes assigned by fuzzy addition)**

II \( \mu_1(x, y) \)

III \( \mu_2(x, y) \)

IV \( \sup_T \min \{ \mu_1(x, y), \mu_2(x, y) \} \)

VI \( \sup_T \min \{ \mu_1(x, y), \mu_2(x, y) \} \)

VII \( \sup_T \min \{ \mu_1(x, y), \mu_2(x, y) \} \)

VII \( \sup_T \min \{ \mu_1(x, y), \mu_2(x, y) \} \)

IX \( \sup_T \min \{ \mu_1(x, y), \mu_2(x, y) \} \)
simply means that any point in a resultant region is assigned either a value \( m \) or \( n \), depending on the membership values of \( \mu_T(x, y) \) and \( \mu_H(x, y) \) at that point. Of course, other user-specified criteria may be used to assign the attributes. For instance, point \((x, y)\) in Region VI resulting from the intersection of Polygons \( T \) and \( H \) has a membership \( \mu_T(x, y) \) if \( \mu_T(x, y) \geq \mu_H(x, y) \). Otherwise, the membership value is \( \mu_H(x, y) \) based on the user-specified criterion mentioned above (Table 1).

The membership values related to Regions II and III resulting from the difference of Polygons \( T \) and \( H \) are those inherited from their corresponding locations in Polygon \( T \) because these two regions have no overlap with any location in Polygon \( H \). The membership values related to Regions II and III may also be assigned according to user-specified criteria, as in the second case of enumeration rules given in Table 1. In this case, the rule ‘‘ if \( \mu_T(x, y) \geq 0.5 \); otherwise \( n' \) is used, meaning \( \mu_T(x, y) = n \) if \( \mu_T(x, y) \geq 0.5 \), otherwise \( \mu_H(x, y) = n \).

4.3.2 Dominance rules

Obtaining attribute values for the resultant regions using the dominance rules can be achieved either through the utilization of fuzzy operator union (OR \( (\cup) \)) or through a set of user-specified criteria. Operations of fuzzy operator union (OR \( (\cup) \)) are defined below. In both cases, there is only one possible instance for each of the five regions (Regions IV, V, VI, IX, and X).

\[
\mu_{A:B}(x, y) = \max \{\mu_A(x, y), \mu_B(x, y)\} \tag{9}
\]

The determination of attributes for the resultant regions based on dominance rules is relatively simple. For the resultant regions shown in Figure 6, the operation ‘fuzzy union’ as indicated in Expression (9) can be used to determine the membership values at each point in the resultant regions. When dominance rules are used to determine the attribute values associated with the resultant regions from the overlay of Polygons \( T \) and \( H \), the membership value \( \mu_R(x, y) \) of the resultant region \( R \) at location \((x, y)\) is simply the value of \( \mu_T(x, y) \) if \( \mu_T(x, y) > \mu_H(x, y) \). Otherwise, it is the value of \( \mu_H(x, y) \) (Table 1).
4.3.3 Contributory rules

Contributory rules can be defined based on fuzzy arithmetic operations (Klir and Yuan 1995, pp. 102–109). The most commonly used fuzzy arithmetic operations in this context can be fuzzy addition and fuzzy subtraction as defined below. In addition, a weighted combination of addition and subtraction of fuzzy membership values from input Polygons $A$ and $B$ may also apply.

\[
\mu_{A+B}(x, y) = \sup_{A+B} \min[\mu_A(x, y), \mu_B(x, y)]
\]

\[
\mu_{A-B}(x, y) = \sup_{A-B} \min[\mu_A(x, y), \mu_B(x, y)]
\]

The attributes of the resultant regions in the example shown in Figure 6 are summarized in Table 1 when they are determined based on contributory rules. Fuzzy addition was used as the contributory rule. For example, as shown in Table 1, for Region IV resulting from the union of Polygons $T$ and $H$, the membership value ($\mu_R(x, y)$) of Region IV is determined by the addition of the two fuzzy sets defining the membership values of Polygons $T$ and $H$ using Expression (10).

4.3.4 Interaction rules

Fuzzy operators such as intersection (AND ($\land$)) and union (OR ($\lor$)) or a set of user specified interaction rules may also be used to determine the attribute values associated with the resultant regions. Here the fuzzy operators, intersection and union, are applied to the two fuzzy sets defining the attribute values of fuzzy polygons $A$ and $B$, which are different from the overlay operations of intersection and union as discussed in Section 4.2. Again, there is only one possible instance for each of the five regions when interaction rules are used. Because operations of fuzzy operator union (OR ($\lor$)) are defined by Expression (9), only operations related to fuzzy operator intersection (AND ($\land$)) are defined below.

\[
\mu_{A \land B}(x, y) = \min[\mu_A(x, y), \mu_B(x, y)]
\]

Similar to the utilization of dominance rules, intersection rules are relatively easy to implement. For the resultant regions shown in Figure 6, attributes can be assigned either using the fuzzy intersection or fuzzy union of the two fuzzy sets defining the membership values of the attributes of Polygons $T$ and $H$. The examples given in Table 1 are situations where the attributes of the resultant regions are assigned using fuzzy intersection (bottom portion of Table 1). For instance, the membership value ($\mu_R(x, y)$) of Region IX resulting from the union of Polygons $T$ and $H$ is $\mu_T(x, y)$ when $\mu_T(x, y) < \mu_H(x, y)$. Otherwise, its value is $\mu_H(x, y)$.

5 Summary and Future Work

In this article, we have outlined operations that can be used to determine the boundaries of resultant regions from the overlay of two simple polygons with indeterminate boundaries (called simple fuzzy polygons) as well as procedures that can be used to assign attribute values associated with each resultant region under different situations. A total of 10 regions can be distinguished in the topological overlay of two simple fuzzy polygons. Any of these 10 regions can either be empty or non-empty. Therefore, there
are a total of 1,024 possible topological configurations of the resultant regions when two simple fuzzy polygons are overlaid with each other.

The boundaries of the 10 regions can be determined using existing topological overlay algorithms because the boundaries of these 10 regions are formed by either the inside or outside edge of the simple fuzzy polygon. The inside and outside edges of the simple fuzzy polygon can be transformed to the boundaries of crisp polygons that can be easily accommodated by algorithms in existing GIS. Depending on the exact Boolean operations in question and the set of rules used, the attributes associated with the 10 resultant regions can be determined using a combination of user-specified rules and any one of the four sets of rules: enumeration rules, dominance rules, contributory rules, and interaction rules.

Although it is an important step towards the development of a comprehensive set of procedures for computing the overlay of polygons with indeterminate boundaries, a simple fuzzy region is a gross approximation of real world phenomena where polygons with indeterminate boundaries can be far more complex than those represented by a simple fuzzy region. Future research will strive to extend the model and operations presented in this article and develop more realistic representations of polygons with indeterminate boundaries and additional computational procedures for the overlay of polygons with indeterminate boundaries.

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