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SCALAR FIELD THEORY IN CURVED SPACE AND THE DEFINITION OF MOMENTUM

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Abstract. Some general remarks are made about the quantum theory of scalar fields and the definition of momentum in curved space. Special emphasis is given to field theory in anti-de Sitter space, as it represents a maximally symmetric space-time of constant curvature which could arise in the local description of matter interactions in the small regions of space-time. Transform space rules for evaluating Feynman diagrams in Euclidean anti-de Sitter space are initially defined using eigenfunctions based on generalized plane waves. It is shown that, for a general curved space, the rules associated with the vertex are dependent on the type of interaction being considered. A condition for eliminating this dependence is given. It is demonstrated that the vacuum and propagator in conformally flat coordinates in anti-de Sitter space are equivalent to those analytically continued from H^4 and that transform space rules based on these coordinates can be used more readily. A proof of the analogue of Goldstone's theorem in anti-de Sitter space is given using a generalized plane wave representation of the commutator of the current and the scalar field. It is shown that the introduction of curvature in the space-time shifts the momentum by an amount which is determined by the Riemann tensor to first order, and it follows that there is a shift in both the momentum and mass scale in anti-de Sitter space.

1. Introduction

Quantum field theory in curved space has been studied for more than twenty years as a preliminary step towards developing a theory of quantum gravity. While it has been assumed that calculations in a classical space-time do not represent a complete quantum theory of matter interacting with the gravitational field, they could be useful in describing the dynamics of particle scattering, as the energy-momentum of any point-particle field induces curvature in the space-time background. Recent considerations in unification of the elementary particle interactions and superstring effective actions suggest that the standard model can be derived from a ten-dimensional theory based on the spinor space $T = \mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$ and the Clifford algebra $R_{1,9}$ [1][2] and that the higher-order curvature terms representing corrections to the standard Einstein-Hilbert action can arranged in a super-Yang-Mills theory with bosonic gauge group SO(1,9), with the torsion in the connection given by the field strength of the anti-symmetric tensor field [3]. At short distances, the gravitational force might then also be described by the exchange of quanta in this theory. Consequently, it is consistent to consider matter interactions on a background which is initially selected to be flat but then is curved by energy-momenta associated with the gravitational and matter quanta.

If it is assumed, in particular, that the energy-momenta of the quantum fields curve the space-time and still maintain maximal symmetry, the local geometry will resemble a manifold such as anti-de Sitter space, while the global geometry would remain essentially flat. Quantum field theory in anti-de Sitter space represents one of the most tractable quantum field theories on a curved manifold, and it has also arisen in the study of gauged supergravity. The techniques of Minkowski space-time quantum field theory shall be adapted to anti-de Sitter space to define appropriate momentum variables and construct Feynman rules, obtain the Lehmann spectral representation of the two-point function, prove the analogue of Goldstone's theorem and investigate the relation between field-theoretic calculations in a local region of space-time with curved geometry and string scattering in a target space with a specified global geometry.

It is shown in this paper how configuration and transform [momentum] space rules can be written for scalar field theory in anti-de Sitter space. In the configuration space rules, factors are obtained for each vertex, propagator, incoming particle and outgoing particle and a symmetry factor may be assigned for each Feynman diagram. Similar rules can be formulated for every vertex, propagator and loop transform variable in transform space, and again, a symmetry factor may be associated with each diagram. The one-loop box diagram for $\lambda\phi^3/3!$ theory is evaluated using these rules.

A general feature of the adaptation of the momentum space rules to curved space is that it is necessary to assign factors associated with each vertex that are dependent on the type of interaction being considered. Independence of the vertex rules with respect to the type of interaction becomes a constraint on the eigenfunctions representing the incoming and outgoing particles. In anti-de Sitter space, it is demonstrated that the maximal symmetry of the space allows for a choice of coordinates in which these constraints may be simplified. These methods are compared to other perturbation theory techniques and comments are made about the renormalizability of the theory.

2. Configuration and Dual Space Rules for Scalar Theory in Euclidean Anti-de Sitter Space

Euclidean anti-de Sitter space is the hyperboloid H^4

$$-x_0^2 - x_1^2 - x_2^2 - x_3^2 + x_4^2 \ = \ 1 \tag{1}$$

embedded in the five-dimensional pseudo-Euclidean space with metric

$$ds^2 \ = \ dx_0^2 + dx_1^2 + dx_2^2 + dx_3^2 - dx_4^2 \tag{2}$$

and it can be represented as the coset space SO(4,1)/SO(4). Using the hyperbolic and spherical angles θ_1, ..., θ_4, defined by

$$\begin{aligned} x_0 &= sinh\ \theta_4\ sin\ \theta_3\ sin\ \theta_2\ sin\ \theta_1 \\ x_1 &= sinh\ \theta_4\ sin\ \theta_3\ sin\ \theta_2\ cos\ \theta_1 \\ x_2 &= sinh\ \theta_4\ sin\ \theta_3\ cos\ \theta_2 \\ x_3 &= sinh\ \theta_4\ cos\ \theta_3 \\ x_4 &= cosh\ \theta_4 \end{aligned} \tag{3}$$

$$0 \leq \theta_1 < 2\pi,\ 0 \leq \theta_2,\ \ \theta_3 < \pi,\ \ 0 \leq \theta_4 < \infty$$

the Laplacian in H^4 is

$$\begin{aligned} \nabla^\mu \nabla_\mu = \frac{1}{sinh^3\theta_4} \frac{\partial}{\partial\theta_4} sinh^3\theta_4 \frac{\partial}{\partial\theta_4} + \frac{1}{sinh^2\theta_4 sin^2\theta_3} \frac{\partial}{\partial\theta_3} sin^2\theta_3 \frac{\partial}{\partial\theta_3} \\ + \frac{1}{sinh^2\theta_4 sin^2\theta_3 sin\theta_2} \frac{\partial}{\partial\theta_2}\ sin\theta_2\ \frac{\partial}{\partial\theta_2} + \frac{1}{sinh^2\theta_4 sin^2\theta_3 sin^2\theta_2} \frac{\partial^2}{\partial\theta_1^2} \end{aligned} \tag{4}$$

which has the following basis of eigenfunctions [4]

$$\begin{aligned} \Theta_K^\sigma(x) &= B_K^\sigma (sinh\ \theta_4)^{-1} \mathcal{P}_{\sigma+1}^{-k_0-1}(cosh\ \theta_4) \\ &\quad \cdot \prod_{j=0,1} C_{k_j - k_{j+1}}^{1-\frac{j}{2}+k_{j+1}}(cos\ \theta_{3-j}^4) sin^{k_{j+1}}\theta_{3-j} e^{ik_2\theta_1} \\ B_K^\sigma &= \frac{2(-1)^{k_0}\Gamma(\sigma-1)}{\Gamma(\sigma-k_0+1)} \\ &\quad \left[\Gamma(2)^{-1} \prod_{j=0,1} \frac{2^{2k_{j+1}-j}(k_j - k_{j+1})!}{\sqrt{\pi}\Gamma(k_j + k_{j+1} + 2 - j)} \cdot (2 - j + 2k_j)\Gamma^2(1 - \frac{j}{2} + k_{j+1}) \right]^{\frac{1}{2}} \\ K &= (k_0, k_1, k_2) \end{aligned} \tag{5}$$

where $\mathcal{P}_l^m(z)$ and $C_m^p(t)$ are the associated Legendre functions and Gegenbauer polynomials respectively. From the relations

$$\sqrt{z^2-1}\frac{d\mathcal{P}_l^m(z)}{dz} \ - \ \frac{mz}{\sqrt{z^2-1}}\mathcal{P}_l^m(z) \ = \ \mathcal{P}_l^{m+1}(z) \tag{6}$$

it follows that the corresponding eigenvalues are $(k-1)(k+2) + (\sigma - k + 1)(\sigma + k + 2) = \sigma(\sigma+3)$. The normalization factor associated with the eigenfunctions $\Theta_K^\sigma(x)$ determined by

$$\frac{1}{(N(\sigma,k))^2} \int_{H^4} dx\ \Theta_K^\sigma(x)\bar{\Theta}_K^\sigma(x) \ = \ 1 \tag{7}$$

is

$$N(\sigma,k) \;=\; \left[\frac{(-1)^{-1-k}}{\pi^2}(\sigma+\frac{1}{2})\frac{\Gamma^2(\sigma-1)}{\Gamma(\sigma-k+1)\Gamma(\sigma+k+3)}\right]^{\frac{1}{2}}, \quad k\equiv k_0 \tag{8}$$

The configuration space rules for scalar field theory can then be given:

1. To each vertex attach a factor $-i\lambda$ and $\int d^4x\ \sqrt{g(x)}$.
2. A propagator $i\Delta_F(x,x')$ for each line from x to x'.
3. A factor of $\frac{\Theta_K^\sigma(x)}{N(\sigma,k)}$ for an incoming particle with transform numbers σ, K.
4. A factor of $\frac{\Theta_{\bar{K}}^{-\sigma-3}(x)}{N(\sigma,k)}$ for an outgoing particle with transform space numbers $-\sigma-3$, $\bar{K}$.
5. A symmetry factor $\frac{1}{g}$ where g is the order of the symmetry group of the diagram for operations that leave the external lines fixed.

An integral transform may be used to generalize momentum-space Feynman rules to H^4. If the coefficients $a_K(\sigma)$ are defined to be $\int_{H^4} dx f(x)\Theta_K^\sigma(x)$, then

$$f(x) \;=\; \frac{i}{16\pi^2\Gamma(2)}\sum_K\int_{(Re\ \sigma_0)-i\infty}^{(Re\ \sigma_0)+i\infty} d\sigma\frac{\Gamma(\sigma+3)}{\Gamma(\sigma)}ctg\ \pi\sigma\ a_K(\sigma)\Theta_{\bar{K}}^{-\sigma-3}(x) \qquad -3 < Re\ \sigma_0 < 1 \tag{9}$$

which follows from the delta-function identity

$$\frac{i}{16\pi^2\Gamma(2)}\sum_K\int_{(Re\ \sigma_0)-i\infty}^{(Re\ \sigma_0)+i\infty} d\sigma\frac{\Gamma(\sigma+3)}{\Gamma(\sigma)}\Theta_K^\sigma(x)\Theta_{\bar{K}}^{-\sigma-3}(x') \;=\; \delta(x-x') \tag{10}$$

Then the Feynman propagator may expressed as

$$\Delta_F(x,x') \;=\; \frac{i}{16\pi^2\Gamma(2)}\sum_K\int_{(Re\ \sigma_0)-i\infty}^{(Re\ \sigma_0)+i\infty} d\sigma\frac{\Gamma(\sigma+3)}{\Gamma(\sigma)}ctg\ \pi\sigma\Delta_F(k,\sigma)\Theta_K^\sigma(x)\Theta_{\bar{K}}^{-\sigma-3}(x') \tag{11}$$

if

$$\Delta_F(K,\sigma) \;=\; \frac{1}{\sigma(\sigma+3)+m^2} \tag{12}$$

where m is the mass of the scalar field. The transform space, or dual space, rules for scalar field theory would be

1. A factor $-i\lambda$ and $f_{\{K\}}^{\{\sigma\}}$ for each vertex.
2. $\frac{i}{\sigma(\sigma+3)+m^2}$ for each propagator.

3. $\frac{i}{16\pi^2\Gamma(2)}\sum_K \int_{Re\ \sigma_0 - i\infty}^{Re\ \sigma_0+i\infty}\ d\sigma \frac{\Gamma(\sigma+3)}{\Gamma(\sigma)}\ ctg\ \pi\sigma$ for each loop transform variable.
4. A factor $\frac{1}{g}$ where g is the order of the symmetry group of the diagram.

3. The $\lambda\phi^3/3!$ Box Diagram in H^4 and Vertex Factors

The factor $f_{\{K\}}^{\{\sigma\}}$, which is necessary for equivalence between the integrals derived from configuration space and dual space rules, depends on the type of interaction being considered. For example, the box diagram (Fig. 1) in $\lambda\phi^3/3!$ theory can be evaluated using either set of rules. Configuration space rules give

$$\frac{(-i\lambda)^4}{4}\int d^4x_1\ d^4x_2\ d^4x_3\ d^4x_4\ \ \sqrt{g(x_1)}\ \sqrt{g(x_2)}\ \sqrt{g(x_3)}\ \sqrt{g(x_3)}$$
$$\cdot\ (i\Delta_F(x_1,x_2))(i\Delta_F(x_2,x_3))(i\Delta_F(x_3,x_4))(i\Delta(x_4,x_1)) \qquad (13)$$
$$\cdot\frac{\Theta_{K_1}^{\sigma_1}(x_1)}{N(\sigma_1,K_1)}\frac{\Theta_{K_2}^{\sigma_2}(x_2)}{N(\sigma_2,K_2)}\frac{\Theta_{\bar{K}_3}^{-\sigma_3-3}(x_3)}{N(\sigma_3,K_3)}\frac{\Theta_{\bar{K}_4}^{-\sigma_4-3}(x_4)}{N(\sigma_4,K_4)}$$

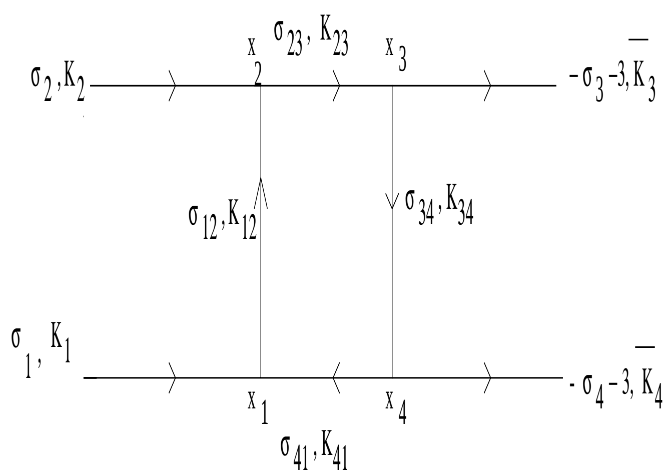

Fig. 1. The $\lambda\frac{\phi^3}{3!}$ box diagram in Euclidean anti-de Sitter space.

while transform space rules give

$$\frac{(-i\lambda)^4}{4}\left(\frac{1}{16\pi^2\Gamma(2)}\right)^4 \sum_{\substack{K_{12},K_{23}\\K_{34},K_{41}}} \int_{(Re\ \sigma_0)-i\infty}^{(Re\ \sigma_0)+i\infty} d\sigma_{12}d\sigma_{23}d\sigma_{34}d\sigma_{41}\frac{\Gamma(\sigma_{12}+3)}{\Gamma(\sigma_{12})}\ ctg\ \pi\sigma_{12}$$
$$\frac{\Gamma(\sigma_{23}+3)}{\Gamma(\sigma_{23})}ctg\ \pi\sigma_{23}\frac{\Gamma(\sigma_{34}+3)}{\Gamma(\sigma_{34})}\ ctg\ \pi\sigma_{34}\frac{\Gamma(\sigma_{41}+3)}{\Gamma(\sigma_{41})}$$
$$ctg\ \pi\sigma_{41}\cdot(i\Delta_F(\sigma_{12}))(i\Delta_F(\sigma_{23}))(i\Delta_F(\sigma_{34}))(i\Delta_F(\sigma_{41}))$$
$$\cdot\prod_{vert.} f_{\{K\}}^{\{\sigma\}}(vert.) \tag{14}$$

Applying the formula (11) to equation (13) implies that

$$\prod_{vert.} f_{\{K\}}^{\{\sigma\}}(vert.) = \int d^4x_1\ \sqrt{g(x_1)}\ \Theta_{\bar{K}_{12}}^{-\sigma_{12}-3}(x_1)\ \Theta_{K_{41}}^{\sigma_{41}}(x_1)\ \frac{\Theta_{K_1}^{\sigma_1}(x_1)}{N(\sigma_1,K_1)}$$
$$\int\ d^4x_2\ \sqrt{g(x_2)}\ \Theta_{K_{12}}^{\sigma_{12}}(x_2)\ \Theta_{\bar{K}_{23}}^{-\sigma_{23}-3}(x_2)\frac{\Theta_{K_2}^{\sigma_2}(x_2)}{N(\sigma_2,K_2)}$$
$$\int\ d^4x_3\ \sqrt{g(x_3)}\ \Theta_{K_{23}}^{\sigma_{23}}(x_3)\ \Theta_{\bar{K}_{34}}^{-\sigma_{34}-3}(x_3)\ \frac{\Theta_{\bar{K}_3}^{-\sigma_3-3}(x_3)}{N(\sigma_3,K_3)} \tag{15}$$
$$\int\ d^4x_4\ \sqrt{g(x_4)}\ \Theta_{K_{34}}^{\sigma_{34}}(x_4)\ \Theta_{\bar{K}_{41}}^{-\sigma_{41}-3}(x_4)\ \frac{\Theta_{\bar{K}_4}^{-\sigma_4-3}(x_4)}{N(\sigma_4,K_4)}$$

In Euclidean space, this product would be

$$\int\ d^4x_1\ e^{-ip_{12}\cdot x_1}e^{ip_{41}\cdot x_1}e^{ip_1\cdot x_1}\int\ d^4x_2\ e^{ip_{12}\cdot x_2}e^{-ip_{23}\cdot x_2}e^{ip_2\cdot x_2}$$
$$\int\ d^4x_3\ e^{ip_{23}\cdot x_3}e^{-ip_{34}\cdot x_3}e^{-ip_3\cdot x_3}\int\ d^4x_4\ e^{ip_{34}\cdot x_4}e^{-ip_{41}\cdot x_4}e^{-ip_4\cdot x_4} \tag{16}$$
$$= \delta(p_{12}-p_1-p_{41})\ \delta(p_2+p_{12}-p_{23})\ \delta(p_3+p_{34}-p_{23})\ \delta(p_4+p_{41}-p_{34})$$

which ensures momentum conservation at each vertex and for the external states in the scattering process.

The appearance of the factors $f_{\{K\}}^{\{\sigma\}}$ for each vertex in the transform space rules is a feature of quantum field theory in a curved space. They arise even for the hyperboloid H^4, which has maximal symmetry. These factors depend on the type of interaction being considered because the number of eigenfunctions in each integral defining $f_{\{K\}}^{\{\sigma\}}$ is equal to the number of lines at each vertex in the Feynman diagram. For a general curved Riemannian space, an integral involving three or more eigenfunctions of the Laplacian would not reduce to a delta-function containing the transform space variables, whereas, in Euclidean space, the reduction, which is independent of the type of interaction, does occur and only implies the physically necessary constraint of momentum conservation. The condition that must be obeyed by the eigenfunctions to obtain a delta-function is that they form a group under pointwise multiplication, as the number of eigenfunctions in the integral then can be reduced until the orthogonality relation (7) may be used. As this condition is satisfied by $e^{ip\cdot x}$, it is useful to consider generalized plane waves in curved spaces, and, specifically for H^4, to clarify their connection with the eigenfunctions (5).

4. Generalized Plane Waves and the Fourier Transform on H^4

The maximal symmetry of H^4 allows for the construction of generalized plane waves and the Fourier transform using group theoretic methods. Horocycles, the analogue of hyperplanes, may be defined by the action of a nilpotent subgroup in the Iwasawa decomposition of SO(4,1). Generalized plane waves are represented by eigenfunctions of the Laplacian operator that are exponential functions of the hyperbolic distance from a chosen origin to the horocycle.

Specifically, the Iwasawa decomposition of a semisimple Lie algebra is $\mathcal{G} = \mathcal{H} \oplus \mathcal{A} \oplus \mathcal{N}$ where $\mathcal{H}$ is the maximal compact subalgebra, $\mathcal{A}$ is a maximal abelian subalgebra in the space spanned by the non-compact generators and $\mathcal{N}$ is a nilpotent subalgebra defined the positive roots of $\mathcal{G}$. Given a point o chosen to be the origin in a Cartan symmetric space G/H, the fundamental horocycle is $\xi_0 = N \cdot o$. All other horocycles may be expressed as $ha \cdot \xi_0,\ h \in H,\ a \in A$. This representation is not unique; defining M to be the centralizer of $\mathcal{A}$ in H, h and h' give the same horocycles if they belong to the same coset in H/M, which is also known as the boundary of the symmetric space [5].

The Fourier transform is given by

$$\tilde{f}(\lambda, b) = \int_{G/H} dx\ f(x)\ e^{-(i\lambda+\rho)(r(x,b))} \tag{17}$$

where $\rho = \frac{1}{2} \sum_{\alpha > 0} \, dim \, \mathcal{G}_\alpha$, with $\mathcal{G}_\alpha$ being the vector space spanned by generators in $\mathcal{G}$ associated with the positive root α, and $r(x,b)$ is the distance from the origin o to the horocycle passing through x with normal b. The inverse transform is

$$f(x) \;=\; \int_{\mathcal{A}^*} \int_{H/M} d\lambda \; db \; \tilde{f}(\lambda, b) \; e^{(i\lambda - \rho)(r(x,b))} \; |\hat{c}(\lambda)|^{-2} \tag{18}$$

where $\mathcal{A}^*$ is the space of functionals on $\mathcal{A}$ and $\hat{c}(\lambda)$ is a spectral function selecting only those values of λ corresponding to irreducible representations of G [6]. In the standard treatise on the Fourier transform on symmetric spaces [7], the exponent ρ appears with a positive sign. However, the negative sign is also appropriate for H^4, considering the eigenvalues of the d'Alembertian, mentioned immediately following equation (6), and this is confirmed by the integral transform for H^{n-1}, where the contour of integration is $-\frac{n-2}{2} + i\lambda$ [4]. The difference between the two transforms can be absorbed into the spectral function.

The action of the nilpotent subgroup in the Iwasawa decomposition of SO(4,1) on the chosen origin $o = (0,0,0,0,1)$ defines the fundamental horocycle on H^4 which is the intersection of a hyperboloid with a hyperplane

$$\begin{aligned} -x_0^2 - x_1^2 - x_2^2 - x_3^2 + x_4^2 &= 1 \\ -x_0 + x_4 &= 1 \end{aligned} \tag{19}$$

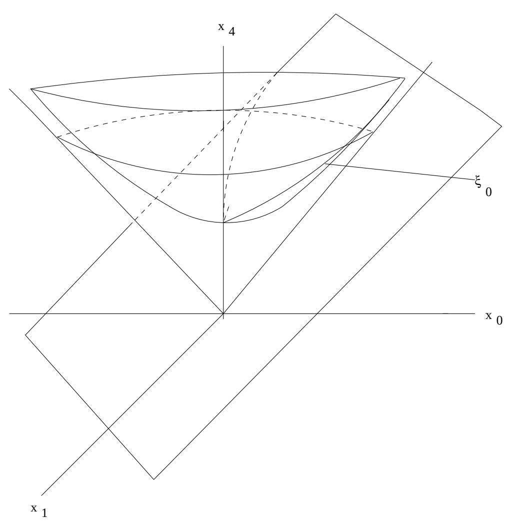

Fig. 2. The fundamental horocycle ξ_0.

It can be shown, either group theoretically or geometrically [6], that the distance from the origin to the horocycle passing through x with normal b is $ln[x, h\xi_0]$, where hM is the coset corresponding to b, ξ_0 represents the null vector (1,0,0,0,1) on the cone $-\xi_0^2 - \xi_1^2 - \xi_2^2 - \xi_3^2 + \xi_4^2 = 0$ and $[x, \xi]$ is the scalar product $-x_0\xi_0 - x_1\xi_1 - x_2\xi_2 - x_3\xi_3 + x_4\xi_4$. The calculation of the hyperbolic distance is given in the appendix. Since $\rho = \frac{3}{2}$, the generalized plane waves on H^4 will be $[x, \xi]^{-i\lambda - \frac{3}{2}}$, which are eigenfunctions of the Laplacian with eigenvalue $-(\lambda^2 + \frac{9}{4})$, and the Fourier transform will be

$$f(\eta, \lambda) \;=\; \int_{H^4} dx \; f(x) \; [x, \xi]^{-i\lambda - \frac{3}{2}} \tag{20}$$

Since the null vectors on the cone are in one-to-one correspondence with points on S^3, these functions can be expanded in terms of basis eigenfunctions on the sphere

$$\begin{aligned} \tilde{f}(\xi, \lambda) \;&=\; \sum_K \; a_K(\lambda) \; \Xi_K(\xi) \\ a_K(\lambda) \;&=\; \int_{S^3} d\eta \; \tilde{f}(\xi, \lambda) \; \bar{\Xi}_K(\eta) \end{aligned} \tag{21}$$

The equivalence of the Fourier coefficients $a_K(\lambda)$ in equation (21) and $a_K(\sigma)$ defined just before equation (9) follows from an identity containing the associated Legendre functions [4]

$$\Theta_K^{\sigma}(x) \;=\; \int_{S^3} d\xi \; [x, \xi]^{\sigma} \; \bar{\Xi}_K(\eta) \tag{22}$$

This relation reveals the plane-wave content in the configuration space and dual space rules based on the eigenfunctions $\Theta_K^{\sigma}(x)$. One may note further that the propagator for a massive scalar field [5][8] may also be written as

$$\Delta_F(\lambda) \;=\; \frac{1}{a^2\lambda^2 \;-\; m^2 \;+\; \frac{9}{4}a^2} \tag{23}$$

so that the variable λ could be interpreted as the norm of the four-momentum with a shift in the zero-point of the squared-mass scale by $-\frac{9}{4}a^2$, where a^{-1} is the radius of curvature of the hyperboloid, which is customarily set equal to 1. Analytic continuation to anti-de Sitter space is achieved by replacing λ by $-i\lambda$ and a by ia.

5. Spectral Representation of the Propagator

In flat-space field theory, non-perturbative results about S-matrix elements can be obtained when the fields satisfy properties such as

Poincare invariance

$$e^{i\alpha\cdot P}\phi(x)e^{-i\alpha\cdot P} \ = \ \phi(x+\alpha)$$
$$U(\Lambda)\phi(x)U(\Lambda^{-1}) \ = \ \phi(\Lambda x)$$

Locality

$$[\phi(x),\phi(y)] \ = \ 0 \quad if \quad (x-y)^2 \ < \ 0$$

Spectral assumption

The eigenvalues of P^0, P^2 are non-negative.

Uniqueness of vacuum

There exists a unique vacuum $|0\rangle$ such that $P_\mu|0\rangle \ = \ 0$, $U(\Lambda)|0\rangle \ = \ |0\rangle$.

Because of the spectral assumption, the positive frequency commutator function can be written as $\Delta_+(x) \ = \ -\frac{i}{(2\pi)^3}\int d^4k \ e^{-ik\cdot x} \ \rho(k^2)\theta(k^0)$ where $\rho(k^2)\geq 0$ and $\rho(k^2)=0$ if $k^2\leq 0$, and the Lehmann representation for the propagator is $\Delta_F(p) \ = \ \int_0^\infty \ d\sigma \ \frac{\rho(\sigma)}{p^2-\sigma+i\delta}$.

The assumptions above can be generalized to H^4, with Poincare invariance being replaced by SO(4,1) invariance for example. However, the spectrum, instead of starting at zero, begins at $-\frac{9}{4}a^2$, so that the Lehmann representation [6][7] becomes

$$\Delta_F(p) \ = \ \int_{-\frac{9}{4}a^2}^{\infty} \ d\sigma \ \frac{\rho(\sigma)}{p^2-\sigma+i\delta} \tag{24}$$

Thus, the generalized plane wave decomposition of Green function can be used to provide a momentum-space Lehmann representation for Feynman propagators in anti-de Sitter space [6], leading to a non-perturbative definition of the mass as a pole in the propagator [8].

6. The Vertex Factor and Products of Generalized Plane Waves

Feynman rules may now be formulated in the configuration and dual space using the generalized plane wave decomposition of the scalar field.

Configuration space rules

1. To each vertex attach a factor $-i\lambda$ and $\int\ d^4x\sqrt{g(x)}$.
2. A propagator $i\Delta_F(x_i, x_j)$ for each line from x_i to x_j.
3. A factor of $e^{-(i\lambda+\frac{3}{2})r(x,b)}$ for each particle traveling towards x with 'momentum' λ on a geodesic orthogonal to the horocycle $\xi(x,b)$.
4. A factor of $e^{(i\lambda-\frac{3}{2})r(x,b)}$ for each particle traveling outwards from x with 'momentum' λ on a geodesic orthogonal to the horocycle $\xi(x,b)$.
5. Symmetry factor $\frac{1}{g}$ where g is the order of the symmetry group of the diagram leaving external lines fixed.

Setting $a\ =\ 1$, the Fourier transform in H^4 implies the following set of rules.

Dual space rules

1. A factor of $-i\lambda$ and $f^{\{\lambda\}}_{\{b\}}$ is assigned to each vertex.
2. A factor of $\frac{i}{\lambda^2-m^2+\frac{9}{4}}$ for each propagator.
3. $\int_\lambda\ d\lambda\ |\hat{c}(\lambda)|^{-2}\ \int_{S^3}\ db$ for each independent loop dual space variable.
4. A symmetry factor $\frac{1}{g}$ where g is the order of the symmetry group of the diagram.

The usefulness of the dual space rules depends on the factor $\prod_{vert.} f^{\{\lambda\}}_{\{b\}}$. The generalized plane waves have a form similar to the plane waves $e^{ip\cdot x}$ in flat space, and their product arises in the vertex factor.

Consider, for example, the product of two generalized plane waves

$$e^{-(i\lambda+\frac{3}{2})r(x,b)}e^{-(i\lambda'+\frac{3}{2})r(x,b')}\ =\ e^{-\frac{[i(\lambda+\lambda')+3]}{2}(r(x,b)+r(x,b'))}e^{-\frac{i(\lambda'-\lambda)}{2}(r(x,b')-r(x,b))} \tag{25}$$

The point x can reached from the origin o by the action of a group element g_x. Since H acts transitively on the boundary B, there always exists an element $h_{b'}$ such that $b' = g_x h_{b'} \cdot b$. It can be demonstrated that the following addition theorem is valid [7].

$$r(x,b) + r(g \cdot o, g \cdot b) = r(g \cdot x, g \cdot b) \tag{26}$$

As $h_{b'} \cdot o = o$, $x = g_x h_{b'} \cdot o$ and

$$r(x,b) + r(x,b') = r(x,b) + r(g_x h_{b'} \cdot o, g_x h_{b'} \cdot o) = r(g_x h_{b'} \cdot x, g_x h_{b'} \cdot b) = r(g_x h_{b'} \cdot x, b') \tag{27}$$

As $x = g_x \cdot o$, $g_x H g_x^{-1}$ is the stability group of x. If $\tilde{h}_{b'} \in H$ is chosen appropriately, $b' = g_x \tilde{h}_{b'} g_x^{-1}$ and

$$\begin{aligned} r(x,b') - r(x,b) &= r(g_x \tilde{h}_{b'} g_x^{-1} \cdot x, g_x \tilde{h}_{b'} g_x^{-1} \cdot b) - r(x,b) = r(g_x \tilde{h}_{b'} g_x^{-1} \cdot o, g_x \tilde{h}_{b'} \cdot b) \\ &= r(g_x \tilde{h}_{b'} g_x^{-1} \cdot o, b') \end{aligned} \tag{28}$$

These formulas could be used iteratively to simplify products of more than two generalized plane wave functions, and this would be necessary for the dual space rules to be useful in the evaluation of diagrams in perturbation theory. The Fourier transform (17) on H^4 and the inverse transform (18) lead to the following identity

$$\tilde{f}(\lambda, b) = \int_{\mathcal{A}^*} \int_{S^3} d\tilde{\lambda}\ d\tilde{b}\ \tilde{f}(\tilde{\lambda}, \tilde{b}) |\hat{c}(\tilde{\lambda})|^{-2} \int_{H^4} dx\ e^{-(i\lambda + \frac{3}{2}) r(x,b)}\ e^{(i\tilde{\lambda} - \frac{3}{2}) r(x,\tilde{b})} \tag{29}$$

which implies an identity

$$\int_{H_4} dx\ e^{-(i\lambda + \frac{3}{2})(r(x,b))} \cdot\ e^{-(i\lambda' + \frac{3}{2}) r(x,b')} = \delta(\lambda + \lambda')\ \delta(b - b') \cdot |\hat{c}(\lambda)|^2 \tag{30}$$

that resembles the delta function relation (16) expressing momentum conservation. The concept of generalized plane waves in the hyperboloid H^4 and the momentum-space Lehmann representation for the propagator in anti-de Sitter space rely on the general theory of the Fourier transform developed for locally symmetric spaces, for which there exist isometry-reversing geodesics at any point [5]. Since it follows that there are space-like slices dividing these manifolds into two parts M^+ and M^- and isometries such that

$\Theta(M^{\pm}) \;=\; M^{\mp}$, it has been noted that this is precisely the condition needed for reflection positivity [9], which is a property of an axiomatic quantum field theory possessing a positive-definite inner product in the Fock space.

7. Goldstone's Theorem in Anti-de Sitter Space

These techniques can also be applied to the proof of a theorem for anti-de Sitter space similar to Goldstone's theorem. The existence of Goldstone particles in de Sitter space has been studied previously [10]. The proof of this theorem begins with the charges

$$Q^a \;=\; \int d^3x \; j^{0a}(x) \qquad \partial^\mu j^a_\mu \;=\; 0 \tag{31}$$

satisfying commutation relations $[Q^a, Q^b] \;=\; i f^a{}_{bc} Q^c$ generating a group G and a set of scalar fields transforming under a nontrivial representation of G, so that
$[Q^a, \phi_r] \;=\; -iT^a{}_{rs}\phi_s$. In Minkowski space-time, the expectation value of the commutators of the currents associated with the original symmetry of the Lagrangian and the scalar fields is

$$F^a_{r\mu}(x) \;=\; \langle 0|\; [j^a_\mu(x), \phi_r] \;|0\rangle \;=\; \partial_\mu D(x) \tag{32}$$

where

$$F^a_{r\mu}(q) \;=\; \int d^4x \; e^{iq\cdot x}\; F^a_{r\mu}(x) \;=\; \int d^4x \; \partial_\mu D(x) \;=\; -iq_\mu \int d^4x \; e^{iq\cdot x} D(X) \;=\; -iq_\mu D(q) \tag{33}$$

and current conservation implies that

$$q^\mu \; F^a_{r\mu}(q) \;=\; -iq^2 \; D(q) \;=\; 0 \tag{34}$$

and $D(q)$ contains $\delta(q^2)$, demonstrating the existence of dim G - dim H massless scalar particles where H is the stability group of $\langle 0|\underline{\phi}|0\rangle \;\neq\; 0$.

In anti-de Sitter space, the analytic continuation to H^4 can be used to obtain the Fourier representation

$$F^a_{r\mu}(\xi, \lambda) \;=\; \int_{H^4} dx \; \langle 0|\; [j^a_\mu(x), \phi_r] \;|0\rangle \; [x, \xi]^{-i\lambda - \frac{3}{2}} \tag{35}$$

and since G acts transitively on the space of horospheres labelled by (ξ, λ),

$$
\begin{aligned}
F^a_{r\mu}(g\cdot\xi, g\cdot\lambda) &= \int_{H^4} dx\ \langle 0|\ [j^a_\mu(x), \phi_r]\ |0\rangle\ [x, g\cdot\xi]^{-i(g\cdot\lambda)-\frac{3}{2}} \\
&= \int_{H^4} dx\ \langle 0|\ [j^a_\mu(x), \phi_r]\ |0\rangle\ [g^{-1}\cdot x, \xi]^{-i(g\cdot\lambda)-\frac{3}{2}} \\
&= \int_{H^4} dx\ \langle 0|\ [j^a_\mu(g\cdot x), \phi_r]\ |0\rangle\ [x, \xi]^{-i(g\cdot\lambda)-\frac{3}{2}} \\
&= (g)^\nu_\mu \int_{H^4} dx\ \langle 0|\ [j^a_\nu(x), \phi_r]\ |0\rangle\ [x, \xi]^{-i(g\cdot\lambda)-\frac{3}{2}}
\end{aligned}
\tag{36}
$$

Since the elements $g \in SO(4)$ translate points on S^3 and leave λ invariant,

$$
F^a_{r\mu}(g\cdot\xi, g\cdot\lambda) = (g)^\nu_\mu \int_{H^4} dx\ \langle 0|\ [j^a_\nu(x), \phi_r]\ |0\rangle\ [x, \xi]^{-i\lambda-\frac{3}{2}} = (g)^\nu_\mu\ F^a_{r\nu}(\xi, \lambda) \tag{37}
$$

Since the expectation value of the commutator transforms covariantly under the action of the subgroup SO(4) of the isometry group H^4, it follows that

$$
2\ F_{[\rho;\lambda]}(g\cdot y, g\cdot y') = 2(g)^\nu_\rho (g)^\sigma_\lambda\ F_{[\nu;\sigma]}(y, y') = 2\ (g)^\nu_\rho\ F_{[\nu;\lambda]}(y, y') \tag{38}
$$

Consequently, $F_{[\mu;\sigma]}(y, y') = 0$ and $F_\mu(y, y') = \partial_\mu D(y, y')$. Although $F^a_{r\mu} = \partial_\mu D(x)$ does not imply $F^a_{r\mu} = -iq_\mu D(q)$, integration by parts can be used to show that

$$
\begin{aligned}
\int_{H^4} d^4x\ \sqrt{g(x)}\ \nabla^\mu \partial_\mu D(x)\ [x, \xi]^{-i\lambda-\frac{3}{2}} &= \int_{H^4} d^4x\ \sqrt{g(x)}\ D(x)\ \nabla^\mu \partial_\mu\ [x, \xi]^{-i\lambda-\frac{3}{2}} \\
&= -(\lambda^2 + \frac{9}{4}) \int_{H^4} dx\ D(x)\ [x, \xi]^{-i\lambda-\frac{3}{2}} \\
&= -(\lambda^2 + \frac{9}{4})\ D(\xi, \lambda)
\end{aligned}
\tag{39}
$$

Replacing λ by $-i\lambda$ for the analytic continuation to anti-de Sitter space and using current conservation implies that $D(\xi, \lambda)$ contains $\delta(\lambda^2 - \frac{9}{4})$. The physical spectrum is given by a discrete series representation of SO(3,2) with the eigenvalues of the Casimir operator equal to $-\omega_0(\omega_0 - 3)$, so that $\omega_0 = \lambda + \frac{3}{2}$. Thus the condition $\lambda = \frac{3}{2}$ is equivalent to $\omega_0 = 3$. As representations of SO(3,2) are denoted as $\mathcal{D}(\omega_0, s)$ where $\omega_0 \geq s + \frac{1}{2}$ is the lowest energy eigenvalue and s is the spin of the field, the Goldstone bosons correspond to the representation $\mathcal{D}(3, 0)$.

8. Equivalence of Vacua in Different Coordinate Systems

A closer analogy with the construction of Feynman diagrams in Minkowski space-time can be achieved by using conformally flat coordinates in anti-de Sitter space, as there are flat three-dimensional sections of the space-time that are spanned by three of the coordinates. Plane waves in three dimensions $e^{i(k_{t'}t' - k_y y - k_z z)}$ are then included in the eigenfunctions leading to an identification of the transform variables with components of the momentum. Conservation of momentum follows from the delta functions that arise in the conversion of the configuration space integrals to the transform space integrals.

To establish the connection between the analytic continuation of the Feynman propagator (11) and the propagator $\langle 0|T\phi(x)\phi(x')|0\rangle$ in these coordinates, it is sufficient to establish equivalence of the vacua in the different coordinate systems. From the embedding of anti-de Sitter space as a hyperboloid in the five-dimensional pseudo-Euclidean space with signature $(+ - - - +)$, one obtains the metric in different intrinsic coordinates and the corresponding eigenfunctions of the d'Alembertian.

A brief list of coordinate systems and eigenfunctions associated with a special choice of eigenvalue is now given.

(i) Globally static coordinates

$$\begin{aligned} z_0 &= (a^2 + r^2)^{\frac{1}{2}}\ sin(\frac{t}{a}) \\ z_1 &= r\ sin\ \theta\ cos\ \phi \\ z_2 &= r\ sin\ \theta\ sin\ \phi \\ z_3 &= r\ cos\ \theta \\ z_4 &= (a^2 + r^2)^{\frac{1}{2}}\ cos(\frac{t}{a}) \end{aligned} \tag{40}$$

$$ds^2 \;=\; \left(1+\frac{r^2}{a^2}\right)dt^2 - \left(1+\frac{r^2}{a^2}\right)^{-1} dr^2 - r^2(d\theta^2 + sin^2\theta d\phi^2) \tag{41}$$

Since the curvature scalar is $-\frac{12}{a^2}$, the conformally invariant scalar field satisfies

$$\left(1+\frac{r^2}{a^2}\right)^{-1}\frac{\partial^2\Phi}{\partial t^2} - \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\left(1+\frac{r^2}{a^2}\right)\frac{\partial\Phi}{\partial r}\right] - \frac{1}{r^2 sin\,\theta}\frac{\partial}{\partial\theta}\left[sin\,\theta\frac{\partial\Phi}{\partial\theta}\right] - \frac{1}{r^2 sin\,\theta}\frac{\partial^2\Phi}{\partial\phi^2} - \frac{2}{a^2}\Phi \;=\; 0 \tag{42}$$

The basis set of solutions are

$$\begin{aligned}
\phi_{\omega lm} &= \frac{f_l(r)}{r}Y_{lm}(\theta,\phi)e^{-i\omega t} \\
f_l^{(1)}(r) &= e^{-i\pi a\omega}2^{-l-1}\pi^{-\frac{1}{2}}\Gamma(l+1-a\omega)(\frac{ir}{a})^{l+1}(1+\frac{r^2}{a^2})^{-\frac{a\omega}{2}} \\
&\quad F(1+\frac{l}{2}-\frac{a\omega}{2},\frac{1}{2}+\frac{l}{2}-\frac{a\omega}{2};l+\frac{3}{2};-\frac{r^2}{a^2}) \\
f_l^{(2)}(r) &= \frac{\Gamma(-\frac{1}{2}-l)(\frac{ir}{a})^{l+1}}{2^{l+1}\pi^{\frac{1}{2}}(1+\frac{r^2}{a^2})^{-\frac{a\omega}{2}}\Gamma(-l+a\omega)}F(\frac{1}{2}+\frac{l}{2}+\frac{a\omega}{2},1+\frac{l}{2}+\frac{a\omega}{2};l+\frac{3}{2};-\frac{r^2}{a^2}) \\
&\quad + \frac{2^l\pi^{-\frac{1}{2}}\Gamma(\frac{1}{2}+l)(\frac{ir}{a})^{-l}}{(1+\frac{r^2}{a^2})^{-\frac{a\omega}{2}}\Gamma(1+l+a\omega)}F(-\frac{l}{2}+\frac{a\omega}{2},\frac{1}{2}-\frac{l}{2}+\frac{a\omega}{2};\frac{1}{2}-l;-\frac{r^2}{a^2})
\end{aligned} \tag{43}$$

The general solution will involve the combination $A_l f_l^{(1)} + B_l f_l^{(2)}$ and the choice of A_l and B_l correspond to the choice of vacuum state. Defining the scalar product in the space of solutions to be

$$(\phi_1,\phi_2) \;=\; i\int d^3x\sqrt{-g}g^{0\nu}(\bar{\phi}_1\partial_\nu\phi_2 \;-\; \phi_2\partial_\nu\bar{\phi}_1) \tag{44}$$

finiteness of the norm at r=0 implies that $B_l = 0$ because the second term in $f_l^{(2)}$ contains a factor of r^{-l}.

These coordinates can be obtained by analytic continuation of globally static coordinates in de Sitter space after mapping $a \to -ia$. The de Sitter solutions are

$$\begin{aligned}
\phi^{dS}_{\omega lm} &= \frac{g_l(r)}{r}Y_{lm}(\theta,\phi)e^{-i\omega t} \\
g_l^{(1)}(r) &= Q_l^{ia\omega}(\frac{a}{r}) \qquad g_l^{(2)}(r) \;=\; \mathcal{P}_l^{ia\omega}(\frac{a}{r})
\end{aligned} \tag{45}$$

Since $(\phi^{dS}_{\omega lm},\phi^{dS}_{\omega lm})$ is finite at r=0 only if $Q_l^{ia\omega}(\frac{a}{r})$ is chosen for $g_l(r)$, analytic continuation to AdS gives the same mode solutions.

A similar set of coordinates exist on S^4 and analytic continuation to AdS can be achieved through the mapping of the radial coordinate $r \to ir$. Again, finiteness of $(\phi^{S^4}_{\omega lm}, \phi^{S^4}_{\omega lm})$ leads to the same solutions after analytic continuation.

The propagator for a massive scalar field can also be continued from S^4 to anti-de Sitter space, and it has been noted that this is not equal to the direct mode sum in anti-de Sitter space, where the modes are subject to supersymmetric boundary conditions [11] . The latter propagator is the sum of a symmetric and an anti-symmetric combination of the analytically continued S^4 propagator and its antipodal counterpart.

It may be recalled that the basis solutions $\phi_{\omega lm}$ can be split into two groups, corresponding to Dirichlet and Neumann boundary conditions. The conformally coupled scalar field modes form the representations $\mathcal{D}(1,0)$ and $\mathcal{D}(2,0)$ respectively. Instead of using $\mathcal{D}(1,0)$ or $\mathcal{D}(2,0)$, the two representations can be combined to give two irreducible representations with different parity assignments [12].

$$\begin{array}{llll}
\mathcal{D}(1,0)^+ : \ \omega = n+l+1, & n = 0,\ 1,\ 2, ..., & l = 0,\ 1,\ 2,\ ... & parity = (-1)^{\omega-1} \\
\quad A_{\omega lm} = \phi_{\omega lm} & B_{\omega lm} = 0 & even\ n & \\
\quad A_{\omega lm} = 0 & B_{\omega lm} = i\phi_{\omega lm} & odd\ n & \\
\mathcal{D}(1,0)^- : \ \omega = n+l+1 & n = 0,\ 1,\ 2, ... & l = 0,\ 1,\ 2, ... & parity = -(-1)^{\omega-1} \\
\quad A_{\omega lm} = 0 & B_{\omega lm} = \phi_{\omega lm} & even\ n & \\
\quad A_{\omega lm} = -i\phi_{\omega lm} & B_{\omega lm} = 0 & odd\ n &
\end{array} \tag{46}$$

where A and B are scalar and pseudo-scalar fields respectively. A study of the spin-$\frac{1}{2}$ field reveals that the imposition of Dirichlet and Neumann conditions again leads to quantization of the frequencies. The basis solutions are $\{\chi^{(+)}_{\omega jm}\}$, $\omega = 2n+j+1$, $j = \frac{1}{2}, \frac{3}{2}, ...$ with parity $(-1)^{j-\frac{1}{2}}$, and $\{\chi^{(-)}_{\omega jm} = i\gamma_5 \chi^{(+)}_{\omega jm}\}$, $\omega = 2n+j+1$, $j = \frac{1}{2}, \frac{3}{2}, ...$ with parity $-(-1)^{j-\frac{1}{2}}$. Again, these two representations can be combined into two irreducible representations with different parity assignments.

$$
\begin{aligned}
\mathcal{D}(\frac{3}{2},\frac{1}{2})^+ :\ \omega &= n+j+1, \quad n = 0,\ 1,\ 2,\ ...,\ j = \frac{1}{2},\ \frac{3}{2},\ \frac{5}{2},\ ... \quad parity = (-1)^{\omega-\frac{3}{2}} \\
\chi^+_{\omega jm} &= i\chi^{(+)}_{\omega jm} \qquad even\ n \\
&\quad\ \chi^{(-)}_{\omega jm} \qquad odd\ n \\
\mathcal{D}(\frac{3}{2},\frac{1}{2})^- :\ \omega &= n+j+1, \quad n = 0,\ 1,\ 2,\ ...,\ j = \frac{1}{2},\ \frac{3}{2},\ \frac{5}{2},\ ... \quad parity = -(-1)^{\omega-\frac{3}{2}} \\
\chi^-_{\omega jm} &= i\chi^{(-)}_{\omega jm} \qquad even\ n \\
&\quad - \chi^{(+)}_{\omega jm} \qquad odd\ n
\end{aligned}
\tag{47}
$$

The representations $\mathcal{D}(1,0)^+$ and $\mathcal{D}(1,0)^-$ can be related to the representations $\mathcal{D}(\frac{3}{2},\frac{1}{2})^+$ and $\mathcal{D}(\frac{3}{2},\frac{1}{2})^-$ by the supersymmetry transformation

$$
\begin{aligned}
\delta A &= \frac{1}{\sqrt{2}}\,\bar{\epsilon}\chi \qquad \delta B = \frac{i}{\sqrt{2}}\bar{\epsilon}\gamma_5\chi \\
\delta\chi &= -\frac{1}{\sqrt{2}}\,[i\gamma^\mu\partial_\mu(A+i\gamma_5 B) \ +\ a(A-i\gamma_5 B)]
\end{aligned}
\tag{48}
$$

For a general spin s, two unitary irreducible representations $\mathcal{D}(s+1,s)^\pm$, $\omega = n + l + s + 1$ can be formed, [12][13] and the sets $\{\mathcal{D}(1,0)^+, \mathcal{D}(\frac{3}{2},\frac{1}{2})^+, \mathcal{D}(2,1)^+,\ ...\}$ and $\{\mathcal{D}(1,0)^-, \mathcal{D}(\frac{3}{2},\frac{1}{2})^-, \mathcal{D}(2,1)^-,\ ...\}$ are separately invariant under supersymmetry transformations.

It has been indicated already that analytic continuation from H^4 or S^4 to anti-de Sitter space could be used to obtain an expression for the Green function. The lack of global hyperbolicity of anti-de Sitter space requires the use of reflective boundary conditions, which are included in the supersymmetric boundary conditions above, and their effect can be reproduced on H^4 by adding a second source on the lower sheet of the hyperboloid. The analyticity properties of the Green function in de Sitter space [14] or Euclidean de Sitter space, S^4, lead to constraints on the parity, which explains the discrepancy between the anti-de Sitter propagator associated with supersymmetric boundary conditions and the propagator continued from S^4.

(ii) Conformal mapping from the Einstein static universe

$$\begin{aligned} z_0 &= a\ sin\ t\ sec\ \rho \\ z_1 &= a\ tan\ \rho\ sin\ \theta\ cos\ \phi \\ z_2 &= a\ tan\ \rho\ sin\ \theta\ sin\ \phi \\ z_3 &= a\ tan\ \rho\ cos\ \theta \\ z_4 &= a\ cos\ t\ sec\ \rho \end{aligned} \tag{49}$$

$$ds^2 = \frac{a^2}{cos^2\rho}\left(dt^2 - d\rho^2 - sin^2\rho\ (d\theta^2 + sin^2 d\phi^2)\right) = \frac{a^2}{cos^2\rho} ds_E^2 \tag{50}$$

where ds_E^2 is the line element for the Einstein static universe $R^1 \times S^3$. The solution to the conformally invariant scalar field equation can be can be found by a conformal mapping of the solution in the Einstein static universe [15].

$$\begin{aligned} \phi^{AdS}_{\omega lm} &= (cos\rho)\phi^{E}_{\omega lm} = N_{\omega l} e^{-i\omega t}(sin\rho)^l cos\rho \\ & \qquad F(\frac{1}{2}(l+1-\omega), \frac{1}{2}(l+1+\omega); l+\frac{3}{2}; sin^2\rho) Y_{lm}(\theta,\phi) \\ \omega &= 2j+l+1,\ 2j+l+2,\ j = 0,1,2,... \end{aligned} \tag{51}$$

where $N_{\omega l}$ is a normalization factor. A second set of solutions to the conformally invariant wave equation would contain $(sin^2\rho)^{-l-\frac{1}{2}} F(\frac{1}{2}(\omega - l), -\frac{1}{2}(\omega + l); \frac{1}{2}; cos^2\rho)$. Finiteness of $(\phi_{\omega lm}, \phi_{\omega lm})$ at $\rho = 0$ excludes this solution.

The first set of solutions in equation (43) and the functions in equation (51) can be shown to be equivalent using the transformation $r \rightarrow a\ tan\ \rho$, $\frac{t}{a} \rightarrow t$ from globally static coordinates to those of the Einstein static universe and the identity $F(a,b;c;x) = (1-x)^{-b} F(b, c-a; c; \frac{x}{x-1})$. Solutions to the general Klein-Gordon equation will have the same form as the solutions in equation (51), except that $\omega = \omega_0 + 2j + l$ where ω_0 depends on the mass.

(iii) Conformally flat coordinates

$$\begin{aligned} z_0 - z_1 &= \rho' + \frac{1}{\rho'}(y^2 + z^2 - t'^2) \\ z_0 + z_1 &= \frac{1}{a^2\rho'} \\ z_2 &= \frac{y}{a\rho'} \\ z_3 - z_4 &= \frac{z + t'}{a\rho'} \\ z_3 + z_4 &= \frac{z - t'}{a\rho'} \end{aligned} \tag{52}$$

$$ds^2 = \frac{1}{a^2\rho'^2}\,[dt'^2 - d\rho'^2 - dy^2 - dz^2] \tag{53}$$

covers half of the space, with $\rho > 0$. The other half is covered by choosing $\rho < 0$. The Klein-Gordon equation

$$-a^2\rho'^2\frac{\partial^2\Phi}{\partial\rho'^2} + 2a^2\rho'\frac{\partial\Phi}{\partial\rho'} - a^2\rho'^2\left(\frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\Phi}{\partial z^2} - \frac{\partial^2\Phi}{\partial t'^2}\right) + m^2\Phi = 0 \tag{54}$$

has basis solutions

$$\begin{aligned} \phi_{k_{t'},k_y,k_z}(t',\rho',y,z) &= \frac{a}{2\sqrt{2}\pi}\sqrt{1 - \frac{(k_y^2+k_z^2)}{k_{t'}^2}}\; e^{-i(k_{t'}t' - k_y y - k_z z)} \\ &\quad \rho'^{\frac{3}{2}}\; J_{\sqrt{\frac{m^2}{a^2}+\frac{9}{4}}}\left(\sqrt{{k_{t'}}^2 - {k_y}^2 - {k_z}^2}\,\rho'\right) \end{aligned} \tag{55}$$

Note that $\phi_{k_{t'},k_y,k_z}(t',\rho',y,z)$ contains a factor representing a plane wave in the three dimensional subspaces spanned by the coordinates (t', y, z).

The mode solutions are associated with a choice of vacuum state for the scalar field, and it is of interest to determine whether the vacuum states are equivalent in the three coordinate systems for anti-de Sitter space. A mixing of positive and negative frequencies in the transformation between coordinate systems would change a vacuum state into one with non-zero particle number.

Since the time coordinate differs by a constant scaling factor for coordinate systems (i) and (ii), it follows trivially that the vacuum states in these coordinates are the same. To compare the vacuum states in coordinate systems (ii) and (iii), it is useful to expand the solutions in equation (55) in terms of the solutions in equation (51), with $\omega = \omega_0 + 2j + l$.

$$e^{-ik_{t'}t'}e^{ik_y y}e^{ik_z z}\rho'^{\frac{3}{2}}J_\nu(-ik\rho')$$
$$= \sum_{\omega lm} \alpha_{\substack{\omega lm \\ k_{t'}k_y k_z}} e^{-i\omega t} Y_{lm}(\theta\phi)(sin\ \rho)^l cos\ \rho f_{\omega l}(\rho) + \sum_{\omega lm} \beta_{\substack{\omega lm \\ k_{t'}k_y k_z}} e^{i\omega t} Y^*_{lm}(\theta,\phi)(sin\ \rho)^l cos\ \rho f_{\omega l}(\rho) \quad (56)$$

Since

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\omega' t}e^{-ik_{t'}t'}e^{ik_y y}e^{ik_z z}\rho'^{\frac{3}{2}}J_\nu(-ik\rho')dt$$
$$= \sum_{lm} \alpha_{\substack{\omega' lm \\ k_{t'}k_y k_z}} Y_{lm}(\theta\phi)(sin\ \rho)^l cos\ \rho f_{\omega' l}(\rho)\theta(\omega') + \sum_{lm} \beta_{\substack{-\omega' lm \\ k_{t'}k_y k_z}} Y^*_{lm}(\theta,\phi)(sin\ \rho)^l cos\ \rho f_{-\omega' l}(\rho)\theta(-\omega') \quad (57)$$

vanishing of the integral implies that $\beta_{\substack{-\omega' lm \\ k_{t'}k_y k_z}} = 0$ for all ω', l, m because the functions $Y^*_{lm}(\theta\phi)(sin\ \rho)^l cos\ \rho f_{-\omega' l}(\rho)$ form a complete basis in three dimensions. This method therefore requires evaluation of the integral

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} dt\ e^{i\omega' arc\ tan\ [\frac{a}{2t'}[\frac{1}{a^2}+\rho'^2+y^2+z^2-t'^2]]}e^{-ik_{t'}t'}e^{ik_y y}e^{ik_z z}\rho'^{\frac{3}{2}}J_\nu(-ik\rho') \quad (58)$$

Equivalence of the vacua in coordinates of the Einstein static universe (t,ρ,θ,ϕ) and the conformally flat coordinates (t',ρ',y,z) can be demonstrated more easily by showing that the domain of holomorphicity of $e^{-i\omega t}$ is mapped into the domain of holomorphicity of $e^{-i\omega t'}$. Let $t = \tau + i\sigma,\ -\pi \le \tau < \pi,\ 0 \le \sigma < \infty$. The time coordinate t' is given by

$$t' = -\frac{cos(\tau + i\sigma)sec\ \rho}{a[sin(\tau + i\sigma)sec\ \rho + tan\ \rho\ sin\ \theta\ cos\ \phi]} \quad (59)$$

When

$$I.\ \ \sigma = 0 \quad (-\pi,\pi) \to (-\infty,\infty)$$
$$II.\ \ \sigma > 0$$
$$Im\ t' = \frac{1}{a}\frac{sinh\ \sigma(cosh\ \sigma\ sec^2\rho + sin\ \tau\ sec\ \rho\ tan\ \rho\ sin\ \theta\ cos\ \phi)}{(sin\ \tau\ cosh\ \sigma\ sec\ \rho + tan\ \rho\ sin\ \theta\ cos\ \phi)^2 + cos^2\tau\ sinh^2\sigma\ sec^2\rho} > 0$$
$$as\ 0 \le \rho \le \frac{\pi}{2},\ 0 \le \theta \le \pi,\ 0 \le \phi \le 2\pi$$
$$III.\ \ \sigma < 0 \quad Im\ t' < 0 \quad (60)$$

From equation (60), it follows that the domains of holomorphicity $e^{-i\omega t}$ and $e^{-i\omega t'}$ are mapped into each other and there is no mixing of positive and negative frequencies in the conformally flat coordinates. Thus, the vacuum state for the scalar field in all three coordinate systems will be equivalent.

Configuration and momentum space Feynman rules can again be formulated in these coordinates, and they closely resemble the flat-space Feynman rules as the eigenfunctions of the Laplacian contain three-dimensional plane waves, which give rise to delta functions representing conservation of momentum in the three dimensions. The factor $\prod_{vert.} f^{\{k\}}(vert.)$ is now determined by the integral of the product of Bessel functions. Identities such as

$$\int_0^\infty d\rho\ \rho^{r-1}\ J_\nu(a\rho)\ J_\nu(b\rho)\ J_\nu(c\rho)\ =\ \frac{2^{r-1}\ (ab)^\nu\ c^{-2\nu-r}\ \Gamma\left(\frac{3\nu+r}{2}\right)}{(\Gamma(\nu+1))^2\ \Gamma\left(1-\frac{\nu+r}{2}\right)} \cdot\ F_4\left(\frac{\nu+r}{2},\frac{3\nu+r}{2};\nu+1,\nu+1;\frac{a^2}{c^2};\frac{b^2}{c^2}\right)$$

$$Re\ (3\nu+r)\ >\ 0 \quad Re\ r\ <\ \frac{5}{2} \quad a\ >\ 0,\ \ b\ >\ 0,\ \ c\ >\ 0,\ \ c\ >\ a+b$$

$$\int_0^\infty d\rho\ \rho^{r-1}\ J_r(cx)\ \prod_{i=1}^n\ \rho_i^{-\mu_i}\ J_{\mu_i}(a_i\rho_i)\ =\ 2^{r-1}\ \Gamma(r)\ c^{-r}\ \prod_{i=1}^n\ [\ b_i^{-\mu_i}\ J_{\mu_i}(a_i b_i)]$$

$$\rho_i\ =\ \sqrt{\rho^2+b_i^2} \quad a_i\ =\ 0 \quad Re\ b_i\ >0 \quad \sum_{i=1}^n\ a_i\ <\ c$$

$$Re\ \left(\frac{1}{2}n\ +\ \sum_{i=1}^n \mu_i\ +\ \frac{3}{2}\right)\ >\ Re\ r\ >\ 0 \tag{61}$$

reveal that the Bessel function integrals can be evaluated, although they no longer contain the delta functions associated with the orthogonality relations for two Bessel functions [16]. Thus, the vertex factor $\prod_{vert.} f^{\{k\}}(vert.)$ will give rise to extra non-trivial functions of the momenta which must be included in the momentum-space integrals.

9. Lehmann Representation for the Propagator in the Presence of a Second Source in H^4

For an interacting scalar field, the two-point function can be expressed in terms of the free-field Green functions

$$\begin{aligned} \langle 0|T\phi(x)\phi(x')|0\rangle &= \sum_{\omega=\omega_{0\pm}}^{\infty} \rho(\omega,\phi) G_\omega(x,x') \\ \omega &= \omega_{0\pm}, \omega_{0\pm}+1, \ldots \end{aligned} \tag{62}$$

where $\omega_{0\pm}$ are the lowest eigenvalues of J_{04}, the energy operator, and

$$\begin{aligned} J_{AB} &= \int d^3x \, \sqrt{-g} \, T^0_\nu K^\nu_{AB} \\ K_{AB} &= y_A \frac{\partial}{\partial y^B} - y_B \frac{\partial}{\partial y^A} \\ \eta_{AB} y^A y^B &= \frac{1}{a^2} \end{aligned} \tag{63}$$

The Green function $G_{\omega\pm}(x,x')$ satisfies the equation

$$\begin{aligned} (\nabla^\mu \nabla_\mu + m^2) G_{\omega_{0\pm}}(x,x') &= \frac{1}{\sqrt{-g}} \delta(x,x') \\ \omega_{0\pm} &= \frac{3}{2} \pm \sqrt{\frac{9}{4} + \frac{m^2}{a^2}} \end{aligned} \tag{64}$$

The curved-space delta function $\tilde{\delta}(x,x') = \frac{1}{\sqrt{-g}} \delta(x,x')$ is defined so that $\int_X d^4x \sqrt{-g} \, \tilde{\delta}(x,x') = 1 = \int d^4x \sqrt{-g} \, \frac{1}{\sqrt{-g}} \delta(x,x')$. The Lehmann spectral representation for the two-point function can be deduced from the Fourier transformation of the analytically continued Green function in H^4. Reflective boundary conditions may be realized in H^4 by adding a second source for the Green function at the anti-podal point $[x^A \to -x^A]$ on the second sheet of the hyperboloid.

$$(\nabla^\mu \nabla_\mu + m^2) \, G^{E.R.B.}_{\omega_{0\pm}}(x,x') = \frac{1}{\sqrt{g}} [\delta(x,x') \pm \delta(x,\hat{x}')] \tag{65}$$

The solution to this equation is

$$G^{E.R.B.}_{\omega_{0\pm}}(x,x') = G^E_{\omega_{0\pm}}(x,x') \pm G^E_{\omega_{0\pm}}(x,\hat{x}') \tag{66}$$

For a conformally coupled massless scalar field with $m^2 = -2a^2$, $\omega_{0+} = 2$ and $\omega_{0-} = 1$, the Green function satisfying reflective boundary conditions is

$$G^{E.R.B.}_{\omega 0\pm}(u_E) = \frac{1}{8\pi^2}\left[\frac{1}{u_E} \pm \frac{1}{2-u_E}\right] \tag{67}$$

and

$$G^{E.R.B.}_{\omega 0+}(x,x') \approx -\frac{1}{4\pi^2}\frac{1}{u_E^2} \qquad G^{E.R.B.}_{\omega 0-}(x,x') \approx \frac{1}{4\pi^2}\frac{1}{u_E} \tag{68}$$

as $u_E = \frac{1}{2}a^2(x^A - x'^A)^2 \to \infty$. Defining z to be $x^A x'_A$, the Fourier representation [4] is

$$G^E_\omega(z) = \frac{i}{2}\int_{-\frac{1}{2}+i\infty}^{\frac{1}{2}+i\infty} d\sigma\ (\sigma+1)\sigma(\sigma-1)\ ctg\ \pi\sigma\ \tilde{G}^E_\omega(\sigma)\frac{\mathcal{P}^{-1}_\sigma(z)}{(z^2-1)^{\frac{1}{2}}} \tag{69}$$

where

$$\tilde{G}^E_\omega(\sigma) = \int_1^\infty dz\ G^E_\lambda(z)(z^2 - 1)^{\frac{1}{2}}\mathcal{P}^{-1}_\sigma(z) \tag{70}$$

and $\frac{\mathcal{P}^{-1}_\sigma(z)}{(z^2-1)^{\frac{1}{2}}}$ being the eigenfunction of $\nabla^\mu\nabla_\mu$ with eigenvalue $-(\sigma(\sigma+1) - 2)$. The integral (70) represents integration over the upper sheet of the hyperboloid. The other possible range of the argument of the Green function corresponds to integration over the second sheet, with $u_E \geq 2$, and $\hat{z} = x^A\ \hat{x}'_A$ ranging from -1 to $-\infty$. It follows that

$$\begin{aligned} G^{E.R.B.}_{\omega 0\pm}(z) &= G^E_{\omega 0\pm}(x,x') \pm G^E_{\omega 0\pm}(x,\hat{x}') \\ &= \frac{i}{2}\int_{-\frac{1}{2}-i\infty}^{-\frac{1}{2}+i\infty} d\sigma\ (\sigma+1)\sigma(\sigma-1)\ ctg\ \pi\sigma\tilde{G}^E_{\omega 0\pm}(\sigma)\frac{\mathcal{P}^{-1}_\sigma(z)}{(z^2-1)^{\frac{1}{2}}} \\ &\pm \frac{i}{2}\int_{-\frac{1}{2}\ -\ i\infty}^{-\frac{1}{2}\ +\ i\infty} d\sigma\ (\sigma+1)\sigma(\sigma-1)\ ctg\ \pi\sigma\ \hat{G}^E_{\omega 0\pm}(\sigma)\frac{\mathcal{P}^{-1}_\sigma(\hat{z})}{(\hat{z}^2-1)^{\frac{1}{2}}} \end{aligned} \tag{71}$$

where

$$\hat{G}^E_{\lambda\pm}(\sigma) = \int_{-1}^{-\infty} d\hat{z}\ G^E_{\omega 0\pm}(\hat{z})(\hat{z}^2-1)^{\frac{1}{2}}\ \mathcal{P}^{-1}_\sigma(\hat{z}) \tag{72}$$

Applying the Klein-Gordon operator to the Green function (71) and requiring that it satisfy the equation (65) implies that

$$\hat{G}^E_{\omega 0\pm}(\sigma) = \frac{1}{\omega_{0\pm}(\omega_{0\pm}-3) - \sigma(\sigma+1) + 2} \tag{73}$$

and thus the propagator in transform space is essentially unchanged from the one given in equation (23).

10. Curvature, Shifts in the Momentum and Ground State Contributions to the String Hamiltonian

The definition of momentum might also be considered within the context of the effect of curvature on the commutation relations for position and momentum operators [17]. If the momentum operators are defined so that they induce the change in geodesic coordinates from x_Q^μ to $x_{Q'}^\mu$, then

$$x_Q^\mu \ \rightarrow \ x_{Q'}^\mu \ = \ x_Q^\mu \ - \ \frac{1}{i\hbar}\alpha^\nu [x_Q^\mu, p_{Q\nu}] \tag{74}$$

Assuming that a local coordinate system has been chosen so that $\Gamma^\mu_{\alpha\beta}(Q) \ = \ 0$, derivatives of the geodesic equation imply that

$$\begin{aligned} x_{Q'}^\mu \ &= \ x_Q^\mu \ - \ \alpha^\mu \ + \ \frac{1}{2}\Gamma^\mu_{\alpha\beta,\nu}(Q) \ \alpha^\nu x_Q^\alpha x_Q^\beta \ + \ \mathcal{O}(\alpha^2, x_Q^3) \\ &= \ x_Q^\mu \ - \ \alpha^\mu \ - \ \frac{1}{6}(R^\mu{}_{\alpha\beta\nu}(Q) \ + \ R^\mu{}_{\beta\alpha\nu}(Q)) \ \alpha^\nu x_Q^\alpha x_Q^\beta \end{aligned} \tag{75}$$

The momentum operator which generates such a translation of the coordinate x^ρ is

$$-i\frac{\partial}{\partial x^\nu} \ - \ \frac{i}{6}\,(R^\rho{}_{\alpha\beta\nu} \ + \ R^\rho{}_{\beta\alpha\nu})\, x^\alpha x^\beta \frac{\partial}{\partial x^\rho} \tag{76}$$

which gives

$$k_\nu \ + \ \frac{1}{6}\,[R^\rho{}_{\alpha\beta\nu} \ + \ R^\rho{}_{\beta\alpha\nu}] \ x^\alpha x^\beta k_\rho \tag{77}$$

after applying the operator to the plane wave $e^{ik\cdot x}$, whose use would be justified by the approximate flatness of the space-time outside the local region with curvature. The result depends on the choice of coordinate x^ρ and the index ρ is not summed even though it occurs more than once in the formula (77). To obtain a directionally-averaged definition of the momentum, one may sum over ρ and divide by four to obtain the following result

$$k'_\mu \ = \ k_\mu \ + \ \frac{1}{24} \ \sum_\rho \ [R^\rho{}_{\alpha\beta\mu} \ + \ R^\rho{}_{\beta\alpha\mu}]\ x^\alpha x^\beta k_\rho \tag{78}$$

For a constant curvature metric, such as the one that is used for anti-de Sitter space, the identity

$$R_{\rho\alpha\beta\nu} \ + \ R_{\rho\beta\alpha\nu} \ = \ -a^2(g_{\rho\beta} \ g_{\alpha\nu} \ + \ g_{\rho\alpha}g_{\beta\nu} \ - \ 2g_{\rho\nu}g_{\alpha\beta}) \tag{79}$$

is valid.

$$\begin{aligned}
k'^2 \;&=\; k^2 \;-\; \frac{a^2}{12}\,(g_{\rho\beta}g_{\alpha\nu} \;+\; g_{\rho\alpha}g_{\beta\nu} \;-\; 2g_{\rho\nu}g_{\alpha\beta})k^\rho k^\nu x^\alpha x^\beta \\
&\quad +\; \frac{a^4}{576}(g_{\rho\beta}g_{\alpha\nu} \;+\; g_{\rho\alpha}g_{\beta\nu} \;-\; 2g_{\rho\nu}g_{\alpha\beta})(g_{\sigma\delta}\delta^\nu_\gamma \;+\; g_{\sigma\gamma}\delta^\nu_\delta \;-\; 2\delta^\nu_\sigma g_{\gamma\delta})x^\alpha x^\beta x^\gamma x^\delta k^\rho k^\sigma \\
\langle k'^2\rangle \;&=\; k^2\;\left[\,1 \;+\; \frac{a^2}{8}\langle x^2\rangle \;+\; \frac{a^4}{192}\;\langle x^4\rangle\,\right]
\end{aligned} \tag{80}$$

If the index ρ in equation (76) is summed, then the shift in (75) is not altered, but each directional derivative would then be contributing to the shifted momentum, in contrast to the coordinate translation, and this could represent an overcounting of the terms containing the curvature tensor. With the summation over the index ρ included, the shifted momentum, denoted by k''_μ has an average squared value of $\langle k''^2\rangle \;=\; k^2\;\left[1 \;+\; \frac{a^2}{2}\langle x^2\rangle \;+\; \frac{a^4}{12}\langle x^4\rangle\right]$ Using either definition of the shifted momentum, the change is directly related to the introduction of curvature in the manifold.

The momentum operator acts differently on wave functions by shifting the argument, so that, for example, $e^{-i\alpha\cdot P}\psi(x) \;=\; \psi(x-\alpha)$. This property would be shared by the fields X^μ representing the coordinates of a string moving in a target space. Choosing a fixed base point X^μ_0 so that $\Delta X^\mu \;=\; X^\mu \;-\; X^\mu_0$, one may define ΔX^A by $\Delta X^\mu \;=\; \frac{\partial x^\mu}{\partial X^A}\Delta X^A$, where X^A represent standard embedding coordinates for the local anti-de Sitter geometry. The action of the momentum operator K is given by

$$(1 - i\alpha\cdot K)\cdot\Delta X^A \;=\; \Delta X^A \;-\; \alpha^A \;-\; \frac{1}{6}(R^A{}_{BCD} \;+\; R^A{}_{CBD})\alpha^D\Delta X^B\Delta X^C \tag{81}$$

so that

$$K'_D \;=\; K_D \;-\; \frac{i}{6}(R^A{}_{BCD} \;+\; R^A{}_{CBD})\Delta X^B\Delta X^C\frac{1}{\Delta X^A} \tag{82}$$

where there is no sum over the index A. However, when squaring the momentum, a special method of summing over the first index of the curvature tensor and the index associated with $\frac{1}{\Delta X^A}$ will be used, and the sum shall be divided by 4, to average over all of the directions.

$$K'_D K'^D = K_D K^D - \frac{i}{12}\sum_A (R^A{}_{BCD} + R^A{}_{CBD})\Delta X^B \Delta X^C \frac{1}{\Delta X^A} K^D$$
$$- \frac{1}{4}\cdot\frac{1}{36}\sum_A (R^A{}_{BCD} + R^A{}_{CBD})\Delta X^B \Delta X^C \frac{1}{\Delta X^A} \tag{83}$$
$$\cdot (R_{AEF}{}^D + R_{AFE}{}^D)\Delta X^E \Delta X^F \frac{1}{\Delta X_A}$$

with an implied summation over the indices A, or equivalently,

$$K'^2 = K^2 - \frac{a^4}{144}\sum_A \Big[(\eta^A{}_C\eta_{BD} + \eta^A{}_B\eta_{CD} - 2\eta^A{}_D\eta_{CB})$$
$$\cdot (\eta_{AF}\eta_E{}^D + \eta_{AE}\eta_F{}^D - 2\eta_A{}^D\eta_{EF})$$
$$\Delta X^B \Delta X^C \Delta X^E \Delta X^F \cdot \frac{1}{\Delta X^A}\frac{1}{\Delta X_A}\Big]$$
$$= K^2 - \frac{a^4}{144}\sum_A (2\Delta X^A \Delta X_D - 2\eta^A{}_D\Delta_B \Delta X^B)(2\Delta X_A \Delta X^D - 2\eta_A{}^D \Delta X_E \Delta X^E)$$
$$\cdot \frac{1}{\Delta X^A}\frac{1}{\Delta X_A} \tag{84}$$

with $\eta_{AD} = g_{\mu\nu}\frac{\partial x^\mu}{\partial X^A}\frac{\partial x^\nu}{\partial X^D}$, since

$$\frac{1}{4}\sum_A \frac{i}{3}a^2(\eta_{AC}\eta_{BD} + \eta_{AB}\eta_{CD} - 2\eta_{AD}\eta_{BC})\Delta X^B \Delta X^C \frac{1}{\Delta X^A} K^D$$
$$= \frac{ia^2}{6}\sum_A \left[\Delta X_A \Delta X_D - \frac{\eta_{AD}}{a^2}\right]\frac{\Delta X^A}{\Delta X_E \Delta X^E} K^D = 0 \tag{85}$$

Using $\eta_{AB}\Delta X^A \Delta X^B = \frac{1}{a^2}$, it follows that

$$\sum_{A,D} \eta^A{}_D \Delta X^D \frac{1}{\Delta X^A} = \sum_{A,D} \delta^\lambda{}_\sigma \frac{\partial X^A}{\partial x^\lambda}\frac{\partial x^\sigma}{\partial X^D}\Delta X^D \frac{1}{\Delta X^A}$$
$$= \sum_A \delta^\lambda{}_\sigma \frac{\partial X^A}{\partial x^\lambda}\frac{1}{\Delta X^A}\Delta X^\sigma = 4$$
$$\sum_{A,B} \eta^A{}_D \frac{1}{\Delta X^A}\eta_A{}^D \frac{1}{\Delta X_A}\Delta X_B \Delta X^B = \sum_{A,B} \delta^\lambda{}_\sigma \delta_\tau{}^\rho \delta^\sigma{}_\rho \left(\frac{\partial X^A}{\partial x^\lambda}\frac{\partial x^\tau}{\partial X^A}\frac{1}{\Delta X^A}\frac{1}{\Delta X_A}\right) \tag{86}$$
$$\Delta X_B \Delta X^B$$
$$= \sum_{A,B} \frac{\partial X^A}{\partial x^\tau}\frac{\partial x^\tau}{\partial X^A}\frac{1}{\Delta X^A}\frac{1}{\Delta X_A}\Delta X_B \Delta X^B$$

The average value of the last sum in (86) is 16, so that the shift in the squared momentum is $-\frac{9}{36}a^2$. Viewing the coordinate fields as a collection of scalar fields, the zero-point of the squared-mass scale would be $-\frac{9}{4}a^2$. Relative to this scale, the squared mass is shifted by $\frac{9}{4}a^2$, so that $m'^2 = m^2 + \frac{9}{4}a^2$ and

$$K'^2 + m'^2 = K^2 + m^2 + 2a^2 \tag{87}$$

This suggests a connection with the bosonic string. The Hamiltonian for the closed bosonic string [17] is given by

$$H = \frac{1}{2} \sum_n [\alpha_{-n} \cdot \alpha_n + \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n] \tag{88}$$

where α_n^μ and $\tilde{\alpha}_n^\mu$ are the coefficients in the expansion of the target space coordinates X^μ that satifsy standard operator commutation relations. After imposing the operator condition $L_0 = \tilde{L}_0$ and normal ordering, it follows that the sum over transverse string oscillators can be assigned the value

$$\sum_{\mu=1}^{D-2} \sum_n : \alpha_{-n}^\mu \alpha_{n\mu} : + (D-2) \sum_{n=1}^{\infty} n \tag{89}$$

Upon use of the vanishing of the vacuum expectation value of the operator product in the first sum, zeta-function regularization of the second sum gives a ground state contribution of $-\frac{24}{12}a^2$, after multiplying by the factor a^2, which is dimensionally and numerically consistent with equation (87). Since the energy-momentum of the string might be expected to curve the space-time through which it propagates, it would be useful to establish the relation between scattering in a local region of constant curvature and string scattering. The coincidence might be explained by viewing the collection of bosonic string coordinates $\{X^\mu\}$ as part of a string field $\Phi(\{X^\mu\})$. Thus, although it is conceivable that the scattering of component fields should be considered in a curved local geometry, the ground state contribution to the string Hamiltonian appears to be compensated by the shift in $K^2 + m^2$, indicating that the field-theoretic results might be included already in the entire string scattering calculations in flat space.

A consideration of the various possiblities for the number of dimensions shows that this coincidence between the shift in $K^2 + m^2$ and the magnitude of the residual contribution to the string Hamiltonian arising after normal ordering can only be achieved in four dimensions. The momentum shift can also be calculated in the space of positive constant curvature, de Sitter space. After replacing a^2 by $-a^2$ in equations (79), (80) and (84), it can be seen that the shift in the Hamiltonian can be represented as $K''^2 + m''^2 \;=\; K^2 + m^2 - 2a^2$. The shift of $-2a^2$ has also been obtained by regularizing the sum over transverse string oscillators in (89). Given this alternative representation of the string Hamiltonian, it might be conjectured that the target space geometry would be locally altered to de Sitter space. Scattering amplitudes in geometries which are locally de Sitter space and globally flat have not yet been computed, although a conformal transformation of the local target space metric to flat space could be used. Two possible models of string propagation in the interaction region therefore exist. First, the energy-momentum associated with the string might curve the background geometry so that it is locally anti-de Sitter space and then the expression for the string Hamiltonian receives the extra contribution (87) which would cancel the normal-ordering effect in (89), consistent with the conventional formulation of scattering amplitudes in flat space. The other alternative is that the overall effect of the string energy-momentum is a momentum shift which reveals a geometry with a locally de Sitter metric.

The latter possibility probably can be eliminated because supersymmetry requires a flat or anti-de Sitter background, whereas theories with de Sitter supersymmetry contain negative-norm states [6]. This is confirmed by a study of string theories in curved space-times, based on coset conformal field theories [18]. Although the bosonic and superstring theories are typically viewed as independent, they can be combined in the heterotic string theory, and since this theory is connected by duality to the other string theories, this indicates that the first description of string propagation in a globally flat space-time is more appropriate.

A calculation of more physical relevance is the scattering of the superstring in ten dimensions. For type II theories, the Hamiltonian is

$$H \;=\; \frac{1}{2\pi p^+} \int_0^\pi d\sigma \; [\pi^2 (P_\tau^i)^2 \;+\; (X^{i\prime})^2 \;-\; iS^1 S^{1\prime} \;+\; iS^2 S^{2\prime}] \tag{90}$$

where S^{1a} and S^{2a} are one-component Majorana-Weyl world-sheet spinors describing right-moving and left-moving degrees of freedom [19]. Expanding

$$\begin{aligned}
X^\mu &= x^\mu + p^\mu \tau + i \sum_{n\neq 0} \frac{1}{n} \alpha_n^\mu \, e^{-in\tau} \, cos \; n\sigma \\
\psi_+^\mu &= \sum_r b_r^\mu \, e^{-2ir(\tau-\sigma)} \qquad (NS) \\
\psi_-^\mu &= \sum_r \tilde{b}_r^\mu \, e^{-2ir(\tau+\sigma)} \qquad (NS) \\
\psi_+^\mu &= \sum_n d_n^\mu \, e^{-2in(\tau-\sigma)} \qquad (R) \\
\psi_-^\mu &= \sum_n \tilde{d}_n^\mu \, e^{-2in(\tau+\sigma)} \qquad (R)
\end{aligned} \tag{91}$$

where

$$\begin{aligned}
[\alpha_m^\mu, \alpha_n^\nu] &= [\bar{\alpha}_m^\mu, \bar{\alpha}_n^\mu] = 0 \\
\{b_r^\mu, \; b_s^\nu\} &= \eta^{\mu\nu} \, \delta_{r,-s} \qquad (NS) \\
\{d_n^\mu, \; d_m^\nu\} &= \eta^{\mu\nu} \delta_{n,-m} \qquad (R)
\end{aligned} \tag{92}$$

The Virasoro operators are

$$\begin{aligned}
L_m &= L_m^{(\alpha)} + L_m^{(b)} \qquad (NS) \\
L_m &= L_m^{(\alpha)} + L_m^{(d)} \qquad (R)
\end{aligned} \tag{93}$$

so that

$$\begin{aligned}
L_0^{(\alpha)} &= \frac{1}{2} \sum_{n=-\infty}^{\infty} : \alpha_{-n} \cdot \alpha_n : \\
L_0^{(b)} &= \frac{1}{2} \sum_r r \; : b_{-r} \cdot b_r : \qquad half-integrally \; moded \\
L_0^{(d)} &= \frac{1}{2} \sum_n n \; : d_{-n} \cdot d_n : \qquad integrally \; moded
\end{aligned} \tag{94}$$

The normal ordering constant from the physical bosonic coordinate is $\frac{1}{24}$ $[\epsilon_B^+ = -\frac{1}{24}]$, while the normal ordering constant from a half-integrally moded fermionic coordinate is $\frac{1}{48}$ $[\epsilon_F^- = -\frac{1}{48}]$ and from an integrally moded fermionic coordinate is $-\frac{1}{24}$, $[\epsilon_F^+ = \frac{1}{24}]$, so that the bosonic contribution to the Hamiltonian is $-\frac{8}{12}$. Now consider superstring scattering in a background where four of the dimensions locally represent anti-de Sitter space and six dimensions are compactified with radius of curvature significantly greater than the dimensions of the string. The dominant contribution to the shift in the momentum arises from four-dimensional anti-de Sitter space. Viewing $\Delta X^\mu = \frac{\partial x^\mu}{\partial X^A}\Delta X^A$ as a vector field on this space, and noting that spin-s fields satisfy the equation

$$[C_2 + [\omega_0(\omega_0 - 3) + s(s+1)]a^2]\chi_s = 0 \tag{95}$$

where $C_2 = \frac{1}{2}J^{AB}J_{AB}$ is the second Casimir invariant, it follows that the squared mass is shifted by $\frac{1}{4}a^2$ for each component of the vector field ΔX^ν. Multiplying this result by 4, associated with the four dimensions, gives $2a^2$. The shift in the vector field ΔX^ν is given by a Lie derivative in the direction of the momentum field, or equivalently the commutator of the two fields. This has already been anticipated in equation (74). From equation (80), and conversion of the product of k^μ and x^μ, the position coordinate also denoted by ΔX^μ, through the replacement of the flat-space momentum operator by $-i\frac{\partial}{\partial x^\mu}$, it can be shown the average value of the squared momentum becomes $\langle k'^2\rangle = k^2 - \frac{a^2}{8}\langle \frac{1}{x^3}\frac{d}{dx}x^3\frac{d}{dx}(x^2)\rangle - \frac{a^4}{192}\langle \frac{1}{x^3}\frac{d}{dx}x^3\frac{d}{dx}(x^4)\rangle = k^2 - a^2 - \frac{a^4}{8}\langle x^2\rangle$. The distribution of measurements of the variable $x = (x_\mu x^\mu)^{\frac{1}{2}}$ will be a normal distribution $N(\mu, \sigma^2)$. Given the dimensions of the anti-de Sitter geometry, the mean value of x can be set equal to $\frac{1}{a}$, while the variance may be chosen initially so that the expectation value of x^2 produces the required shift in the squared momentum, $k'^2 = k^2 - \frac{4}{3}a^2$. This can be achieved with an expectation value $\langle x^2\rangle = \frac{8}{3a^2}$ and variance $\sigma^2 = \frac{5}{3a^2}$. The underlying reasons for the occurrence of the normal distribution $N(\frac{1}{a}, \frac{5}{3a^2})$ have yet to be determined, although the number of variables x^μ and any correlations between these coordinates would affect the variance $\sum_\mu Var(x^\mu) + 2 \sum_{\mu<\nu} Cov(x^\mu, x^\nu)$, as the covariance $Cov(x^\mu, x^\nu)$ measures the correlation between x^μ and x^ν. Given this momentum shift, that the $K^2 + m^2$ is shifted by $\frac{2}{3}a^2$, compensating the bosonic ground state contribution to the Hamiltonian calculated earlier. The fermionic contribution can be determined in the identical manner using the normal ordering of the corresponding oscillators. The difference in the interpretation of the coordinate fields $\{\Delta X^\mu\}$ of the bosonic string and superstring might be traced to the inclusion of the fermions, which necessarily must combine to produce a worldsheet vector field representing the projection of a target-space vector field mediating the interactions.

Although the definition mentioned above is motivated by physical considerations such as position-momentum commutation relations, it is not necessarily the only one to use in a quantum theory. Other possibilities include the transform space variables such as λ in equation (23), the flat-space version of momentum used in the adiabatic expansion of Green functions [20][21][22], and even the generator $aJ_{\mu 4},\ \mu\ =\ 0,1,2,3$, which does not form a commutative subalgebra but does have the property that it is conserved along the particle's worldline. The last expression for the momentum is

$$P_\mu\ =\ p_\mu\ -\ \frac{ia}{m}\,p^\nu\ J_{\nu\mu} \tag{96}$$

for a particle of mass m [22], and the square is

$$\begin{aligned} P_\mu\ P^\mu\ &=\ p_\mu\ p^\mu\ -\ 2\frac{ia}{m}\,p^\mu\ p^\nu\ J_{\nu\mu}\ -\ \frac{a^2}{m^2}\,p^\nu\ p^\sigma\ J_{\nu\mu}\ J_\sigma{}^\mu \\ &=\ p_\mu\ p^\mu\ -\ \frac{a^2}{m^2}\,p^\nu\ p^\sigma\ J_{\nu\mu}\ J_\sigma{}^\mu \end{aligned} \tag{97}$$

where p_μ is a flat-version of the momentum defined to be $m\tilde{x}_\mu$, with the four-vectors $\tilde{x}_\mu$ satisfying $\tilde{x}_0\ >\ 0$ and $\tilde{x}^\mu\ \tilde{x}_\mu\ =\ 1$ [23]. In contrast to the other definitions of momentum, these quantities do not represent dual space variables with respect to position coordinates and therefore cannot be used directly in the calculation of loop diagrams and renormalized energy-momentum tensors. From the calculations of the shifts in the squared momentum and squared mass in anti-de Sitter space, relations between the different definitions of the momenta have been deduced. It would be of interest to determine whether any of these connections are maintained for general curved spaces. The conclusions may be relevant for recent work in curved space quantum field theory regarding the improved calculation of propagators [24] for fields of spin 0, $\frac{1}{2}$ and 1, which may be useful for determining particle-production rates, and localized renormalization theory [25].

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Appendix

It is possible to find the analogue of hyperplanes in homogeneous spaces $X = G/H$, for which there exists an isometry reversing geodesics, or equivalently a translation-invariant Riemann tensor [5].

Any semi-simple Lie group may be decomposed into a compact and non-compact part. The Lie algebra may be written as a direct sum of the vector spaces associated with the compact and non-compact generators: $\mathcal{G} = \mathcal{H} \oplus \mathcal{P}$, the Cartan decomposition. Furthermore, there exists an isometry Θ which maps $\mathcal{H}$ into $\mathcal{H}$ and $\mathcal{P}$ into $-\mathcal{P}$. The classification of locally symmetric spaces can then be reduced to the listing of all compact Lie algebras together with the isometry Θ.

A Cartan symmetric space is of the form $X = G/H$ where H is the maximal compact subgroup. There exists another decomposition, known as the Iwasawa decomposition, which is more useful for our purposes. Let $u = \mathcal{H} + \mathcal{P}_0$ where $\mathcal{P}_0 = i\mathcal{P}$ is a compact form of $\mathcal{G}$, and let $\mathcal{A}$ be a maximal abelian subalgebra of $\mathcal{P}_0$. Consider the subspace

$$\mathcal{G}_\alpha = \{\mathbf{g} \in \mathcal{G} | [\mathbf{g}, \mathbf{a}] = \alpha(\mathbf{a})\mathbf{g} \ for\ all\ \mathbf{a} \in \mathcal{A}\} \tag{98}$$

where $\alpha(\mathbf{a})$ is called the restricted root. Thus, $\mathcal{G} = \sum_\alpha \mathcal{G}_\alpha + \mathcal{G}_0$ and $\mathcal{A} \subset \mathcal{G}_0$. The subset of generators $\{\mathbf{a}\}$ in $\mathcal{A}$ for which all the restricted roots are non-zero has a connected component $\mathcal{A}^+$, the Weyl chamber.

A root α is positive if $\alpha(\mathbf{a}) > 0$ for all $\mathbf{a} \in \mathcal{A}^+$. Now let $\mathcal{N} = \sum_{\alpha>0} \mathcal{G}_\alpha$. Then, the Iwasawa decomposition of the Lie algebra $\mathcal{G}$ is

$$\mathcal{G} = \mathcal{H} \oplus \mathcal{A} \oplus \mathcal{N} \tag{99}$$

and exponentiating this expression, $g = han$, $g \in G$, $h \in H$, $a \in A$ and $n \in N$. The nilpotency of N as subgroup of G follows from the existence of n such that $N^{(n)} = 0$, with the definition

$$N^{(0)} = N, \quad N^{(1)} = [N, N], \quad N^{(2)} = [N, [N, N]], ... \tag{100}$$

This decomposition can be used to define generalized plane waves or horocycles in X. Consider a point o, whose stability subroup is H, to be the origin of X. The simplest horocycle is the orbit $\xi_0 = N \cdot o$. All horocycles in X can be written as $\xi = gNg^{-1}\tilde{g} \cdot o$, $g, \tilde{g} \in G$. Since $\tilde{g}^{-1}g = han$,

$$\xi = \tilde{g}(\tilde{g}^{-1}g)N(\tilde{g}^{-1}g)^{-1} \cdot o = \tilde{g}hanNn^{-1}a^{-1}h^{-1} \cdot o = \tilde{g}haNa^{-1} \cdot o = \tilde{g}hN \cdot o \quad (101)$$

since $aNa^{-1} \subset N$, and therefore,

$$\xi = \tilde{g}h \cdot \xi_0 = [k(\tilde{g}h)][a(\tilde{g}h)][n(\tilde{g}h)]N \cdot o = [k(\tilde{g}h)][a(\tilde{g}h)]N \cdot o \equiv \tilde{h}\tilde{a} \cdot \xi_0 \quad (102)$$

Every horocycle may be written in the form $\xi = ha \cdot \xi_0$ where ξ_0 is the fundamental horocycle.

Since a is an element of a group, it can be expressed as $exp\ r$ where $r \in \mathcal{A}$ represents the distance from the origin to the horocycle ξ. However, a horocycle is not specified by a particular choice of h and a. In fact, if we define the centalizer of $\mathcal{A}$ in H by $M = \{h \in H | Ad(h)\mathbf{a} = \mathbf{a}\ for\ all\ \mathbf{a} \in \mathcal{A}\}$, then h and h' give the same horocycle if they belong to the same coset H/M. If $h = h'm$,

$$\xi = haN \cdot o = h'maN \cdot o = h'aNm \cdot o = h'a \cdot \xi_0 \quad (103)$$

since M stabilizes AN. Finally, defining the boundary of the symmetric space to be $B = H/M$, only one horocycle passes through $x \in X$ with normal $b \in B$.

Although this formalism cannot be adapted to anti-de Sitter space, SO(3,2)/SO(3,1), with a non-compact stability group, $H^4 = SO(4,1)/SO(4)$ is a Cartan symmetric space. Results obtained in H^4 may then be analytically continued back to anti-de Sitter space, by analogy with a Wick rotation from a Euclidean field theory to a Lorentzian field theory.

Since the Lie algebra so(p,q) is given by

$$\left\{ \begin{pmatrix} X_1 & X_2 \\ X_2^t & X_3 \end{pmatrix} \middle| X_1 = -X_1^t,\ X_3 = -X_3^t \right\} \quad (104)$$

Since $\mathcal{G} = so(4,1)$, $\mathcal{K} = so(4)$ and $so(4,1) = so(4) + \mathcal{P}$ and the maximum abelian subalgebra $\mathcal{A}$ in $\mathcal{P}$ is

$$\mathbb{R}\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{105}$$

from the commutation relations of $\mathcal{P}$ with an arbitrary element of the Lie algebra $\mathcal{G}$, it can be shown that the roots $\{\alpha\}$ assume the values 1 or -1. The decomposition of the so(4,1) is then

$$\begin{aligned} \mathcal{G} &= \mathcal{G}_{+1} + \mathcal{G}_{-1} + \mathcal{G}_0 \\ &= \mathbb{R}\begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \end{pmatrix} + \mathbb{R}\begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \end{pmatrix} + \mathbb{R}\begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 \end{pmatrix} \\ &+ \mathbb{R}\begin{pmatrix} 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix} + \mathbb{R}\begin{pmatrix} 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} + \mathbb{R}\begin{pmatrix} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \\ &+ \mathbb{R}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \mathbb{R}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} + \mathbb{R}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \\ &+ \mathbb{R}\begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned} \tag{106}$$

The nilpotent subalgebra is $\mathcal{N} = \mathcal{G}_{+1}$ and

$$\begin{aligned} N_1^2 = N_2^2 = N_3^2 &= \begin{pmatrix} -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 \end{pmatrix} \\ N_1^3 = N_2^3 = N_3^3 &= 0 \\ N_1 N_2 = N_1 N_3 = N_2 N_3 &= 0 \end{aligned} \tag{107}$$

The general element of the nilpotent group is given by

$$\begin{aligned} exp(n_1N_1+n_2N_2+n_3N_3) \\ &= 1 + n_1N_1 + n_2N_2 + n_3N_3 + \frac{1}{2}(n_1^2N_1^2+n_2^2N_2^2+n_3^2N_3^2) \\ &= \begin{pmatrix} 1-\frac{1}{2}(n_1^2+n_2^2+n_3^2) & -n_1 & -n_2 & -n_3 & \frac{1}{2}(n_1^2+n_2^2+n_3^2) \\ n_1 & 1 & 0 & 0 & -n_1 \\ n_2 & 0 & 1 & 0 & -n_2 \\ n_3 & 0 & 0 & 1 & -n_3 \\ -\frac{1}{2}(n_1^2+n_2^2+n_3^2) & -n_1 & -n_2 & -n_3^2 & 1+\frac{1}{2}(n_1^2+n_2^2+n_3^2) \end{pmatrix} \end{aligned} \tag{108}$$

Since H rotates the first four coordinates, the point $o = (0,0,0,0,1)$ is chosen to be the origin and the fundamental horocycle is given by

$$N\cdot o = \begin{pmatrix} -\frac{1}{2}(n_1^2+n_2^2+n_3^2) \\ -n_1 \\ -n_2 \\ -n_3 \\ 1+\frac{1}{2}(n_1^2+n_2^2+n_3^2) \end{pmatrix} \qquad n_1,\ n_2,\ n_3 \in \mathbb{R} \tag{109}$$

To determine the distance from the origin to the horocycle passing through x with normal $b = hM$, it may be noted that if $x \in \xi = haN\cdot o$, then $an\cdot o = h^{-1}\cdot x$ for some $n \in N$. Using $\sum_{i=0}^{3}(h^{-1}\cdot x)_i^2 = \sum_{i=0}^{3} x_i^2$, it can be shown that the matrix equation

$$\begin{pmatrix} cosh\ r & 0 & 0 & 0 & sinh\ r \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ sinh\ r & 0 & 0 & 0 & cosh\ r \end{pmatrix} \begin{pmatrix} -\frac{1}{2}(n_1^2+n_2^2+n_3^2) \\ -n_1 \\ -n_2 \\ -n_3 \\ 1+\frac{1}{2}(n_1^2+n_2^2+n_3^2) \end{pmatrix} = \begin{pmatrix} (h^{-1}\cdot x)_0 \\ (h^{-1}\cdot x)_1 \\ (h^{-1}\cdot x)_2 \\ (h^{-1}\cdot x)_3 \\ x_4 \end{pmatrix} \tag{110}$$

has solutions $n_i = -(h^{-1}\cdot x)_i$ and

$$\begin{aligned} cosh\ r &= \frac{1}{2}\left[\frac{1}{x_4-(h^{-1}\cdot x)_0} + x_4-(h^{-1}\cdot x)_0\right] \\ r &= ln[x_4-(h^{-1}\cdot x)_0] = ln[h^{-1}\cdot x,\xi] = ln[x,h\xi] \qquad \xi = (1,0,0,0,1) \end{aligned} \tag{111}$$

This distance is independent of $(h^{-1}\cdot x)_i,\ i=1,2,3$ as the rotation of these three coordinates corresponds to the SO(3) subgroup M which stabilizes $\mathcal{A}$. Indeed, since $\mathcal{A} = J_{04}$, M is generated by $J_{12},\ J_{13},\ J_{23}$ since these are the only compact generators which commute with J_{04}. Consequently, the boundary is B=SO(4)/SO(3).

The group-theoretical definition of horocycles, or equivalently horospheres, agrees with the geometric one. The geometric definition is given as follows: consider a point x_0 in H^4 and draw all geodesics through it. A sphere of radius r with center at x_0 is the set of points which are a distance r from x_0 on the geodesics, $\{exp\ rX | X \in T_{x_0}(M)\}$. A horosphere is a sphere with center at infinity that still passes through a specified point in H^4.

Lobachevskii space can be identified with the set of lines through the origin with the metric

$$cosh\ r \ = \ \frac{[x,y]}{[x,x][y,y]} \qquad x,\ y\ are\ arbitrary\ points\ on\ two\ different\ lines \tag{112}$$

Instead of lines, a representative point can be chosen from each line, for example the intersection with $[x,x] = 1$. Then, $cosh\ r = [x,y]$. A point at infinity may be represented by a null vector on the cone, $-\xi_0^2 - \xi_1^2 - \xi_2^2 - \xi_3^2 + \xi_4^2 = 0$. Thus, the equation of the horosphere is $[x,\xi] = const.$ and by rescaling ξ, one obtains $[x,\xi] = 1$, defining a horosphere of the first kind. In imaginary Lobachevskii space (or de Sitter space in 4 dimensions), horospheres of both the first and second kind exist [26], where the latter type is given by the equation $[x,\xi] = 0$.

Let ξ be a particular null vector and a be a point on the hyperboloid. Then, it can be shown that $ln\ [a,\xi]$ equals the distance from a to the horosphere $[x,\xi] = 1$, defined to be the distance to the closest point on that horosphere. Since the scalar product is invariant under the action of the group,

$$\begin{aligned} [a,\xi] \ &= \ [ga,g\xi] \ = \ [a',\xi'] \ = \ t \\ a' \ &= \ (0,0,0,0,1) \qquad \xi' \ = \ (t,0,0,0,t) \end{aligned} \tag{113}$$

The distance from a to the horosphere $[x,\xi] = 1$ equals the distance from a' to the horosphere $[x,\xi'] = tx_0 + tx_4 = 1$. The distance from a' to $[x,\xi'] = 1$ is the radius of the sphere centered at a' and tangent to $[x,\xi'] = 1$, and every sphere is the intersection of the $[x,x] = 1$ hyperboloid with the hyperplane $x_4 = constant$. The sphere is tangent with $[x,\xi'] = 1$ at the point y, where y_4 assumes its minimum value. Since $y_4^2 - y_0^2 = 1 + y_1^2 + y_2^2 + y_3^2$, minimizing y_4 implies that $y_1 = y_2 = y_3 = 0$, and since $y_0 = -t^{-1} + y_4$, $y_{4min} = \frac{1}{2}(t + t^{-1})$. Again the distance is given by

$$\begin{aligned} cosh\ r &= [a', y] = \frac{1}{2}(t + t^{-1}) \\ r &= ln\ t = ln\ [a, \xi] \end{aligned} \tag{114}$$

Thus, choosing $\xi = (1,0,0,0,1)$, it follows that the distance from $h^{-1} \cdot x$ to the horocycle $[z, \xi] = 1$ equals $ln\ [h^{-1} \cdot x, \xi] = ln\ [x, h\xi]$, which is the distance from x to $[z, h\xi] = 1$. Since h does not change the last coordinate of ξ, $(0,0,0,0,1)$ is still a point on this horocycle.

The geometrical definition of $r(x, b)$ is therefore the distance from x to the horocycle passing through (0,0,0,0,1) with normal $b = hM$, whereas the group-theoretical definition of $r(x, b)$ is the distance from $(0,0,0,0,1)$ to the horocycle passing through x with normal b.

The equivalence between these two definitions is analogous to the equality in Euclidean space between the distance from the origin o to the hyperplane passing through x with normal w and the distance from x to the hyperplane through the origin $(z, w) = 0$ (Fig. 3).

The equality of the two distances for horocycles will now be shown. Let y be the point on the horocycle through $(0,0,0,0,1)$ closest to x and z be the point on the horocycle through x closest to $(0,0,0,0,1)$. The isometry which maps $(0,0,0,0,1)$ into z also maps the horocycles into each other. This means that there must be a point y' on the horocycle at the same distance from x as the distance between $(0,0,0,0,1)$ and z. The point y is obtained by drawing a geodesic intersecting $[x, \xi] = 1$ orthogonally. If the geodesic is continued to the boundary, it will reach the same boundary point as the geodesic from $(0,0,0,0,1)$ to z. As there is an isometry mapping the two geodesics into each other, while keeping the horocycles fixed, the distances must be the same. Essentially, the geodesics and the horocycles form an orthogonal coordinate system.

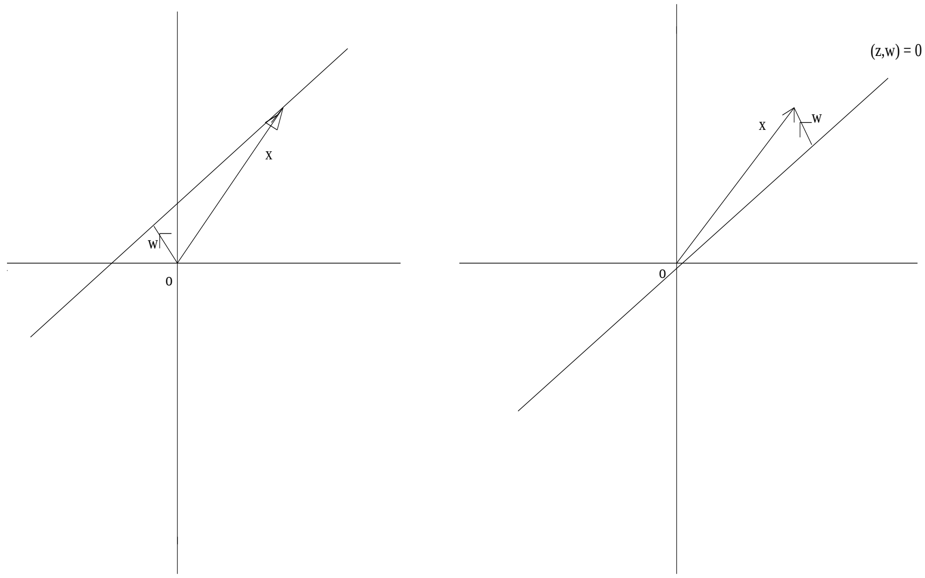

Fig. 3 Equivalence between the distance from the origin to the hyperplane passing through x with normal w and the distance from x to the hyperplane through the origin.

Since $r(x,b) = ln\,[x,\xi]$, generalized plane waves in H^4 have the form $[x,\xi]^\sigma$. It can be verified that $[x,\xi]^\sigma$ is an eigenfunction of the Laplacian with eigenvalue $\sigma(\sigma+3)$. Using coordinates $\eta,\ \hat{x},\ \hat{y},\ \hat{z}$ defined by

$$\begin{aligned} X_0 &= \frac{\eta^{-1} - \eta}{2} + \frac{1}{2}\eta^{-1}(\hat{x}^2+\hat{y}^2+\hat{z}^2) \\ X_1 &= \eta^{-1}\hat{x} \qquad X_2 = \eta^{-1}\hat{y} \qquad X_3 = \eta^{-1}\hat{z} \\ X_4 &= \frac{\eta+\eta^{-1}}{2} + \frac{1}{2}\eta^{-1}(\hat{x}^2+\hat{y}^2+\hat{z}^2) \end{aligned} \tag{115}$$

so that $ds^2 = \frac{1}{\eta^2}\,[d\eta^2 + d\hat{x}^2 + d\hat{y}^2 + d\hat{z}^2]$. Then,

$$\begin{aligned} [x,\xi]^\sigma &= -\frac{1}{2}\eta^{-1}\xi_0[1 - \eta^2 - (\hat{x}^2 + \hat{y}^2 + \hat{z}^2)] - \eta^{-1}\xi_1 x - \eta^{-1}\xi_2 y - \eta^{-1}\xi_3 z \\ &\quad + \frac{1}{2}\eta^{-1}\xi_4[1 + \eta^2 + \hat{x}^2 + \hat{y}^2 + \hat{z}^2] \end{aligned} \tag{116}$$

Since

$$\Box = \frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}g^{\mu\nu}\partial_\nu) = \eta^2\frac{\partial^2}{\partial\eta^2} - 2\eta\frac{\partial}{\partial\eta} + \eta^2\left(\frac{\partial^2}{\partial\hat{x}^2} + \frac{\partial^2}{\partial\hat{y}^2} + \frac{\partial^2}{\partial\hat{z}^2}\right) \tag{117}$$

$$
\begin{aligned}
\Box\ [x,\xi]^\sigma\ &= \eta^2\ \sigma(\sigma-1)\ [x,\xi]^{\sigma-2}\{\frac{1}{2}\xi_0[\eta^{-2}+1+\eta^{-2}(\hat{x}^2+\hat{y}^2+\hat{z}^2)] \\
&\quad +\ \eta^{-2}(\xi_1\hat{x}+\xi_2\hat{y}+\xi_3\hat{z})-\ \frac{1}{2}\xi_4\ [\eta^{-2}-1+\eta^{-2}(\hat{x}^2+\hat{y}^2+\hat{z}^2)]\}^2 \\
&\quad -\ \eta^2\sigma[x,\xi]^{\sigma-1}\{\xi_0\eta^{-3}[1-(\hat{x}^2+\hat{y}^2+\hat{z}^2)] \\
&\quad +\ 2\eta^{-3}(\xi_1\hat{x}+\xi_2\hat{y}\ +\xi_3\hat{z})\ -\ \xi_4\eta^{-3}\ [1+\hat{x}^2+\hat{y}^2+\hat{z}^2]\} \\
&\quad -\ \eta\sigma\ [x,\xi]^{\sigma-1}\{\xi_0\ [\eta^{-2}+1-\eta^{-2}(\hat{x}^2+\hat{y}^2+\hat{z}^2)] \\
&\quad +\ 2\eta^{-2}(\xi_1\hat{x}+\xi_2\hat{y}+\xi_3\hat{z})\ -\ \xi_4\ [\eta^{-2}-1+\eta^{-2}(\hat{x}^2+\hat{y}^2+\hat{z}^2)]\} \\
&\quad +\eta^2\sigma(\sigma-1)\ [x,\xi]^{\sigma-2}\eta^{-2}\ [(\xi_0+\xi_4)\hat{x}-\xi_1]^2+\eta^2\sigma\ [x,\xi]^{\sigma-1}\eta^{-1}(\xi_0+\xi_4) \\
&\quad +\eta^2\ \sigma(\sigma-1)\ [x,\xi]^{\sigma-2}\eta^{-2}[(\xi_0+\xi_4)\hat{y}-\xi_2]^2+\eta^2\sigma[x,\xi]^{\sigma-1}\eta^{-1}(\xi_0+\xi_4) \\
&\quad +\eta^2\ \sigma(\sigma-1)\ [x,\xi]^{\sigma-2}\eta^{-2}[(\xi_0+\xi_4)\hat{z}-\xi_3]^2+\eta^2\sigma[x,\xi]^{\sigma-1}\eta^{-1}(\xi_0+\xi_4) \\
&= \eta^{-2}\ \sigma(\sigma-1)\ [x,\xi]^{\sigma-2}[\{\frac{1}{2}\xi_0\ [1+\eta^2-(\hat{x}^2+\hat{y}^2+\hat{z}^2)]\ +\ (\xi_1\hat{x}+\xi_2\hat{y}+\xi_3\hat{z}) \\
&\quad -\ \frac{1}{2}\xi_4\ [1+(\hat{x}^2+\hat{y}^2+\hat{z}^2)]\ -\ \frac{1}{2}(\xi_0+\xi_4)\eta^2\}^2+\eta^2\{(\xi_0+\xi_4)\hat{x}-\xi_1\}^2 \\
&\quad +\ \eta^2\{(\xi_0+\xi_4)\hat{y}-\xi_2\}^2\ +\ \eta^2\{(\xi_0+\xi_4)\hat{z}\ -\ \xi_3\}^2]+4\sigma\ [x,\xi]^\sigma \\
&= \sigma(\sigma-1)\ [x,\xi]^{\sigma-2}\ [x,\xi]^2\ +\ 4\sigma[x,\xi]^\sigma\ =\ \sigma(\sigma+3)\ [x,\xi]^\sigma
\end{aligned}
\tag{118}
$$

using the property that $-\xi_0^2\ -\ \xi_1^2\ -\ \xi_2^2\ -\ \xi_3^2\ +\ \xi_4^2\ =\ 0$.

The hyperboloid H^4 is a two-point homogeneous space, so that any two pairs of points separated by the same distance may be mapped into each other by an isometry. As any two-point homogeneous space has rank 1, the vector space of G-invariant differential operators, $D(G/K)$ has one generator, the d'Alembertian. The Dirac operator, for example, is not an invariant differential operator, and the generalized plane waves are not eigenfunctions, in contrast to flat space, where $(i\gamma\cdot\partial-m)e^{-ip\cdot x}\ =\ (\gamma\cdot p-m)e^{-ip\cdot x}$. Instead of considering functions on the manifold $X\ =\ G/K$, it is preferable to consider the action of the Dirac operator on the space of sections of bundles on X, which can also be used to obtain a decomposition of higher-spin fields analogous to the generalized plane wave expansion of scalar fields.

References

[1] G. Dixon, Il Nuovo Cimento 105*B* (1990) 349 - 364

[2] S. Davis, 'Connections and Generalized Gauge Transformations', University of Cambridge preprint (1996) DAMTP-R/94/49, hep-th/9601178

[3] M. de Roo, H. Suelmann and A. Weidmann, Phys. Lett. *B*280 (1992) 39 - 46
H. Suelmann, Int. J. Mod. Physics D, Vol. 3, No. 1 (1994) 285 - 288

[4] N. Ya. Vilenkin, *Special Functions and the Theory of Group Representations*, Translations of Mathematical Monographs, Vol. 22 (Providence: American Mathematical Society, 1968)

[5] S. Helgason, 'Lie Groups and Symmetric Spaces' in *Batelle Rencontres* eds. C. De Witt and J. Wheeler (New York: W. A. Benjamin, 1986) 1-71
S. Helgason, *Groups and Geometric Analysis* (New York: Academic Press, 1984)

[6] S. Davis, 'Quantum Fields in Anti-de Sitter Space', Knight Prize Essay, University of Cambridge (1984)
S. Davis, *Supersymmetry in Anti − de Sitter Space*, Ph. D. thesis, University of Cambridge (1985)

[7] S. Helgason, *Geometric Analysis on Symmetric Spaces*, Mathematical Surveys and Monographs, Vol. 39 (Providence: American Mathematical Society, 1994)

[8] D. W. Dusedau and D. Z. Freedman, Phys. Rev. *D*33 (1986) 389 - 394

[9] A. Uhlmann, Czec. J. Phys. 29 (1979) 117 - 126

[10] L. Castell, Nuovo Cim. A 61 (1969) 585 - 592

[11] C. P. Burgess and C. A. Lutken, Phys. Lett. *B*153 (1985) 137 -141

[12] P. Breitenlohner and D. Z. Freedman, Ann. Phys. 144 (1982) 249 - 281

[13] W. Heidenreich, Phys. Lett. _B110_ (1985) 461 - 464

[14] P. Candelas and D. J. Raine, Phys. Rev. _D12_ (1975) 965 - 974

[15] S. J. Avis, C. J. Isham and D. Storey, Phys. Rev. _D18_ (1978) 3565 - 3576

[16] I. S. Gradshteyn and I. M. Ryzhik, _Table of Integrals, Series and Products_ (New York: Academic Press, 1980)

[17] A. Kempf, 'On Path Integration on Noncommutative Geometries', University of Cambridge preprint, DAMTP/96-10, hep-th/9603115

[18] I. Bars, 'Curved space-time strings and black holes', Strings and Symmetries 1991 (River Edge, NJ: World Scientific Publishing, 1992) 135 - 145
I. Bars, 'Curved space-time geometry for strings and affine non-compact algebras' _Perspective in Mathematical Physics_, Conf. Proc. Lecture Notes Math. Phys. III (Cambridge, MA: International Press, 1994) 51 - 76

[19] M. B. Green, J. H. Schwarz and E. Witten, _Superstring Theory_ (Cambridge: Cambridge University Press, 1987)

[20] T. S. Bunch and L. Parker, Phys. Rev. _D20_ (1979) 2499 - 2510

[21] T. S. Bunch, Gen. Rel. Grav. Vol. 13, No. 7 (1981) 711 - 723

[22] N. Birrell and P. C. W. Davies, _Quantum Fields in Anti − de Sitter Space_ (Cambridge: CUP, 1982)

[23] C. Fronsdal, Rev. Mod. Phys. _37_ (1965) 221 - 224
C. Fronsdal, Phys. Rev. _D10_ (1974) 589 - 598

[24] F. Antonsen and K. Bormann, 'Propagators in Curved Space', hep-th/9608041

[25] R. Brunetti and K. Fredenhagen, 'Interacting Quantum Fields in Curved Space: Renormalizability of ϕ^4', Talk given at Conference on Operator Algebras and Quantum Field Theory, Rome, Italy, 1 - 6 July 1996, gr-qc/9701048

[26] I. M. Gel'fand, M. I. Graev and N. Ya. Vilenkin, *Generalized Functions*, Vol. 5: Integral Geometry and Representation Theory (New York: Academic Press, 1966)