Distributed Optimal Control for Multi-agent Trajectory Optimization

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Abstract

This paper presents a novel optimal control problem, referred to as distributed optimal control, that is applicable to multiscale dynamical systems comprised of numerous interacting agents. The system performance is represented by an integral cost function of the macroscopic state, and is optimized subject to a hyperbolic partial differential equation known as the advection equation. The microscopic control laws are derived from the optimal macroscopic description using a potential function approach. The optimality conditions and computational complexity of the distributed optimal control problem are first derived analytically and, then, demonstrated numerically through a multi-agent trajectory optimization problem.

Key words: Optimal Control, Distributed Control, Robotic Navigation, Multilevel Control, Large-scale Systems

1 Introduction

Many complex systems ranging from renewable resources [24] to very large scale robotic systems (VLSR) [23] can be described as a multiscale dynamical system comprised of many dynamical systems or agents that, on small spatial and temporal scales, can each be described by a small system of ordinary differential equations (ODEs), referred to as microscopic or detailed equation. On larger spatial and temporal scales, the agents’ dynamics and interactions give rise to macroscopic coherent behavior or coarse dynamics that can be modeled by partial differential equations (PDEs) [10]. In many cases, the macroscopic PDE model can be derived by mapping the microscopic states of the agents to a macroscopic description using an appropriate restriction operator, such as the distribution of the agents or its lower-order moments [10]. This paper presents a distributed optimal control (DOC) problem formulation and optimality conditions applicable to multiscale dynamical systems in which the restriction operator is given by a time-varying probability density function (PDF) of the microscopic state, and the macroscopic PDE is given by the advection equation.

Several approaches have been proposed for the control of cooperative multi-robot systems [5]. In particular, it was recently shown that optimizing the trajectories of \(N\) coupled dynamical systems, or agents, is PSPACE-hard [21]. Therefore, when \(N\) is very large, the problem is typically decoupled into independent components for which solutions can be found quickly at the expense of optimality and completeness. These approaches include prioritized planning techniques [16, 20, 28], and path-coordination methods [12, 15], which plan the trajectories independently and then adjust the control laws to avoid collisions. Behavior-based control specifies a set of simple behaviors for each robot, and their relative importance, in order to achieve a desired macroscopic behavior [23]. Swarm intelligence methods, such as foraging and schooling [8, 17], view each robot as an interchangeable unit with local objectives and constraints that allow the swarm to converge to a desired distribution.

The DOC approach presented in this paper does not rely on decoupling the problem, or the agents’ dynamics and control laws \textit{a priori}. Instead, DOC optimizes the macroscopic behavior of the system subject to coupled microscopic agent dynamics, and relies on the macroscopic evolution equation and restriction operator that characterize the multiscale system to reduce the computational complexity of the optimal control problem. As a result the computation required is far reduced compared to classical optimal control, and the macroscopic

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behavior is optimized globally over large spatial and time scales. The DOC optimality conditions are derived using calculus of variations, and validated using numerical solutions obtained via a direct method. Simulations are presented to illustrate the performance of the DOC approach on a trajectory-optimization problem involving hundreds of agents, and multiple cooperative objectives.

2 Problem Formulation and Assumptions

This paper considers the problem of computing the optimal state and control trajectories for a multiscale dynamical system comprised of $N$ dynamical systems, referred to as agents, that can each be described by a small system of ODEs, referred to as the detailed equation,

$$\dot{x}_i(t) = f[x_i(t), u_i(t), t], \quad x_i(T_0) = x_{i0}, \quad (1)$$

where $x_i \in \mathcal{X} \subset \mathbb{R}^n$ and $u_i = c[x_i(t), t] \in \mathcal{U} \subset \mathbb{R}^m$ denote the microscopic state and control of the $i^{th}$ agent, respectively. On larger spatial and temporal scales, the interactions of the $N$ agents give rise to macroscopic coherent behavior or coarse dynamics modeled by PDEs. The macroscopic description of the multiscale system, denoted by $X \in \mathbb{R}^N$, can be deduced from the microscopic ones, either by deriving it from first principles, or by equation-free multiscale modeling [10].

Typically, the state variables that capture the macroscopic system dynamics and performance consist of lower-order moments of the microscopically evolving agent distribution [10]. Thus, from the distribution, it is possible to determine a restriction operator $\Phi_t$ that maps the microscopic states to the macroscopic description, i.e., $X(t) = \Phi_t[x_i(t), t]$. Since $x_i$ is a time-varying continuous vector, $\Phi_t$ can be assumed to be a time-varying probability density function (PDF), $\phi_{X_i} : \mathcal{X} \times \mathbb{R} \rightarrow \mathbb{R}$, such that the probability of event $x_i(t) \in B$ is,

$$P(x_i(t) \in B) = \int_B \phi_{X_i}[x_i(t), t]dx_i \quad (2)$$

for any subset $B \subset \mathcal{X}$. Where, $\phi_{X_i}$ is a non-negative function that satisfies the normalization property,

$$\int_{\mathcal{X}} \phi_{X_i}[x_i(t), t]dx_i = 1 \quad (3)$$

and is abbreviated to $\phi$ in the remainder of this paper.

The macroscopic system performance is a function of the agent distribution and control, and can be expressed as an integral cost function of $X$ and $u_i$,

$$J = \phi[x_i(T_f), T_f] + \int_{T_0}^{T_f} \int_{\mathcal{X}} \mathcal{L}[\phi[x_i(t), t], u_i(t), t]dxdtdt \quad (4)$$

where $X(t) = \phi[x_i(t), t]$, $\mathcal{L}$ is the Lagrangian, and $\phi$ is the terminal cost. DOC seeks to determine the macroscopic state and microscopic control trajectories that minimize $J$ over a (large) time interval $[T_0, T_f]$, subject to the coarse dynamics and to admissibility constraints. In multiscale dynamical systems, the goal typically is not to optimize the expected value of (4) but, instead, a functional that represents lower-order moments or information theoretic functions of the agent distribution.

Since the time-rate of change of the agent state is known from the detailed equation (1), if we assume that the agents are never created nor destroyed, the macroscopic evolution equation describing the coarse system dynamics can be obtained in closed form from the continuity equation. Through the admissibility constraints, it is possible to guarantee that the macroscopic state remains in $\mathcal{X}$ at any time $t \in (T_0, T_f)$. Then, the evolution equation is given by a hyperbolic PDE, known as advection equation, that describes the kinematics of a conserved scalar quantity transported by a known velocity field [4]. According to the advection equation, the time-rate of change of the agent distribution $\phi$ can be written in terms of the divergence of a vector field, $\nu \in \mathbb{R}^n$, that represents the time-rate of change of the agent state $x_i$. Since, the divergence of a vector field is given by the dot product with the gradient operator $\nabla = \partial/\partial x_i$, denoted by $\nabla \cdot \nu$, the macroscopic evolution equation is,

$$\frac{\partial \phi}{\partial t} = -\nabla \cdot \{\phi f[x_i(t), u_i(t), t]\} \quad (5)$$

where, the arguments of $\phi$ are omitted for brevity. The gradient is defined as a row vector of partial derivatives, and $\cdot$ denotes the dot product.

Assuming the initial agent distribution is a known PDF $g_0$, the macroscopic evolution equation (5) is subject to the following initial and boundary conditions,

$$\phi[x_i(T_0), T_0] = g_0(x_i) \quad (6)$$
$$\phi[x_i(t), t] \in \partial \mathcal{X}, t = 0, \forall t \in (T_0, T_f), \quad (7)$$

by which agents are assumed to lie in the interior of $\mathcal{X}$ at all times. Additionally, $\phi$ must obey the normalization condition (3), and the admissibility constraint

$$\phi[x_i(t) \notin \mathcal{X}, t] = 0, \forall t \in (T_0, T_f) \quad (8)$$

Then, the DOC problem consists of finding the optimal agent distribution, $\phi^*$, and agent control law, $u_i^*$, that minimize the macroscopic cost function (4) subject to the dynamic constraint (5), and to the equality constraints (6)-(8). Since the DOC problem does not obey the classical optimal control formulation [27], new optimality conditions are derived in the next section, and then validated numerically in Section 8 using the direct method presented in Section 5.
3  DOC Optimality Conditions

Necessary conditions for optimality are derived using calculus of variations to determine agent distribution and control histories that minimize the integral cost \( J \). Since the optimization of \( J \) is subject to a set of dynamic and equality constraints, the integral to be minimized is found by adjoining the dynamic constraints to \( J \) using a Lagrange multiplier [7]. By this approach, necessary conditions for optimality are found from the first-order effects of control variations that must be zero at all times for the integral functional to be stationary.

Then, higher-order sensitivity to control variations can be tested to discriminate between cases in which the integral is a minimum, a maximum, or is neither [7].

From the distributive property of the dot product, the advection equation (5) is rewritten as the time-varying equality constraint,

\[
\frac{\partial \phi}{\partial t} + (\nabla \phi) \cdot \mathbf{f} + \phi (\nabla \cdot \mathbf{f}) = 0 \quad (9)
\]

where the functions’ arguments are omitted for brevity. Since (9) is a dynamic constraint that must be satisfied at all times, a time-varying Lagrange multiplier, \( \lambda(t) \), is used to adjoin the equality constraint (9) to the integral cost (4). Then, the augmented cost function,

\[
J_A = \phi[\phi(x_i(t), T_f)] + \int_{T_0}^{T_f} \left\{ \mathcal{L}(\phi, \mathbf{u}_i, t) + \lambda \left[ \frac{\partial \phi}{\partial t} + (\nabla \phi) \cdot \mathbf{f} + \phi (\nabla \cdot \mathbf{f}) \right] \right\} \; dx_i \; dt
\]

(10) is to be minimized with respect to the functional forms of the time-varying agent distribution \( \phi \) and control \( \mathbf{u}_i \), and subject to the equality constraints (3),(6)-(8).

The integrand of (10) must satisfy stationarity conditions throughout \( (T_0, T_f) \) in order for \( J_A \) to be stationary [7]. This is proven by introducing the Hamiltonian,

\[
\mathcal{H} = \mathcal{L}(\phi) + \lambda \phi (\nabla \cdot \mathbf{f}) = \mathcal{H}[\phi(x_i, t), \mathbf{u}_i(t), \lambda(t), t] \quad (11)
\]

which is a function of the agent distribution, the control, and the Lagrange multiplier, and is analogous to the Hamiltonian from Pontryagin’s minimum principle [27]. The augmented cost function (10) is then re-written in terms of the Hamiltonian, and simplified using integration by parts, and by noting that \( \int_x \frac{\partial \phi}{\partial t} \; dx_i = 0 \) from (3) and Leibniz integral rule, such that

\[
J_A = \phi[\phi] + \int_{T_0}^{T_f} \left\{ \mathcal{H}[\phi(x_i, t), \mathbf{u}_i(t), \lambda(t), t] + \lambda(t) \frac{\partial \phi(x_i, t)}{\partial t} + \frac{\partial \phi}{\partial t} \mathbf{f} \right\} \; dx_i \; dt
\]

(12)

By the fundamental theorem of calculus of variations [7], an integral with fixed end points, \( T_0 \) and \( T_f \), is stationary for weak variations if the first order effect of variations in the function, or curve, to be optimized are zero throughout \((T_0,T_f)\). Thus, for \( J_A \) to be stationary, the first-order effect of control variations \( \delta \mathbf{u}_i (t) \) on (12) must be zero for all \( t \in (T_0, T_f) \). By the causality of the dynamic equation (1), control perturbations lead to state perturbations, and thus the first variation of \( J_A \) is

\[
\delta J_A = \int_{T_0}^{T_f} \left\{ \frac{\partial \mathcal{H}[\phi]}{\partial \mathbf{u}_i} \cdot \delta \mathbf{u}_i + \frac{\partial \mathcal{H}[\phi]}{\partial \lambda} \cdot \delta \lambda \right\} \; dx_i \; dt
\]

(13)

For an extremum, we must have \( \delta J_A = 0 \) for all \( \delta x_i, \delta \mathbf{u}_i, \) and each part of \( \delta J_A \) must equal zero separately near the optimal solution. Thus, the equations,

\[
\frac{\partial \mathcal{H}[\phi]}{\partial \mathbf{u}_i} = \frac{d}{dt} (\lambda \nabla \phi) \quad \text{or} \quad \lambda(t) \nabla \phi = \frac{\partial \mathcal{H}[\phi]}{\partial \mathbf{u}_i} \quad (14)
\]

and,

\[
\frac{\partial \mathcal{H}[\phi]}{\partial \lambda} = 0 \quad \text{or} \quad \frac{\partial \mathcal{H}[\phi]}{\partial \lambda} + \lambda(t) \nabla \phi \mathbf{G} = 0 \quad (15)
\]

must be satisfied for \( T_0 \leq t \leq T_f \), subject to the terminal conditions

\[
\lambda(T_f) \nabla \phi(x_i(T_f), T_f) = -\nabla \phi_{|t=T_f} \quad (16)
\]

where \( \mathbf{F} \equiv \partial \mathcal{T}/\partial \mathbf{x}_i \) and \( \mathbf{G} \equiv \partial \mathcal{T}/\partial \mathbf{u}_i \) denote Jacobian

3
matrices obtained from (1).

Equations (14)-(16) constitute necessary conditions for optimality for the DOC problem in Section 2. Thus, the optimal agent distribution \( \varphi^* \) must satisfy (14)-(16) along with the initial and boundary conditions (6)-(7), and the admissibility constraint (8). If these conditions are satisfied, the extremals can be tested using higher-order variations to verify that they lead to a minimum of the augmented cost function \( J_A \) in (10). In particular, sufficient conditions for optimality could be derived from the second-order derivatives of the Hamiltonian (11) with respect to \( u_i \), or Hessian matrix that is positive definite for a convex Hamiltonian. In this paper, we consider admissible solutions of (14)-(16) to be optimal if perturbations at any \( t \in (T_0, T_f) \) only increase the value of \( J_A \).

The microscopic control laws are determined from the optimal macroscopic description \( \varphi^* \) by defining an attractive potential that allows the agents to follow the optimal time-varying distribution. A potential function is defined as the difference between a time-shifted optimal macroscopic description and the actual agent distribution \( y \), as follows:

\[
U[\mathbf{x}(t), t] = -\varphi^*[\mathbf{x}(t + t_d), t + t_d] - \varphi[\mathbf{x}(t), t] \quad (17)
\]

The time-shift parameter \( t_d \) is used to prevent the agents from lagging behind \( \varphi^* \), which is varying in time. The actual distribution \( y \) can be computed by kernel density estimation using actual agent positions as the input data set [25]. The optimal microscopic control law is computed from the negative gradient of \( U \) based on the detailed equation (1), such that \( u_i^* = c[\varphi(\mathbf{x}_i, t)] \). An example is shown in Section 7 for unicycle robots.

### 4 Conservation Law Analysis

In this section, we prove that the dynamics of the closed-loop DOC problem have a Hamiltonian structure. The Hamiltonian structure provides a constant of motion for the trajectories of the controlled system dynamics [9]. Optimal trajectories thus correspond to trajectories that have vanishing variations along these constants of motion according to the maximum principal of optimal control [13]. Because the coarse dynamics are described by the advection equation (5), the open-loop system is inherently conservative [26]. In this section, we show that the DOC problem satisfies Hamilton equations,

\[
\frac{\partial \psi}{\partial q_i} = -\frac{\partial p_i}{\partial t}, \quad \frac{\partial \psi}{\partial p_i} = \frac{d q_i}{d t} \quad (18)
\]

where \( \psi(x(t), q_i, t) \) is the Hamiltonian function, \( q = q(t) \in \mathbb{R}^n \) are the generalized coordinates, and \( p = p(t) \in \mathbb{R}^n \) are the generalized momenta.

For simplicity, the proof is presented for \( n = 2 \), where \( \mathbf{x}_i = [x_i, y_i]^T \) denotes the position of the \( i \)th agent in \( \mathbb{R}^2 \). Then, the Hamiltonian function is determined by recasting the detailed equation (1) into a three-dimensional time-invariant ODE. Letting \( \mathbf{x}_i = [x_i, y_i, t]^T \) and \( \mathbf{u}_i(\mathbf{x}_i) = u_i(t) \), (1) can be written as,

\[
\begin{bmatrix} \dot{x}_i(\mathbf{x}_i, \mathbf{u}_i) \\ \dot{y}_i(\mathbf{x}_i, \mathbf{u}_i) \\ \dot{t} \end{bmatrix} = \mathbf{f}(\mathbf{x}_i, \mathbf{u}_i) \quad (19)
\]

where, \( \mathcal{X} \) is transformed into the time-space domain \( \mathcal{X} = \mathcal{X} \times (T_0, T_f) \). It also follows that the macroscopic evolution equation (5) can be rewritten as,

\[
\frac{\partial \psi(\mathbf{\hat{x}}_i)}{\partial t} + \frac{\partial [\psi(\mathbf{\hat{x}}_i) \dot{x}_i(\mathbf{\hat{x}}_i, \mathbf{\hat{u}}_i)]}{\partial x_i} + \frac{\partial [\psi(\mathbf{\hat{x}}_i) \dot{y}_i(\mathbf{\hat{x}}_i, \mathbf{\hat{u}}_i)]}{\partial y_i} = 0 \quad (20)
\]

Now, let \( \mathbf{A} = [A_x A_y A_t] = \mathbf{A}(\mathbf{x}_i) \) denote the vector potential of the product \( \varphi(\mathbf{u}_i) \), i.e.:

\[
\varphi(\mathbf{\hat{x}}_i) \mathbf{u}_i(\mathbf{\hat{x}}_i) = \nabla \times \mathbf{A}(\mathbf{x}_i) \quad (21)
\]

By performing a coordinate transformation to a canonical reference frame defined such that \( A_x = 0 \), \( \mathbf{A} \) can be used to relate the two-dimensional time-varying system to the three-dimensional time-invariant form, such that the Hamiltonian functions for the two forms are equivalent [1, 26]. The coordinate transformation is then given by \( \mathcal{F} : \mathbf{\hat{x}}_i \rightarrow \mathbf{x}_i \), where \( \mathbf{\hat{x}}_i = [x_i, p_i, t]^T \), and,

\[
p_i = -A_x [x_i, y_i(x_i, p_i, t), t] \quad (22)
\]

The resulting vector potential is \( \mathbf{A} = [A_x(x_i, y_i(x_i, p_i, t), t) \quad 0 \quad A_t(x_i, y_i(x_i, p_i, t), t)] \), which is governed by

\[
\varphi \dot{x}_i = \frac{\partial A_t}{\partial y_i}, \quad \varphi \dot{y}_i = \frac{\partial A_x}{\partial t} - \frac{\partial A_t}{\partial x_i}, \quad \varphi = -\frac{\partial A_x}{\partial y_i} \quad (23)
\]

where, the function \( y_i(x_i, p_i, t) \) is implicitly defined in (22). Then, the equivalent system is,

\[
\frac{d \mathbf{x}_i}{d t} = \mathbf{\hat{f}}(\mathbf{x}_i) = \begin{bmatrix} \partial A_t \\ -\frac{\partial A_t}{\partial x_i} \end{bmatrix} \quad (24)
\]

and the time scales in the physical and canonical forms are also equivalent.

Finally, choosing the Hamiltonian function,

\[
\psi(x_i, p_i, t) = A_t[x_i, y_i(x_i, p_i, t), t] \quad (25)
\]

Hamilton equations in (18) are satisfied as follows,

\[
\frac{\partial \psi}{\partial x_i} = -\frac{d p_i}{d t}, \quad \frac{\partial \psi}{\partial p_i} = \frac{d x_i}{d t} \quad (26)
\]
and are equivalent to a two-dimensional time-varying system in canonical space \( \bar{X} = F(X) \), with Hamiltonian function \( \psi \). Furthermore, this Hamiltonian formulation is unconditionally valid for any system governed by (1) and (5), and is mathematically equivalent to Lagrangian fluid transport for unsteady flow in two dimensions [26].

5 Numerical Solution of DOC Problem

This section presents a direct method of solution for the DOC problem presented in Section 2. Similarly to direct methods for classical optimal control [3], this method discretizes the continuous DOC problem about a finite set of collocation points, and then transcribes it into a finite-dimensional nonlinear program (NLP) that can be solved using sequential quadratic programming (SQP).

A finite Gaussian mixture model provides a parametric approximation of \( \varphi^* \) obtained from the superposition of \( z \) components with Gaussian PDFs \( f_1, \ldots, f_z \), and corresponding mixing proportions or weights \( w_1, \ldots, w_z \). The \( n \)-dimensional multivariate Gaussian PDF,

\[
f_j(x_i(t), t) = \frac{1}{(2\pi)^{n/2} |\Sigma_j|^{1/2}} \exp\left(-\frac{1}{2}(x_i - \mu_j)^T \Sigma_j^{-1} (x_i - \mu_j)\right)
\]

is referred to as component density of the mixture, and is characterized by a time-varying mean vector \( \mu_j \in \mathbb{R}^n \), and a time-varying covariance matrix \( \Sigma_j \in \mathbb{R}^{n \times n} \), with \( j = 1, \ldots, z \). We assume that, at any \( t \in (T_0, T_f) \), the agent distribution can be represented as follows,

\[
\varphi(x_i(t), t) = \sum_{j=1}^{z} w_j(t) f_j(x_i(t), t)
\]

where, \( 0 \leq w_j \leq 1 \) \( \forall j \), and \( \sum_{j=1}^{z} w_j = 1 \) [19]. In this paper, it is assumed that \( z \) is fixed, and chosen by the user. Then, an optimal agent distribution \( \varphi^* \) can be obtained by determining the optimal trajectories of the mixture model parameters from the DOC problem.

The mixture model parameters to be optimized over time are the weights \( w_j \), the elements of \( \mu_j \), and the variances and covariances in \( \Sigma_j \), with \( j = 1, \ldots, z \). In addition to satisfying the DOC constraints and optimality conditions, the mixture model parameters must be determined such that the component densities \( f_1, \ldots, f_z \) are nonnegative and obey the normalization condition for all \( t \in (T_0, T_f) \). This is accomplished by discretizing the continuous DOC problem in state space and time, about a finite set of collocation points in \( X \times (T_0, T_f) \). Let \( \Delta t \) denote a constant discretization time interval, and \( k \) denote the discrete time index, such that \( \Delta t = (T_f - T_0)/K \), and thus \( t_k = k\Delta t \), for \( k = 0, \ldots, K \). It is assumed that the microscopic control inputs, \( u_i \), are piecewise-constant during every time interval, and that,

\[
\varphi_k = \varphi(x_i(t_k), t_k) = \sum_{j=1}^{z} w_j(t_k) f_j(x_i(t_k), t_k)
\]

(29)

\[
\equiv \sum_{j=1}^{z} \frac{1}{(2\pi)^{n/2} |\Sigma_j|^{1/2}} \exp\left(-\frac{1}{2}(x_i - \mu_j)^T \Sigma_j^{-1} (x_i - \mu_j)\right)
\]

represents the agent distribution at \( t_k \). Then, the set of weights \( \{w_jk\} \), and the elements of \( \mu_jk \) and \( \Sigma_jk \) \( \forall j,k \), are all grouped into a vector \( \chi \) that represents the trajectories of the mixture model parameters in discrete time.

Since \( \varphi \) is a conserved quantity of a Hamiltonian system (Section 4), the evolution equation (5) can be discretized using a finite volume (FV) approach [11]. The FV approach partitions the state space \( X \) into FVs defined by a constant discretization interval \( \Delta x \in \mathbb{R}^n \), and each centered about a collocation point \( x_i \in X \subset \mathbb{R}^n \), with \( i = 1, \ldots, X \). Let \( \varphi_i \) and \( u_{i,k} \) denote the finite-difference approximations of \( \varphi(x_i, t_k) \) and \( e_c(\varphi(x_i, t_k)) \), respectively. Then, the finite-difference approximation of the evolution equation (5) is obtained by applying the divergence theorem to (5) for every FV, such that, \( \varphi_{k+1} = \varphi_k + \Delta t \rho_k \), where,

\[
\rho_k = -\int_S [\varphi_k f(\varphi_{i,k}, u_{i,k}, t_k)] \cdot \hat{n} dS
\]

(30)

and \( S \) and \( \hat{n} \) denote the FV boundary and unit normal, respectively. To ensure numerical stability, the discretization intervals \( \Delta t \) and \( \Delta x \) are chosen to satisfy the Courant-Friedrichs-Lewy condition [11].

Then, letting \( \Delta x_{(j)} \) denote the \( j \)-th element of \( \Delta x \), the discretized DOC problem can be written as the finite-dimensional NLP,

\[
\min J_D = \sum_{j=1}^{n} \Delta x_{(j)} \sum_{i=1}^{X} \left[ \phi_{i,k} + \Delta t \sum_{l=1}^{K} \mathcal{L}(\varphi_{i,k}, u_{i,k}, t_k) \right]
\]

s.t. \( \varphi_{k+1} - \varphi_k - \Delta t \rho_k = 0, \ k = 1, \ldots, K \)

\[
\sum_{j=1}^{n} \Delta x_{(j)} \sum_{i=1}^{X} \varphi_{i,k} - 1 = 0, \ k = 1, \ldots, K
\]

\[
\varphi_{i,0} = g_0(x_i), \ \forall x_i \in X
\]

\[
\varphi_{i,k} = 0, \ \forall x_i \in \partial X, \ k = 1, \ldots, K
\]

where \( \phi_{i,K} = \phi(\varphi_{i,K}) \) is the terminal constraint.

From (29) it can be seen that \( \phi_{i,k} \) and \( u_{i,k} \) are functions solely of the mixture model parameters \( \chi \), which constitute the NLP variables. Also, since \( \varphi \) is modeled by a Gaussian mixture, the admisibility constraint (8) is always satisfied and needs not be included in the constraints. The solution \( \chi^* \) of the NLP in (31) is obtained
using an SQP algorithm that solves the Karush-Kuhn-Tucker (KKT) optimality conditions by representing (31) as a sequence of unconstrained quadratic programming (QP) subproblems with objective function

\[ J_S(\chi) = J_D(\chi) + \sum_j \lambda_j \xi_j(\chi), \]

where \( \xi_j \) denotes the \( j \)th constraint in (31), and \( \lambda_j \) denotes a vector of multipliers of proper dimensions.

At each major iteration \( \ell \) of the SQP algorithm, the Hessian matrix \( H = \partial J_S/\partial \chi \) is approximated using the Broyden-Fletcher-Goldfarb-Shanno (BFGS) method

\[ H_{\ell+1} = H_\ell + \frac{d\ell}{d\chi} H^T d\ell - \frac{H^T \Delta \chi^T \Delta \chi H_\ell}{\Delta \chi^T H_\ell \Delta \chi} \]

(32)

Where \( \Delta \chi_\ell = \chi_\ell - \chi_{\ell-1} \), and \( d\ell \) is the change in the gradient \( \nabla J_S = \partial J_S/\partial \chi \) at the \( \ell \)th iteration [22]. The Hessian approximation (32) is then used to generate a QP subproblem,

\[
\begin{align*}
\min \ h(d_\ell) &= (1/2) d_\ell^T H_\ell d_\ell + \nabla J_S^T d_\ell \\
\text{sbt to} \quad &\nabla \xi_j^T d_\ell + \xi_j = 0, \quad \forall j
\end{align*}
\]

(33)
in \( d_\ell \), the search direction. The optimal search direction \( d_\ell^* \) is computed from the above QP using an off-the-shelf QP solver [18], such that \( \chi_{\ell+1} = \chi_\ell + \alpha d_\ell^* \).

The step-length \( \alpha \ell \) is determined by an approximate line search in the direction \( d_\ell^* \), aimed at producing a sufficient decrease in the merit function,

\[ \Psi(\chi_\ell) = J(\chi_\ell) + \sum_j r_{i,j}^T \xi_j(\chi_\ell) \]

(34)

based on the Armijo condition, and a penalty parameter \( r_{i,j} \) defined in [22]. The algorithm terminates when the KKT conditions are satisfied within a desired tolerance.

6 Computational Complexity Analysis

The computational complexity of numerical DOC is compared to that of classical optimal control by analyzing one major iteration of the SQP algorithm described in the previous section. Analogous direct method can be applied to obtain an NLP representation of a classical optimal control problem involving \( N \)-coupled agents described by (1), and an integral cost function of the agents’ microscopic states and controls [2]. The Hessian update (32), the solution of the QP subproblem (33), and the line-search minimization of the merit function (34) are the most computationally-expensive steps of the SQP algorithm described in Section 5. Therefore, the computational complexity of these three steps is analyzed for both the classical optimal control problem, and the DOC problem (4)-(8), as shown in Table 1.

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<th>DOC</th>
<th>Classical OC</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hessian update</td>
<td>( O(z \cdot X^2) )</td>
<td>( O(n \cdot m^2 N^2 K^2) )</td>
</tr>
<tr>
<td>QP subproblem</td>
<td>( O(z^2 \cdot X^3) )</td>
<td>( O(n \cdot m^2 N^3 K^3) )</td>
</tr>
<tr>
<td>Line search</td>
<td>( O(XK) )</td>
<td>( O(n \cdot N \cdot K) )</td>
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</tbody>
</table>

In both cases, the QP subproblem is the dominant computation, which is carried out by a QR decomposition of the active constraints using Householder Triangularization [22]. Then, the computation required by the classical optimal control problem exhibits cubic growth with respect to \( K \) and \( N \), and becomes prohibitive when \( N \) is very large. The computation required by the DOC problem exhibits cubic growth only with respect to \( K \), and quadratic growth with respect to \( z \). Thus, for systems with \( X \ll nN \) and \( z < mN \), the DOC approach can bring about considerable computational savings.

7 Multi-agent Trajectory Optimization

The effectiveness of the DOC methodology presented in the previous section is demonstrated through a multi-agent trajectory optimization problem. Consider a system of \( N \) cooperative unicycle robots traveling through an obstacle-populated compact space \( W \subset \mathbb{R}^2 \), referred to as workspace, and occupied by \( M \) obstacles \( B_1, \ldots, B_M \), where \( B_i \subset W \). The dynamics of each robot are described by the unicycle model,

\[ \dot{x}_i = v_i \cos \theta_i, \quad \dot{y}_i = v_i \sin \theta_i, \quad \dot{\theta}_i = \omega_i \]

(35)

where \( q_i = [x_i, y_i, \theta_i]^T \) is the configuration of agent \( i \), which contains the \( x, y \)-coordinates, \( x_i \) and \( y_i \), and heading angle, \( \theta_i \), with \( i = 1, \ldots, N \). The microscopic control vector of agent \( i \) is \( u_i = [v_i, \omega_i]^T \), where \( v_i \) and \( \omega_i \) are the linear and angular velocities, respectively.

The macroscopic state of the system is described by the time-varying PDF, or restriction operator, \( \varphi : \mathcal{X} \times \mathbb{R} \rightarrow \mathbb{R} \), such that the probability of \( x_i = [x_i, y_i]^T \) is given by (2), in terms of \( \varphi \). It follows that \( \mathcal{W} = \mathcal{X} \), and \( \varphi \) can be regarded as the density of agents in \( \mathcal{W} \) at time \( t \in [T_0, T_f] \). Given an initial distribution \( g_0(x_i) \), the agents must travel in \( \mathcal{W} \) to meet a goal distribution \( g(x_i) \), while avoiding obstacles, and minimizing energy consumption. The goal distribution is assumed to be time-invariant, and all \( M \) obstacles’ positions and geometries are assumed known without error. All trajectory optimization objectives can be expressed in terms of \( \varphi \), as follows. A measure of the difference between \( \varphi \) and the goal distribution \( g \) is given by the instantaneous Kullback-Leibler (KL) divergence at time \( t \)

\[ D(\varphi \parallel g) = \int_{\mathcal{X}} \varphi(x_i(t), t) \log_2 \frac{\varphi(x_i(t), t)}{g(x_i)} \, dx_i \]

(36)
Where, by definition, the support set of \( \varphi \) is contained by the support set of \( g \), and we set \( 0 \log_2(0/0) = 0 \) for continuity [6]. Although the KL divergence is not a true distance function because it is not symmetric, it is a suitable objective function because its value increases when the difference between \( \varphi \) and \( g \) increases, and vice versa. Also, the KL divergence of \( \varphi \) and \( g \) is zero when the two distributions are equal.

Additional objectives may also be included, for example, by specifying desired characteristics of \( \varphi \), such as the mean, and higher-order moments. For example, consider the objective of holding an equilateral triangular distribution pattern, with a constant distance \( a \) between the centers of \((z = 3)\)-mixture components in (28). Such an objective may be used to maintain communications or a desired formation. The center of the \( j \)th component density (27) is defined by the first moment or mean vector

\[
\mu_j(t) = \int_{W} x_i f_j(x_i(t), t) dx_i
\]

Then, the equilateral pattern at time \( t \) may be enforced by the objective function,

\[
Q(\varphi) = \sum_{j \neq l} ||\mu_j(t) - \mu_l(t)|| - a, \quad j, l = 1, \ldots, z,
\]

where \( a > 0 \) is a known scalar specified by the user.

A repulsive potential \( U_{\text{rep}} \) can be generated from the obstacles’ geometries \( B_1, \ldots, B_M \) in \( W \), as shown in [14], such that the obstacle avoidance objective can be represented by the product \( \varphi U_{\text{rep}} \). The energy consumption is modeled as a quadratic function of the control. Then, the DOC cost function to be minimized is,

\[
J = \int_{t_0}^{T_f} \left[ w_d D(\varphi \cdot g) + w_q Q(\varphi) + \int_{W} (w_r \varphi U_{\text{rep}}
+ w_c u_T^T R u_t \ dx_t) \right] dt
\]

where, \( w_d, w_q, w_r, \) and \( w_c \) are user-defined weights that represent the desired tradeoff between competing objectives, and \( R \) is a diagonal positive-definite matrix. By this formulation of the cost function, the divergence of \( \varphi \) and \( g \) is minimized throughout \((T_0, T_f)\).

Once an optimal agent distribution is obtained from the DOC problem (35)-(39), the microscopic control laws are obtained from the negative gradient of the potential function in (17). For robots described by the unicycle model (35), the microscopic control law is,

\[
u_i = [v_c \ Q(\hat{\theta}_i, -\nabla U)]^T
\]

where,

\[
Q(\cdot) = \{a(\hat{\theta}_i) - a[\Theta(-\nabla U)]\} \text{sgn} \{a[\Theta(-\nabla U)] - a(\hat{\theta}_i)\}
\]

is the minimum differential between the agent’s actual heading angle \( \hat{\theta}_i \), and the desired heading angle \( \Theta(-\nabla U) \), \( v_c \) is the agent’s speed, \( \text{sgn}(\cdot) \) is the sign function, and \( a(\cdot) \) is an angle wrapping function [14].

8 Simulation Results

The DOC solution of the multi-agent trajectory optimization problem presented in the previous section is illustrated through two examples. In the first example, \( N = 500 \) agents with unicycle dynamics (35) must travel from a given initial distribution \( g_0 \) to a goal distribution \( g \) (plotted in Fig. 1), during a time interval \((0, 22) \) hr. The initial microscopic states \( x_k \) are obtained by sampling \( g_0 \). Subsequently, the agents are controlled using DOC to travel in a workspace \( W \) with three obstacles shown in Fig. 1. The cost function weights are \( w_d = 20 \), \( w_r = 0.1 \), and \( w_c = 1 \), and \( z = 6 \). Time is discretized in intervals of \( \Delta t = 1 \) hr, such that \( K = 22 \), and the state space is discretized using \( X = 900 \) FVs.

The optimal agent distribution, \( \varphi^* \), and the values of the agents’ state, \( x_k \), are plotted in Fig. 2 at four sample moments in time \( t = 5 \) hr (a), \( t = 10 \) hr (b), \( t = 15 \) hr (c), and \( t = 22 \) hr (d). The microscopic state \( x_i \), plotted by a circle for every agent in Fig. 2, is simulated by integrating the detailed equation (35) using the microscopic control law (40), which is a function of the optimal distribution \( \varphi^* \). It can be seen that, as specified by the objectives in the cost function (39), over time \( \varphi^* \) meets the goal distribution \( g \), while agents avoid obstacles in \( W \). The time-histories of the DOC microscopic state and control for three agents chosen at random are plotted in Fig. 3.

In the second example, the trajectories of \( N = 300 \) unicycle agents are optimized via DOC in a workspace with one obstacle (Fig. 4), and with the additional objective of maintaining an equilateral formation of agents. The initial and goal distributions, \( g_0 \) and \( g \), are shown in Fig. 4, and all parameters are the same as in the first example, except here \( z = 3 \), \( T_f = 22 \) hr, \( w_d = 15 \), \( w_q = 100 \), \( w_r = 0.15 \), and \( w_c = 1.5 \). The optimal agent distribution and the microscopic state values obtained by implementing the microscopic control law in (40) are plotted in Fig. 5 at four sample moments in time \( t = 0 \) hr (a), \( t = 8 \) hr (b), \( t = 15 \) hr (c), and \( t = 21 \) hr (d). It can be seen that the agents move from an initial distribution that does not obey the desired formation (Fig. 5.a), to an equilateral formation that also is able to avoid the obstacle and reach the goal distribution before \( T_f \).

The optimal agent distributions obtained via direct method were also used to show that any perturbations
Fig. 1. Initial (a) and goal (b) agent distributions for a workspace with three obstacles (solid black).

from the optimal Gaussian mixture parameters increase the error in the optimality conditions (Section 3) and, therefore, the cost. As an example, Fig. 6 shows the effects of perturbations on the covariances of two mixture components at $t = 17$ hr, for the optimal distribution in Fig. 5. Here, the $j^{th}$ component’s covariance is modified such that $\Sigma_j = \Sigma_j^* + c_j I_2$, where $c_j$ is the perturbation parameter varied in Fig. 6, and $e_1$ and $e_2$ denote the mean-squared errors for the optimality conditions (14) and (15), respectively. These results are representative of an extensive set of simulations in which the means, covariances, and component weights were perturbed from optimal at various times. In all cases, the optimality conditions were validated numerically by showing that $e_1$ and $e_2$ were at a minimum for the mixture model parameters $\chi^*$ computed via SQP.

9 Conclusion

This paper presents a novel DOC problem formulation that extends the capabilities of classical optimal control to multiscale dynamical systems. The DOC optimality conditions and computational complexity are derived analytically for the case in which the restriction

Fig. 2. Optimal evolution of agent distribution and microscopic state (white circles) for $N = 500$ microscopic agents, at four instants in time.
Fig. 3. Microscopic state and control histories for 3 agents randomly chosen from the example in Fig. 2.

Fig. 4. Initial (a) and goal (b) agent distributions for a workspace with one obstacle (solid black).

operator is characterized by the agent distribution, and the macroscopic dynamics are modeled by the advection equation. The DOC system is shown to have a Hamiltonian structure that can be exploited to simplify the direct method of solution via finite volume approach. Nu-

Fig. 5. Optimal evolution of agent distribution and microscopic state (white circles) for $N = 300$ microscopic agents, at four instants in time.
numerical simulations are used to validate the optimality conditions and to demonstrate the effectiveness of the approach for multi-agent trajectory optimization.

References


Fig. 6. Numerical error for optimality conditions (14) (a) and (15) (b) as a function of covariance perturbation parameters.