A sampled-data observer with time-varying gain for a class of nonlinear systems with sampled-measurements

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Abstract—In this paper a new exponential observer for a class of nonlinear systems with sampled measurements is presented. This observer uses a time-varying gain, which is solution of an ordinary differential equation between two sampling instants. The proposed algorithm can be viewed as a generalization of the observer developed in [1]. The exponential convergence of the proposed observer is proved by using small gain arguments and a bound of the maximum allowable sampling period is provided. Performances of our observer and comparisons with existing observers are also presented.

Index Terms—Nonlinear observers, Sampled measurements, Impulsive systems, Small gain theorem.

I. INTRODUCTION

In this paper, we consider the design of an observer for a class of uniformly observable systems with sampled measurements. In the last years, the observation problem for continuous systems with sampled measurements has received a great attention. This interest is greatly motivated by many industrial applications where the output is only available at discrete-time instants with an important sampling period. For linear systems the observation problem can be solved by using the discrete time model of the continuous system. This is not the case for nonlinear systems because the exact discrete time model is generally not computable. In this case, two main approaches are used in the literature. The first one consists in the design of a discrete observer by using a consistent approximation of the exact discretized model. This approach provides a semi-global practical stability of the observation error for relatively small sampling periods. More details on this method can be found in [2] and its references. The second one is based on both continuous and discrete design. This idea has been inspired by Jazwinski [3] who introduced the continuous-discrete Kalman filter for stochastic continuous-discrete time systems. This method is constituted by two steps. In the first one (which is called the prediction step), the observer is a copy of the model system, whereas in the second step, the value of the state estimate is updated using the newly available sampled measure. The exponential convergence of the observation error is then ensured under some sufficient conditions on the sampling period through the stability analysis of impulsive systems. In [4] the authors use this approach to write a discrete-continuous version of the well known high gain observer [5]. In [6], observers for a Multi input-Multi-output class of state affine systems where the dynamical matrix depends on the inputs have been designed when the inputs are regularly persistent. This work was used in [7] to design an adaptive observer. In [8], a similar method has been used for a larger class of systems and applied to the observation of an emulsion copolymerization process. In [9], the authors presented an observer for a class of systems with output injection and recently in [10], a continuous-discrete observer which uses a high gain framework with constant observation gains has been proposed. In [11], the authors developed a continuous-discrete time observer for the class of systems considered in [12]. In [13], a hybrid observer has been proposed for a class of nonlinear systems. An inter-sample time predictor which estimates the output between two sampling instants is used between two sampling instants. This algorithm has been extended to some networked control systems with a class of scheduling protocols in [14]. An explicit bound of the maximum allowable sampling period guaranteeing the exponential convergence of the observer is provided by using a Lyapunov approach. On the other hand, an impulsive observer was proposed in [15] for a class of nonlinear systems. Sufficient conditions guaranteeing the asymptotic convergence of the observer have been derived by using a Lyapunov Krasovskii approach. This idea has been also used in [16] for a class of nonlinear uniformly observable systems with nonuniformly sampled measurements. Recently, in [1], a delay-dependent gain has been used in the design of an exponential observer for a class of nonlinear systems with delayed measurements. This result, which is proved by using Razumikhin theorem, improves the quality of convergence compared to the one of [17] and it can be also applied to sampled-data case which is an obvious particular case of the delayed measurements case. In this paper, we present a new class of observers for a class of sampled-data systems which use new kinds of time-varying gains. The exponential convergence of our observer is proved by using small-gain arguments and an analysis of performances with respect to measurement errors is deeply discussed. As we will see below, the introduction of the time-varying gain provides an attenuation of the measurement errors effects and tends to augment the bound of the maximum allowable sampling period compared to some existing works in the literature ([15], [16], [13]). The observer presented in the present paper can better attenuate the effects of measurement errors than the one presented in [1]. It has also to be emphasized that the results presented in this paper can be easily extended to several classes of nonlinear observers and for the case.
of networked control systems with some protocols as Uniformly Globally Exponentially Stable (UGES) protocols. The present paper is organized as follows: after introducing our notations in section 2, we describe in the section 3 the class of systems considered here. In the section 4, we present a new observer which uses a new dynamic gain and provide a comparison between it and some observers existing in the literature which use constant gains. Finally we illustrate our results on an academic example through some simulations.

II. NOMENCLATURE

In this paper, the following notations are used: Let \( \mathbb{R} \triangleq (-\infty, \infty) \), \( \mathbb{R}^+ \triangleq (0, \infty) \), \( \mathbb{R}_0^+ \triangleq [0, \infty) \), and let \( \| \cdot \| \) be the Euclidean norm. For \( p, q, n, m \in \mathbb{N} \), \( \mathbb{R}^{p \times q} \) represents the set of real matrices of order \( p \times q \) and \( \mathbb{I}_p \in \mathbb{R}^{p \times p} \) stands for the identity matrix of order \( p \times p \). The notation \( \| P \| \), for \( P \in \mathbb{R}^{p \times q} \), represents the \( L_2 \)-norm of \( P \) and \( X' \) represents the transposed vector of \( X \). We say that \( \alpha \mathbb{I}_n \leq S \leq \beta \mathbb{I}_n \) where \( S \in \mathbb{R}^{n \times n} \) if \( \lambda_{\text{min}}(S) \geq \alpha \) and \( \lambda_{\text{max}}(S) \leq \beta \) where \( \lambda_{\text{min}}(S) \) and \( \lambda_{\text{max}}(S) \) denote respectively the smallest and the biggest eigenvalues of the matrix \( S \). In all this study, the initial time is called \( t_0 \in \mathbb{R} \).

III. PROBLEM STATEMENT

Consider the well known class of nonlinear systems:

\[
\begin{align*}
\dot{x} &= Ax + f(x) \\
y &= Cx
\end{align*}
\]  
(1)

where \( x = (x^1, x^2, \ldots, x^q)^T \in \mathbb{R}^q \) with \( x^k \in \mathbb{R}^p \) and \( p, q = n \),\( y \in \mathbb{R}^p \).

\[
A = \begin{pmatrix}
0_p & \mathbb{I}_p & 0_p & \ldots & 0_p \\
0_p & 0_p & \mathbb{I}_p & \ldots & 0_p \\
0_p & \ldots & 0_p & \ldots & 0_p \\
0_p & \mathbb{I}_p & \ldots & \ldots & 0_p
\end{pmatrix}
\]  
(2)

and

\[
C = \begin{pmatrix} \mathbb{I}_p & 0_p & \ldots & 0_p \end{pmatrix}
\]  
(3)

Throughout this paper, we assume that the following hypotheses are satisfied:

**Hypothesis 1**: The vector \( f(x) \) has a block triangular structure

\[
f(x) = (f_1(x^1), \ldots, f_s(x^1, \ldots, x^q), f_{s+1}(x^s), \ldots, f_q(x))^T
\]  
(4)

**Hypothesis 2**: The functions \( f_s \) are globally Lipschitz, i.e:

\[
\exists \beta_0 > 0 \quad \text{such that} \quad \forall (x_1, x_2) \in \mathbb{R}^{p \times s} \times \mathbb{R}^{p \times s}
\]

\[
|f_s(x_1) - f_s(x_2)| \leq \beta_0 \|x_1 - x_2\|, \quad (s = 1, \ldots, q).
\]  
(5)

We also suppose that the measures of \( y \) are available for the observer only at instants \( t_k \). The notation \((t_k)_{k \geq 0}\) represents a strictly increasing sequence such that \( \lim_{k \to \infty} t_k = \infty \). The sampling intervals are bounded with \( 0 \leq t_k - t_{k-1} \leq T \) for all \( k = 1, 2, \ldots, \infty \).

Our goal is to design a new observer which provides a continuous state estimate for systems (1) and attenuates the measurement errors effects compared to existing works. We will also provide an upper bound of the maximum allowable sampling period \( T_{\text{max}} \) guaranteeing that the observation error converges globally exponentially towards zero.

IV. OBSERVER DESIGN AND STABILITY STUDY

We propose the following observer:

\[
\begin{align*}
\dot{\hat{x}} &= A\hat{x} + f(\hat{x}) - \theta \Delta^{-1} K \Phi(t)(C\hat{x}(t_k) - y(t_k)) \quad t \in [t_k, t_{k+1}) \\
\phi(t) &= -\eta \phi(t)^{\alpha} \quad t \in [t_k, t_{k+1}) \\
\phi(t_k) &= 1,
\end{align*}
\]  
(6)

where \( \eta \) and \( \alpha \) are two constants satisfying \( \eta > 0 \) and \( 0 < \alpha \leq 1 \). The matrix gains \( K = \begin{pmatrix} K^1 \\ \vdots \\ K^q \end{pmatrix} \) where the matrices \( K^q(p, p) \) are chosen so that the matrix \( (A - KC) \) is Hurwitz and satisfies the following Lyapunov function:

\[
P(A - KC) + (A - KC)^T P \leq -\mu \mathbb{I}_n
\]  
(7)

where \( \mu > 0 \) is a free positive constant and \( P \) is a symmetric positive definite matrix. The matrix \( \Delta \) is defined by:

\[
\Delta = \text{Diag} \left( \mathbb{I}_p, \frac{1}{\theta} \mathbb{I}_p, \ldots, \frac{1}{\theta^{q-1}} \mathbb{I}_p \right)
\]  
(8)

The function \( \phi(t) \) represents the time-varying gain. At each sampling instant it is re-initialized such that its value will be equal to 1. This is done in order to recover the continuous-time observer when the measure is available. The function \( \phi(t) \) will be also forced to be strictly positive by choosing appropriate bound of maximum allowable sampling period. In these conditions, this function will usually decrease in each interval \([t_k, t_{k+1})\). As we can see below, this property will attenuate the effects of measurement errors.

**Remark 1**: It should be noticed that if \( a = 1 \), then \( \phi(t) = e^{-\eta(t-t_k)} \), for all \( t \in [t_k, t_{k+1}) \). This function corresponds exactly to the one used in [1] by considering the delay \( \tau = t - t_k \), for all \( t \in [t_k, t_{k+1}) \). This fact means that the time-varying gain used in [1] is a particular case of the class of time-varying gains used in the present paper.

**Theorem 1**: Consider the class of systems (1) and suppose that Hypotheses (1-2) hold. Then for all \( \theta > \theta_0 \), for all \( 0 < \sigma < \sigma_0/2 \), and for all \( T \in (0, T_{\text{max}}) \) with:

\[
\begin{align*}
\theta_0 &= \sup \left\{ \frac{1}{4\lambda_{\text{max}}(P)}, \frac{\sqrt{\beta_0 \|P\|}}{\mu} \right\} \\
\sigma_0 &= \frac{\mu \theta}{4\lambda_{\text{max}}(P)} \\
\sigma_1 &= \frac{\theta \|P\|}{\sqrt{\lambda_{\text{min}}(P)}}
\end{align*}
\]  
(9)

and \( T_{\text{max}} \) satisfying the following inequality:

\[
T_{\text{max}} e^{\sigma T_{\text{max}}} \leq \frac{\sigma_0 \sqrt{\lambda_{\text{min}}(P)}}{2\lambda_1 (\theta + \beta_0 + \Theta |K| + \eta)}
\]  
(10)

The system (6) is a global exponential observer of systems (1).

**Proof** :
Let us consider the observation error $\tilde{x} = \hat{x} - x$, then we get:

\[
\begin{align*}
\dot{\tilde{x}} &= A\tilde{x} + f(\tilde{x}) - f(x) - \theta\Delta^{-1}K\phi(t)(C\tilde{y}(t) - y(t)) \quad t \in [t_k, t_{k+1}) \\
\phi(t) &= -\eta\theta(t)\tilde{x} \quad t \in [t_k, t_{k+1}) \\
\phi(t_k) &= 1.
\end{align*}
\]

(11)

Let us introduce the change of coordinates $\tilde{x} = \Delta\tilde{x}$, then $\Delta\Delta^{-1} = \theta A, C\Delta = C\Delta^{-1} = C$. From this we deduce:

\[
\begin{align*}
\dot{\tilde{x}} &= \theta A\tilde{x} + \Delta f(\tilde{x}) - f(x) - \theta\Delta\tilde{x} - \theta \Delta^{-1}K\phi(t)(\Delta\tilde{y}(t) - \tilde{y}(t)) \\
\phi(t) &= -\eta\theta\tilde{x} \\
\phi(t_k) &= 1.
\end{align*}
\]

(12)

where $\tilde{y} = C\tilde{x}$.

Now, considering the error $\tilde{z}(t) = \tilde{y}(t) - f(t)\tilde{y}(t_k)$ then it is clear that $\tilde{z}(t_k) = 0$. Combining this with system (12), then we derive the following error system:

\[
\begin{align*}
\dot{\tilde{z}} &= \theta(A - KC)\tilde{z} + \Delta f(\tilde{z}) - f(x) + \theta K\tilde{z}(t) \\
\tilde{z}(t_k) &= -\eta\theta\tilde{x} \\
\tilde{z}(t_k) &= 1.
\end{align*}
\]

(13)

Now, let us consider the Lyapunov function $V = \tilde{x}^TP\tilde{x}$, then its time derivative satisfies:

\[
\begin{align*}
\dot{V} &\leq -\mu\|\tilde{x}\|^2 + 2\tilde{x}^TP\Delta(f(\tilde{x}) - f(x)) \\
&\quad + 2\theta\tilde{x}^TPK\tilde{z}(t)
\end{align*}
\]

(14)

Using Hypotheses (1-2), then we get:

\[
\begin{align*}
\dot{V} &\leq -\mu\|\tilde{x}\|^2 + 2\sqrt{\theta}\|\tilde{x}\|^2 + 2\theta\tilde{x}^TPK\tilde{z}(t)
\end{align*}
\]

(15)

Considering the function $W = \sqrt{V}$, then we have

\[
\begin{align*}
\dot{W} &\leq -\mu\|\tilde{x}\|^2 + 2\sqrt{\theta}\|\tilde{x}\|^2 + 2\theta\|\tilde{x}\|/\|PK\|/\|\tilde{z}(t)\|
\end{align*}
\]

(16)

Using the fact that, $\lambda_{\text{min}}(P)||\tilde{x}||^2 \leq V \leq \lambda_{\text{max}}(P)||\tilde{x}||^2$, then we get

\[
\begin{align*}
\dot{W} &\leq -\mu\|\tilde{x}\|^2 W + \sqrt{\theta}\|\tilde{x}\|/\lambda_{\text{min}}(P)/\|\tilde{z}(t)\|
\end{align*}
\]

(17)

Choosing the parameter $\theta$ such that $\mu\theta > \lambda_{\text{min}}(P)/\lambda_{\text{max}}(P)/\sqrt{\theta}$, or equivalently $\theta > \theta_0 = \sup\{1/\lambda_{\text{min}}(P)/\sqrt{\theta}/\lambda_{\text{max}}(P)/\mu\}$, we deduce that for all $\theta > \theta_0$:

\[
\begin{align*}
\dot{W} &\leq -\mu\|\tilde{x}\|^2 W + \theta/\lambda_{\text{max}}(P)/\|\tilde{x}(t)\|
\end{align*}
\]

(18)

Let us set

\[
\begin{align*}
\sigma_0 &= \mu\theta/\lambda_{\text{max}}(P) \\
\sigma_1 &= \theta/\lambda_{\text{max}}(P)
\end{align*}
\]

(19)

Integrating (18), then

\[
W(t) \leq e^{-\sigma_0(t-t_0)}W(t_0) + \sigma_1 e^{-\sigma_1}(t-t_0)\int_{t_0}^{t} e^{\sigma_1}s\|\tilde{z}(s)\|ds
\]

(20)

Taking $0 < \sigma < \sigma_0/2$, then we can write:

\[
e^{\sigma_1}(t-t_0)\int_{t_0}^{t} e^{\sigma_1}s\|\tilde{z}(s)\|ds \leq M(t_0) + \sigma_1 e^{\sigma_1}(t-t_0)\int_{t_0}^{t} e^{\sigma_1}s\|\tilde{z}(s)\|ds
\]

(21)

where $M(t_0) := e^{\sigma_0}W(t_0)$. Moreover,

\[
e^{\sigma_1}(t-t_0)\int_{t_0}^{t} e^{\sigma_1}s\|\tilde{z}(s)\|ds \leq M(t_0) + \sigma_1 e^{\sigma_1}(t-t_0)\int_{t_0}^{t} e^{\sigma_1}s\|\tilde{z}(s)\|ds
\]

(22)

or

\[
e^{\sigma_1}(t-t_0)\int_{t_0}^{t} e^{\sigma_1}s\|\tilde{z}(s)\|ds \leq M(t_0) + \sigma_1 e^{\sigma_1}(t-t_0)\int_{t_0}^{t} e^{\sigma_1}s\|\tilde{z}(s)\|ds
\]

(23)

from this we deduce that

\[
e^{\sigma_1}(t-t_0)\int_{t_0}^{t} e^{\sigma_1}s\|\tilde{z}(s)\|ds \leq M(t_0)
\]

(24)

Using the fact that $0 < \sigma < \sigma_0/2$, then we deduce that

\[
\begin{align*}
\sup_{t \leq t \leq t_0}(e^{\sigma_1}(t-t_0)\|\tilde{z}(s)\|) &\leq M(t_0) + \sigma_1 e^{\sigma_1}(t-t_0)\int_{t_0}^{t} e^{\sigma_1}s\|\tilde{z}(s)\|ds
\end{align*}
\]

(25)

Now, from (13), we can say that for all $t \in [t_k, t_{k+1})$ and $\forall k \geq 0$,

\[
\begin{align*}
\tilde{z}(t_k) &= \int_{t_k}^{t} \left(\theta\tilde{x}(s) + f^1(\tilde{x}(s)) - f^1(\tilde{x}(s))\right)ds \\
&\quad - \int_{t_k}^{t} \left(\theta CK\tilde{y}(t_k) - \tilde{y}(t_k)\phi(s)\right)\phi(s)ds
\end{align*}
\]

(26)

and

\[
\begin{align*}
\|\tilde{z}(t_k)\| &\leq \int_{t_k}^{t} (\theta\|\tilde{x}(s)\|) + ||f^1(\tilde{x}(s)) - f^1(\tilde{x}(s))||ds \\
&\quad + \int_{t_k}^{t} (\theta CK(\|\phi(s)\| + \eta\|\phi(s)\|))\|\tilde{y}(t_k)\|ds
\end{align*}
\]

(27)

Using the facts that $\|\phi(s)\| \leq 1, ||C|| = 1$, and Hypothesis 2, then we have for all $t \in [t_k, t_{k+1})$:

\[
e^{\sigma_1}(t-t_0)\|\tilde{z}(s)\| \leq T e^{\sigma_1}(t-t_0 + \theta + \theta||K|| + \eta)\sup_{t \leq t \leq t_0}(e^{\sigma_1}(t-t_0)\|\tilde{z}(s)\|)
\]

(28)

since $\sup_{t \leq t \leq t_0}(e^{\sigma_1}(t-t_0)\|\tilde{z}(s)\|) \leq \sup_{t \leq t \leq t_0}(e^{\sigma_1}(t-t_0)\|\tilde{z}(s)\|)$, then we
also have:

$$\sup_{0 \leq s \leq t} (e^{\sigma s}||\xi(s)||) \leq Te^{\sigma T} (\theta + \beta_0 + \theta|K| + \eta) \sup_{0 \leq s \leq t} (e^{\sigma s}||\bar{x}(s)||)$$

(29)

Since $t \geq t_j$ for all $j = 0, \ldots, -1, 1, 0$, for the values of the parameter $a$, we have considered three values $a = 1, a = 0.5$ and $a = 0.25$.

Combining (30) and (25), then

$$\sup_{0 \leq s \leq t} (e^{\sigma s}||\bar{x}(s)||) \leq \frac{M(t_0)}{1 - \gamma_0} \sqrt{\lambda_{\min}(P)}$$

(31)

with

$$\gamma_0 = Te^{\sigma T} (\theta + \beta_0 + \theta|K| + \eta) \frac{2\sigma_1}{\sigma_0 \sqrt{\lambda_{\min}(P)}}$$

(32)

Then, under the following small-gain condition:

$$Te^{\sigma T} (\theta + \beta_0 + \theta|K| + \eta) \frac{2\sigma_1}{\sigma_0 \sqrt{\lambda_{\min}(P)}} < 1$$

(33)

we can write

$$\sup_{0 \leq s \leq t} (e^{\sigma s}||\bar{x}(s)||) \leq \frac{1}{(1 - \gamma_0) \sqrt{\lambda_{\min}(P)}} M(t_0)$$

(34)

Using the fact that $e^{\sigma t}||\bar{x}(t)|| \leq \sup_{0 \leq s \leq t} (e^{\sigma s}||\bar{x}(s)||)$, then we derive that

$$e^{\sigma t}||\bar{x}(t)|| \leq \frac{1}{(1 - \gamma_0) \sqrt{\lambda_{\min}(P)}} M(t_0)$$

(35)

and

$$||\bar{x}(t)|| \leq \frac{1}{(1 - \gamma_0) \sqrt{\lambda_{\min}(P)}} M(t_0) e^{-\sigma t}$$

(36)

From this we deduce that the observation error converges exponentially to zero. This ends the proof.

**Remark 2:** It should be emphasized that the bound (10) is only sufficient and not necessary. Note that this is the case for all bounds existing in the literature. The existence of this bound means only that system (6) is an exponential observer for systems (1) for sufficiently small sampling period. The real bound guaranteeing exponential convergence will be determined practically or in simulations.

**Remark 3:** It is well known that the solution of the differential equation $\dot{\phi} = -\eta \phi(t)^a$ for $0 < a < 1$ converges to zero in finite time $t_f$

$$t_f = \frac{1}{(1 - a) \eta}$$

(37)

By solving the above differential equation of the time varying gain for $0 < a < 1$, and if the sampling period is smaller than $\frac{1}{(1 - a) \eta}$, then we have:

$$\begin{cases} 
\phi(t) = (1 - \eta (1 - a) (t - t_k))^{1/a}, t \in [t_k, t_{k+1}) \\
\phi(t_k) = 1.
\end{cases}$$

(38)

However if the sampling period is larger than $\frac{1}{(1 - a) \eta}$, then the time-varying gain $\phi(t)$ will be equal to zero between $[t_k + t_j, t_{k+1})$ and in this case, the observer will be a copy of the system (1). This means that the effect of measurement errors in the observation error will be removed in the intervals $[t_k + t_j, t_{k+1})$.

**Remark 4:** The reader should notice that the same analysis of convergence of observer (6) can be done with $a \geq 1$. The difference is that the time-varying gain will decrease slowly than in the case $0 < a < 1$ studied here. This means that the gain derived from our method will be always smaller than the one derived from [1]. A consequence of this fact is that the attenuation of measurement errors will be more important with the algorithm presented here. It has also to be emphasized that the case $a = 0$ is not considered in order to guarantee that the time-varying is always $\phi \geq 0$.

**A. Comparison with other observers**

When the output of system (1) is corrupted by a noise $v(t)$, the measure available for observer at instants $t_k = y(t_k) = Cx(t_k) + v(t_k)$. The observation error (12) will be rewritten as follows:

$$\begin{cases} 
\dot{x} = \theta A x + \Delta(f(x) - f(x_1) - \theta K \phi(t)y(t_k) + v(t_k)), t \in [t_k, t_{k+1}) \\
\phi(t) = -\eta \phi(t)^a, t \in [t_k, t_{k+1}) \\
\phi(t_k) = 1.
\end{cases}$$

(39)

As we can see, system (39) is disturbed by the term $\theta K \phi(t)v(t_k)$. Notice that for observers developed in [15] and [16] the observation error is disturbed by the term $\theta K v(t_k)$. Since $|\phi(t)| < 1$ for all $t \neq t_k$, then $|\theta K \phi(t)v(t_k)| < |\theta K v(t_k)|$ for all $t \neq t_k$. Then compared to [15] and [16] the effects of measurement errors will be certainly attenuated by the time-varying gain $\phi(t)$ for all $t \neq t_k$.

**V. AN ACADEMIC EXAMPLE**

In order to illustrate our main result, we consider the following nonlinear system:

$$\begin{cases} 
\dot{x}_1 = x_2 \\
\dot{x}_2 = -0.1 x_1 + 0.1 tanh(x_1 - x_2) \\
y = x_1
\end{cases}$$

(40)

The observer (6) has the following form:

$$\begin{cases} 
\dot{x}_1 = \dot{x}_2 - \theta k_1 \phi(t) (\dot{x}_1(t_k) - y(t_k)), t \in [t_k, t_{k+1}) \\
\dot{x}_2 = -0.1 \dot{x}_1 + 0.1 tanh(\dot{x}_1, \dot{x}_2) - \theta^2 k_2 \phi(t) (\dot{x}_1(t_k) - y(t_k)), \\
t \in [t_k, t_{k+1}) \\
\phi(t) = -\eta \phi(t)^a, t \in [t_k, t_{k+1}) \\
\phi(t_k) = 1.
\end{cases}$$

(41)

and the observer corresponding to [16] :

$$\begin{cases} 
\dot{x}_1 = \dot{x}_2 - \theta k_1 (\dot{x}_1(t_k) - y(t_k)), t \in [t_k, t_{k+1}) \\
\dot{x}_2 = -0.1 \dot{x}_1 + 0.1 tanh(\dot{x}_1, \dot{x}_2) - \theta^2 k_2 (\dot{x}_1(t_k) - y(t_k)), \\
t \in [t_k, t_{k+1}) \\
\phi(t) = -\eta \phi(t)^a, t \in [t_k, t_{k+1}) \\
\phi(t_k) = 1.
\end{cases}$$

(42)

The simulations of observer (41) are performed with $\theta = 1, k_1 = 1, k_2 = 2, \eta = 5$ and with the initial conditions respectively for system and observer $(-1,1)$ and $(0,0)$. For the values of the parameter $a$, we have considered three values $a = 1, a = 0.5$ and $a = 0.25$. 

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In Fig.(1), we present the observation error on the variable state $x_2$ of observer (41) for the above three values of parameter $a$ when there is no noise on the measured variable $x_1$.

In Fig.(2), we present the behavior of the observation error on the variable state $x_2$ of observer (41) when the measure of $x_1$ is corrupted by a noise and with $a = 0.5$ and a constant gain (42). The simulations confirm clearly that the effect of measurement errors is attenuated with the adaptive gain compared to the one of [16]. The different values of the mean square error (MSE), computed with 10000 values in the steady state, are given table (I). It is clear that the MSE is smaller with the adaptive gain and decreases as the parameter $a$ increases. The simulations also showed that the maximum allowable sampling period tends to increase compared to the one of [16]. Indeed, for this example, in the case of [16], the maximum allowable sampling period is $T_e = 1s$ whereas it can reach $T_e = 8s$ with the adaptive gain. This property can be explained by the fact that $|\theta K \phi(t)| < |\theta K|$ for all $t \neq t_k$. The counterpart of this augmentation is that the speediness of convergence of our observer tends to decrease compared to observer derived in [16].

<table>
<thead>
<tr>
<th>observer in [16]</th>
<th>$a = 1$</th>
<th>$a = 0.5$</th>
<th>$a = 0.25$</th>
</tr>
</thead>
<tbody>
<tr>
<td>MSE</td>
<td>0.0026</td>
<td>0.0010</td>
<td>0.00077</td>
</tr>
</tbody>
</table>

**TABLE I**
VALUES OF THE MEAN SQUARE ERRORS WITH $T_e = 0.5s$ FOR OBSERVER (41)

VI. CONCLUSIONS

In this paper, we have presented a new sampled-data observer for a class of nonlinear uniformly observable systems. Our contributions can be summarized in what follows:

- We have generalized the idea of [1] by using a large class of dynamic gains.
- We have showed that the effects of measurement errors are largely attenuated compared to the case of constant gains and to the case of [1].
- For observer developed here, the maximum allowable sampling period $T_{max}$ tends to augment compared to the case of constant gains. This is a natural consequence of the decreasing nature of the time-varying gains between two sampling instants.

It has to be noticed that the observer presented in the present paper can be combined with a feedback controller to construct an output feedback controller ensuring the stabilization or tracking for system (1) by using only the sampled measures of the output.

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Fig. 2. Observation errors $\hat{x}_2$, with noise on the measurement, with $T_e = 0.5s$, for observer (41)


