Edge-Maximal $C_{2k+1}$-vertex disjoint Free Graphs

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**ABSTRACT:** Let $k \geq 1$ be a positive integer and $G(n; V_{2k+1})$ the class of graphs on $n$ vertices containing no $2k+1$ vertex disjoint cycles. Let $f(n; V_{2k+1}) = \max \{\varepsilon(G) : G \in G(n; V_{2k+1})\}$. In this paper we determine $f(n; V_{2k+1})$ and characterise the edge maximal members in $G(n; V_{2k+1})$ for $k = 1$ and $2$.

1. INTRODUCTION

First, we recall some notation and terminology. For our purposes a graph $G$ is finite, undirected and has no loops or multiple edges. We denote the vertex set of $G$ by $V(G)$ and the edge set of $G$ by $E(G)$. The cardinalities of these sets are denoted by $v(G)$ and $\varepsilon(G)$, respectively. The cycle on $n$ vertices is denoted by $C_n$. Let $G$ be a graph and $u \in V(G)$. The degree of a vertex $u$ in $G$, denoted by $d_G(u)$, is the number of edges of $G$ incident to $u$. The neighbour set of a vertex $u$ of $G$ in a subgraph $H$ of $G$, denoted by $N_H(u)$, consists of the vertices of $H$ adjacent to $u$; we write $d_H(u) = |N_H(u)|$.

Let $G_1$ and $G_2$ be graphs. The union $G_1 \cup G_2$ of $G_1$ and $G_2$ is a graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. Two graphs $G_1$ and $G_2$ are vertex disjoint if and only if $V(G_1) \cap V(G_2) = \emptyset$; $G_1$ and $G_2$ are edge disjoint if

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E(G_1) \cap E(G_2) = \emptyset. The intersection \( G_1 \cap G_2 \) of graphs \( G_1 \) and \( G_2 \) is defined similarly, but in this case we need to assume \( V(G_1) \cap V(G_2) \neq \emptyset \). The join \( G \vee H \) of two vertex disjoint graphs \( G \) and \( H \) is the graph obtained from \( G + H \) by joining each vertex of \( G \) to each vertex of \( H \). For vertex disjoint subgraphs \( H_1 \) and \( H_2 \) of \( G \), we let
\[
E(H_1, H_2) = \{xy \in E(G) : x \in V(H_1), y \in V(H_2)\}
\]
and
\[
\varepsilon(H_1, H_2) = |E(H_1, H_2)|.
\]
For a proper subgraph \( H \) of \( G \) we write \( G[V(H)] \) and \( G - V(H) \) simply as \( G[H] \) and \( G - H \) respectively.

In this paper we consider the Turán-type extremal problem [6] with the odd vertex disjoint cycles being the forbidden subgraph. Since a bipartite graph contains no odd cycles, we only consider non-bipartite graphs. For a positive integer \( n \) and a set of graphs \( \mathcal{F} \), let \( \mathcal{G}(n; \mathcal{F}) \) denote the class of non-bipartite \( \mathcal{F} \)-free graphs on \( n \) vertices, and \( f(n; \mathcal{F}) = \max \{ \varepsilon(G) : G \in \mathcal{G}(n; \mathcal{F}) \} \). An important problem in extremal graph theory is that of determining the values of the function \( f(n; \mathcal{F}) \) [6]. Further, characterize the extremal graphs \( \mathcal{G}(n; \mathcal{F}) \) of where \( f(n; \mathcal{F}) \) is attained. This problem has been studied by a number of authors [3, 4, 7, 8, 9, 11]. Jia [10] proved that \( f(n; C_5) = \left\lfloor \frac{n^2}{4} \right\rfloor + 2 \) for \( n \geq 9 \), and he characterizes the extremal graphs as well. Jia [10] conjectured that \( f(n; C_{2k+1}) \leq \left\lfloor \frac{n^2}{4} \right\rfloor + 3 \) for \( n \geq 4k+2 \). Recently, Bataineh [1] confirm positively the above conjecture for \( n > 36k \). Most recently, Bataineh and Jaradat [2] proved that for large \( n \), \( \varepsilon(G) \leq \left\lfloor \frac{n^2}{4} \right\rfloor + r - 1 \) where \( G \) is a graph that contains no \( r \) edge disjoint copies of \( C_{2k+1} \).

Let \( \mathcal{G}(n; V_{2k+1}) \) denote the class of graphs on \( n \) vertices containing no vertex disjoint cycles of length \( (2k+1) \). Let \( f(n; V_{2k+1}) = \max \{ \varepsilon(G) : G \in \mathcal{G}(n; V_{2k+1}) \} \). In this
paper we determine \( f(n; V_{2k+1}) \) and characterise the edge maximal members in \( \mathcal{G}(n; V_{2k+1}) \) for \( k = 1 \) and \( 2 \).

Now, we state a number of results, which we use to prove our main results.  

**Lemma 1.1 (Bondy & Murty)** Let \( G \) be a graph on \( n \) vertices. If \( \varepsilon(G) > \frac{n^2}{4} \), then \( G \) contains a cycle of length \( r \) for each \( r \), where \( 3 \leq r \leq \left\lceil \frac{n+1}{2} \right\rceil \).

**Theorem 1.1 (Brandt)** A non-bipartite graph \( G \) of order \( n \) and more than \( \frac{(n-1)^2}{4} + 1 \) edges. Then \( G \) contains all cycles of length between 3 and the length of the longest cycle.

**Theorem 1.2 (Jia)** Let \( G \in \mathcal{G}(n; C_3) \), \( n \geq 9 \). Then

\[
f(n; C_3) \leq \left\lceil \frac{1}{4} (n-2)^3 \right\rceil + 3.
\]

Furthermore, equality holds if and only if \( G \in \mathcal{G}_s^*(n) \) for \( n \geq 10 \) where \( \mathcal{G}_s^*(n) \) denote the class of graphs obtained by adding a triangle, two vertices of which are new, to the complete bipartite graph \( K_{\left\lceil \frac{1}{2} (n-2) \right\rceil \left\lceil \frac{1}{2} (n-2) \right\rceil} \).

### 2. Edge-Maximal \( C_3 \)-vertex disjoint Free Graphs

Let \( \mathcal{G}(n; V_3) \) denote the class of graphs on \( n \) vertices containing no vertex disjoint cycles of length 3. Let

\[
f(n; V_3) = \max \{ \varepsilon(G) : G \in \mathcal{G}(n; V_3) \}.
\]

In this section we determine \( f(n; V_3) \) and characterise the edge maximal members in \( \mathcal{G}(n; V_3) \). We begin with the following construction. Let \( \Omega(n) = K_{\left\lceil \frac{n-1}{2} \right\rceil \left\lceil \frac{n-1}{2} \right\rceil} \).
Observe that $\Omega(n) \subseteq G(n; V_3)$ and the graph $\Omega(n)$ contains $\left\lfloor \frac{(n-1)^2}{4} \right\rfloor + n - 1$ edges. Thus, we have established that

$$f(n; V_3) \geq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + n - 1.$$ 

In the following theorem we establish that equality holds and we determine edge maximum members in $G(n; V_3)$.

**Theorem 2.1** Let $G \in G(n; V_3)$. For $n \geq 10$,

$$f(n; V_3) \leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + n - 1.$$

Furthermore, equality holds if and only if $G = \Omega(n)$.

**Proof:** Let $G \in G(n; V_3)$. Suppose $G$ contains a $K_5$ as a subgraph. Let $x \in V(G - K_5)$, if $x$ is adjacent to $K_5$ by two edges, then $G$ would have two vertex disjoint cycles of length 3. Thus,

$$\varepsilon(G - K_5, K_5) \leq n - 5.$$

Further, observe that $G - K_5$ cannot have cycles of length 3 as otherwise $G$ would have two vertex disjoint cycles of length 3. Thus, by Lemma 1.1, we have

$$\varepsilon(G - K_5) \leq \left\lfloor \frac{(n-5)^2}{4} \right\rfloor.$$

Now,

$$\varepsilon(G) = \varepsilon(G - K_5, K_5) + \varepsilon(G - K_5) + \varepsilon(K_5) \leq n - 5 + \left\lfloor \frac{(n-5)^2}{4} \right\rfloor + 10 < \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + n - 1.$$
for $n \geq 10$. So, we need to consider the second case when $G$ contains no $K_5$. Suppose $G$ contains a $K_4$ as a subgraph. Now, define $A = \{x \in G - K_4 : e(x, K_4) = 3\}$. If $|A| \leq 1$ then, we have
$$e(G - K_4, K_4) \leq 2(n - 4) + 1.$$ Observe that $G - K_4$ contains no cycles of length 3 as otherwise $G$ would have two vertex disjoint cycles of length 3. Thus, by Lemma 1.1, we have
$$e(G - K_4) \leq \left\lfloor \frac{(n - 4)^2}{4} \right\rfloor.$$ Now,
$$e(G) = e(G - K_4, K_4) + e(G - K_4) + e(K_4)$$
$$\leq 2(n - 4) + 1 + \left\lfloor \frac{(n - 4)^2}{4} \right\rfloor + 6$$
$$< \left\lfloor \frac{(n - 1)^2}{4} \right\rfloor + n - 1.$$ for $n \geq 10$. So, we need to consider the case when $|A| \geq 2$. Let $v$ and $w$ be two vertices in $A$. Let $T = G[v, w, K_4]$ and $G_i = G - T$. Let $g \in V(G_i)$, if $g$ is adjacent to $T$ by 4 edges, then $G$ would have two vertex disjoint cycles of length 3. Thus, we have $e(G_i, T) \leq 3(n - 6)$. Observe that $G_i$ cannot have cycles of length 3 as otherwise $G$ would have two vertex disjoint cycles of length 3. Thus, by Lemma 1.1, we have
$$e(G_i) \leq \left\lfloor \frac{(n - 6)^2}{4} \right\rfloor.$$
Now, 
\[ \varepsilon(G) = \varepsilon(G_1, T) + \varepsilon(G_i) + \varepsilon(T) \]
\[ \leq 3(n-6) + \left\lceil \frac{(n-6)^2}{4} \right\rceil + 10 \]
\[ < \left\lceil \frac{(n-1)^2}{4} \right\rceil + n-1. \]
for \( n \geq 10 \). So, we need to consider the case when \( G \) contains no \( K_4 \) as a subgraph.

Suppose \( G \) contains a \( K_3 \) as a subgraph. Let \( T = G[K_3] \) and \( G_i = G - T \). Let \( g \in G_i \), if \( g \) is adjacent to \( T \) by more than 2 edges, then \( G \) would have \( K_4 \) as a subgraph. Thus, we have \( \varepsilon(G_1, T) \leq 2(n-3) \). Observe that \( G_i \) cannot have cycles of length 3 as otherwise \( G \) would have two vertex disjoint cycles of length 3. Thus, by Lemma 1.1, we have
\[ \varepsilon(G_i) \leq \left\lceil \frac{(n-3)^2}{4} \right\rceil. \]

Now, 
\[ \varepsilon(G) = \varepsilon(G_1, T) + \varepsilon(G_i) + \varepsilon(T) \]
\[ \leq 2(n-3) + \left\lceil \frac{(n-3)^2}{4} \right\rceil + 3 \]
\[ = \left\lceil \frac{(n-1)^2}{4} \right\rceil + n-1. \]
So, we need to consider the case when \( G \) contains no cycles of length 3. By Lemma 1.1, we have
\[ \varepsilon(G) \leq \left\lceil \frac{n^2}{4} \right\rceil \]
\[ < \left\lceil \frac{(n-1)^2}{4} \right\rceil + n-1. \]
This completes the proof.
We now characterize the extremal graphs. Through the proof, we notice that the only time we have equality is in case where \( G \) have a \( K_3 \) as a subgraph, \( G - K_3 \) is a complete a bipartite graph \( K_{\frac{n-3}{2}, \frac{n-3}{2}} \) and \( \varepsilon(K_3, G - K_3) = 2(n - 3) \).

This gives rise to the graph \( \Omega(n) = K_{\left\lfloor \frac{n-1}{2} \right\rfloor, \left\lfloor \frac{n-1}{2} \right\rfloor} \).

In the following section we determine edge maximum members in \( G(n; V_5) \).

### 3. Edge-Maximal \( C_5 \)-vertex disjoint Free Graphs

Let \( k \geq 2 \) be a positive integer. Let \( G(n; V_{2k+1}) \) denote the class of graphs on \( n \) vertices containing no \( 2k+1 \) vertex disjoint cycles. Let \( f(n; V_{2k+1}) = \max \{ \varepsilon(G) : G \in G(n; V_{2k+1}) \} \).

In this section, we determine \( f(n; V_5) \) and characterise the edge maximal members in \( G(n; V_5) \). Let \( \Omega(n) = K_{\left\lfloor \frac{n-1}{2} \right\rfloor, \left\lfloor \frac{n-1}{2} \right\rfloor} \). Observe that \( \Omega(n) \subseteq G(n; V_5) \) and the graph \( \Omega(n) \) contains \( \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + n - 1 \) edges. Thus, we have established that

\[
f(n; V_5) \geq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + n - 1.
\]

In this section, we prove that equality holds. In the following theorem we determine edge maximum members in \( G(n; V_5) \).

**Theorem 3.1** Let \( G \in G(n; V_5) \). If \( \delta(G) \geq 40 \), then

\[
f(n; V_5) \leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + n - 1.
\]

Furthermore, equality holds if and only if \( G = \Omega(n) \).
Proof: Suppose $G$ contains no two vertex-disjoint cycles of length 3. Then by the Theorem 2.1, we have
\[ \varepsilon(G) \leq \left\lfloor \frac{(n-1)^2}{4} \right\rfloor + n - 1. \]
So, we need to consider the case when $G$ has at least two vertex-disjoint cycles of length 3. Let $C_3 = x_1, x_2, x_3, x_4$ and $C_3' = y_1, y_2, y_3, y_4$ be two vertex-disjoint cycles of length 3. We consider two cases:

Case 1: $C_3$ form a cycle of length 5 in $G - C_3'$ or $C_3'$ form a cycle of length 5 in $G - C_3$. Without loss of generality, assume $C_3$ form a cycle of length 5 in $G - C_3'$. Let $C_5 = z_1, z_2, z_3, z_4, z_5$ be the cycle of length 5 in $G - C_3'$. Define $H = (G - C_3') - C_5$. Note that the vertices in $G$ have degree more than or equal to 40 in $G$. So, for $j = 1, 2, 3$, let $A_j$ be a set that consist of 4 neighbours of $y_j$ in $H$, selected so that $A_i \cap A_j = \phi$, for $l \neq j$. Let $T_1 = G \left[ \bigcup_{j=1}^{3} y_j, \bigcup_{j=1}^{3} A_j \right]$. The situation as shown below:

Let $u \in V(H)$. If $u$ is adjacent to a vertex in $A_j$, for $j = 1, 2, 3$, then $u$ can not be adjacent to any vertex in $A_{j+2} \cup A_{j-2}$, and to $x_{j+1}$ and $x_{j-1}$. Thus, $\varepsilon(\{u\}, T_1) \leq 5$. Consequently, $\varepsilon(H, T_1) \leq 5(n-20)$. Further, $\varepsilon(T_1, C_5) \leq 25$.
By Theorem 1.2, we have $\varepsilon(T_1) \leq \left\lfloor \frac{13^2}{4} \right\rfloor + 3$. Note that,

$$\varepsilon(H, C_5) + \varepsilon(C_5) \leq 5(n - 20) + 10.$$ Now,

$$\varepsilon(G) = \varepsilon(H) + \varepsilon(H, T_1) + \varepsilon(T_1) + \varepsilon(H, C_5) + \varepsilon(C_5) + \varepsilon(T_1, C_5)$$

$$\leq \left\lfloor \frac{(n - 20)^2}{4} \right\rfloor + 5(n - 20) + \left\lfloor \frac{13^2}{4} \right\rfloor + 3 + 5(n - 20) + 10 + 25 \quad \text{(Lemma 1.1)}$$

$$\leq \left\lfloor \frac{(n - 1)^2}{4} \right\rfloor + n.$$  

**Case 2**: $C_3$ does not form a cycle of length 5 in $G - C_3'$ and $C_3'$ does not form a cycle of length 5 in $G - C_3$.

Define $H = (G - C_3') - C_3$. Note that the vertices in $G$ have degree more than or equal to 40 in $G$. So, for $j = 1, 2, 3$, let $A_j$ be a set that consist of 4 neighbours of $y_j$ in $H$, selected so that $A_i \cap A_j = \emptyset$, for $l \neq j$. Let $T_1 = G \left( \bigcup_{j=1}^{3} y_j, \bigcup_{j=1}^{3} A_j \right)$. The situation is shown below:

Let $u \in V(H)$. If $u$ is adjacent to a vertex in $A_j$, for $j = 1, 2, 3$, then $u$ can not be adjacent to any vertex in $A_{j-2} \cup A_{j-2}$, and to $x_{j-1}$ and $x_{j-1}$. 

[Diagram of graph with vertices $H$, $T_1$, and sets $A_1$, $A_2$, $A_3$]
Thus, \( \varepsilon(\{u\}, T_1) \leq 5 \). Consequently, \( \varepsilon(H, T_1) \leq 5(n - 18) \). Further, \( \varepsilon(T_1, C_3) \leq 15 \).

By Theorem 1.2, we have \( \varepsilon(T_1) \leq \frac{13^2}{4} + 3 \). Note that, \( \varepsilon(H, C_3) + \varepsilon(C_3) \leq 3(n - 20) + 3 \). Now,

\[
\varepsilon(G) = \varepsilon(H) + \varepsilon(H, T_1) + \varepsilon(T_1) + \varepsilon(H, C_3) + \varepsilon(C_3) + \varepsilon(T_1, C_3)
\leq \frac{(n - 18)^2}{4} + 5(n - 18) + \frac{13^2}{4} + 3 + 3(n - 18) + 3 + 15. \tag{Lemma 1.1}
\leq \frac{(n - 1)^2}{4} + n.
\]

This completes the proof.

We now characterize the extremal graphs. Through the proof, we notice that the only time we have equality is in case where \( G \) have a \( K_3 \) as a subgraph, \( G - K_3 \) is a complete a bipartite graph \( K_{\frac{n-3}{2}, \frac{n-3}{2}} \) and \( \varepsilon(K_3, G - K_3) = 2(n - 3) \). This gives rise to the graph \( \Omega(n) = K_{\frac{n-1}{2}, \frac{n-1}{2}} \).

**REFERENCES**


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