Geodetic and Steiner geodetic sets in 3-Steiner distance hereditary graphs

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Abstract

Let \( G \) be a connected graph and \( S \subseteq V(G) \). Then the Steiner distance of \( S \), denoted by \( d_G(S) \), is the smallest number of edges in a connected subgraph of \( G \) containing \( S \). Such a subgraph is necessarily a tree called a Steiner tree for \( S \). The Steiner interval for a set \( S \) of vertices in a graph, denoted by \( I(S) \), is the union of all vertices that belong to some Steiner tree for \( S \). If \( S = \{u, v\} \), then \( I(S) \) is the interval \([u, v]\) between \( u \) and \( v \). A connected graph \( G \) is 3-Steiner distance hereditary (3-SDH) if, for every connected induced subgraph \( H \) of order at least 3 and every set \( S \) of three vertices of \( H \), \( d_H(S) = d_G(S) \). The eccentricity of a vertex \( v \) in a connected graph \( G \) is defined as \( e(v) = \max\{d(v, x) | x \in V(G)\} \). A vertex \( v \) in a graph \( G \) is a contour vertex if for every vertex \( u \) adjacent with \( v \), \( e(u) \leq e(v) \). The closure of a set \( S \) of vertices, denoted by \( I[S] \), is defined to be the union of intervals between pairs of vertices of \( S \) taken over all pairs of vertices in \( S \). A set of vertices of a graph \( G \) is a geodetic set if its closure is the vertex set of \( G \). The smallest cardinality of a geodetic set of \( G \) is called the geodetic number of \( G \) and is denoted by \( g(G) \). A set \( S \) of vertices of a connected graph \( G \) is a Steiner geodetic set for \( G \) if \( I(S) = V(G) \). The smallest cardinality of a Steiner geodetic set of \( G \) is called the Steiner geodetic number of \( G \) and is denoted by \( sg(G) \). We show that the contour vertices of 3-SDH and HHD-free graphs are geodetic sets. For 3-SDH graphs we also show that \( g(G) \leq sg(G) \). An efficient algorithm for finding Steiner intervals in 3-SDH graphs is developed.

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1. Introduction

For graph terminology we follow [6]. All graphs considered here are connected, finite, simple, unweighted and undirected. The distance between a pair of vertices \( u, v \) of \( G \) is the length of a shortest \( u-v \) path (also called a \( u-v \) geodesic) in \( G \) and is denoted by \( d_G(u, v) \) or \( d(u, v) \) if \( G \) is clear from context. We begin with an overview of convexity notions in graphs and discuss how these are related to several invariants, that are the focus of this paper. Moreover, we discuss how questions from graph convexity led to the definition of ‘contour vertices’. For an overview of other abstract convex structures see [20].

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Let $V$ be a finite set and $\mathcal{M}$ a collection of subsets of $V$. Then $\mathcal{M}$ is an alignment of $V$ if and only if $\mathcal{M}$ is closed under taking intersections and contains both $V$ and the empty set. If $\mathcal{M}$ is an alignment of $V$, then the elements of $\mathcal{M}$ are called convex sets and the pair $(V, \mathcal{M})$ is called an aligned space. If $S \subseteq V$, then the convex hull of $S$, denoted by $C H(S)$, is the smallest convex set that contains $S$. Suppose $X \in \mathcal{M}$. Then, $x \in X$ is an extreme point for $X$ if $X - \{x\} \notin \mathcal{M}$. The collection of all extreme points of $X$ is denoted by $\text{ex}(X)$. A convex geometry on a finite set is an aligned space with the additional property that every convex set is the convex hull of its extreme points. This property is referred to as the Minkowski–Krein–Milman (MKM) property. Several abstract convexities associated with the vertex set of a graph are well-known (see [12]). Their study is of interest in Computational Geometry and has some direct applications to other areas such as, for example, Game Theory (see [4]).

The interval between a pair $u, v$ of vertices in a graph $G$ is the collection of all vertices that lie on some $u-v$ geodesic in $G$ and is denoted by $I_G[u, v]$ or $I[u, v]$ if $G$ is understood. Intervals in graphs have been studied extensively (see [3,17,18]) and play an important role in the study of several classes of graphs such as the Ptolemaic graphs (see [16]) or block graphs. A subset $S$ of vertices of a graph is said to be $g$-convex if it contains the interval between every pair of vertices in $S$. It is not difficult to see that the collection of all $g$-convex sets is an alignment of $V$. We thus refer to the $g$-convex sets simply as convex sets. A vertex in a graph is simplicial if its neighbourhood induces a complete subgraph. It can readily be seen that $p$ is an extreme point for a convex set $S$ if and only if $p$ is simplicial in the subgraph induced by $S$. Of course the convex hull of the extreme points of a convex set $S$ is contained in $S$, but equality holds only in special cases. In [12] it is shown that a graph has the MKM property if and only if it has no induced cycles of length bigger than 3 and has no induced 3-fan (see Fig. 1). For another more recent text containing material on graph convexity see [5].

If a graph $G$ has the MKM property and $S$ is a convex set of $V(G)$, then we can rebuild the set $S$ from its extreme vertices using the convex hull operation. This cannot be done with every graph, using only the extreme vertices of a given convex set $S$. In [8] it was shown that the set of extreme vertices of $S$ can be extended to a set that allows us to rebuild $S$ using the vertices in this extended set and the convex hull operation. Let $S$ be a set of vertices in a graph $G$. Then the eccentricity, in $S$, of a vertex $u \in S$ is given by $\text{ecc}_S(u) = \max\{d(u, v) : v \in S\}$ and a vertex $v \in S$ for which $d(u, v) = \text{ecc}_S(u)$ is called an eccentric vertex for $u$ in $S$. In case $S = V(G)$, we denote $\text{ecc}_S(u)$ by $\text{ecc}(u)$. A vertex $u \in S$ is said to be a contour vertex of $S$ if $\text{ecc}_S(u) \geq \text{ecc}_V(u)$ for each neighbour $v$ of $u$ in $S$. The set of all contour vertices of $S$ is called the contour set of $S$ and is denoted by $\text{Ct}(S)$. If $S = V(G)$, the subgraph induced by the contour set of $S$ is called the contour of $G$ and is denoted by $\text{Ct}(G)$. It was shown in [8] that the convex hull of the contour vertices of any convex set in a graph is the set itself.

In order to find the convex hull of a set $S$ one begins by taking the union of the intervals between pairs of vertices of $S$, taken over all pairs of vertices in $S$. We denote this set by $I_G[S]$ or $I[S]$, i.e., $I_G[S] = \bigcup_{[u, v] \subseteq S} I[u, v]$ and call it the geodetic closure of $S$. This procedure is then repeated with the new set and continued until, for the first time, a set $T$ is obtained whose geodetic closure is the set itself, i.e., $T = I[T]$. This set $T$ is the convex hull of $S$. The minimum number of times that the closure operation is applied to get the convex hull of a set $S$ is called the geodetic iteration number of $S$ and is denoted by $\text{gi}(S)$. If $\text{gi}(S) = 1$, we say that the set $S$ is a geodetic set for its convex hull. The notion of a geodetic set for the vertex set of a graph was first defined in [7]. The smallest cardinality of a geodetic set for the vertex set of a graph $G$ is called the geodetic number of $G$ and is denoted by $g(G)$. The problem of finding the geodetic number of a graph is NP-hard [1].

Even though the convex hull of the contour of a graph is the vertex set of the graph, the contour need not be a geodetic set (see [8]). In the same paper the question was posed whether the geodetic iteration number of the contour of any graph is at most 2. This remains an open problem. In this paper we study classes of graphs for which the geodetic iteration number of the contour is 1.

If the contour of a graph is a geodetic set, then the number of contour vertices of the graph is an upper bound on the geodetic number. It was shown in [8] that the contour of every distance hereditary graph is a geodetic set. (A connected
graph $G$ is *distance hereditary* if for every connected induced subgraph $H$ of $G$ and every pair $u, v$ of vertices in $H$, $d_H(u, v) = d_G(u, v)$, see [2,14,15]). In this paper we show that this result can be extended to two larger classes of graphs that contain the distance hereditary graphs. In particular we show that this result holds for the class of ‘3-Steiner distance hereditary graphs’ (defined below) and for the house-hole-domino $HHD$-free graphs. A hole is an induced cycle of length at least 5; a house is a 5-cycle with exactly one chord and a domino is a 6-cycle with exactly one chord that joins two vertices distance 3 apart on the cycle. The $HHD$-free graphs are characterized as those graphs for which every cycle of length at least 5 contains at least two chords (see [5, p. 39]).

Let $G$ be a connected graph and $S$ a set of vertices of $G$. Then the Steiner distance of $S$, denoted by $d_G(S)$ or $d(S)$, is the smallest number of edges in a connected subgraph of $G$ that contains $S$. Such a subgraph is necessarily a tree called a Steiner tree for $S$. For an integer $k \geq 2$, a connected graph $G$ is $k$-Steiner distance hereditary ($k$-SDH) if for every connected induced subgraph $H$ of $G$ and every set $S$ of $k$ vertices of $H$, $d_H(S) = d_G(S)$.

A structural characterization of 3-SDH graphs is given in [11]. Suppose $C: v_1, v_2, \ldots, v_l, v_1$ is a cycle in a graph $G$. An edge of $G$ that joins two vertices of $C$ that are not adjacent on $C$ is called a diagonal or a chord of $C$. Two chords $e_1$ and $e_2$ of $C$ are skew or crossing, if $C + e_1 + e_2$ is homeomorphic to $K_4$.

**Theorem 1.** A graph $G$ is 3-SDH if and only if it is distance hereditary or if the following conditions hold:

1. Every cycle $C: v_1, v_2, \ldots, v_l, v_1$ of length $l \geq 6$
   - (a) has at least two skew diagonals, or, if $l = 6$, then $v_1, v_3, v_5, v_1$ or $v_2, v_4, v_6, v_2$ is a cycle in $\langle V(C) \rangle$ (called an internal triangle) and
   - (b) has no two adjacent vertices neither of which is incident with a diagonal of $C$.
2. $G$ does not contain an induced subgraph isomorphic to any of the graphs of Fig. 2, where any subset of the dotted edges may belong to the subgraph.

The Steiner interval of a set $S$ of vertices in a connected graph $G$, denoted by $I(S)$, is the union of all vertices of $G$ that lie on some Steiner tree for $S$. A set $S$ whose Steiner interval is $V(G)$ is called a Steiner geodetic set of $G$. The smallest cardinality of a Steiner geodetic set of $G$ is called the Steiner geodetic number of $G$ and is denoted by $sg(G)$. It was shown in [19] that in general there is no relationship between $g(G)$ and $sg(G)$, however, for a distance hereditary graph $G$, $g(G) \leq sg(G)$. In the same paper it is shown that contour vertices play an important role in finding the unique smallest Steiner geodetic set of any distance hereditary graph. In Section 3 we show that the relationship between $g(G)$ and $sg(G)$, which held for distance hereditary graphs, extends to 3-SDH graphs. In Section 4 we develop an efficient algorithm for finding Steiner intervals for sets of vertices in 3-SDH graphs.

2. **Contour vertices in 3-SDH and HHD-free graphs**

In order to show that the contour of a 3-SDH and a HHD-free graph is a geodetic set for the graph, we first prove two useful results. The first of these gives another characterization of HHD-free graphs. The second one shows that graphs with certain cycle structures have the property that every vertex has an eccentric vertex that is a contour vertex.
Proposition 1. A graph $G$ is HHD-free if and only if every cycle of length at least 5 does not contain adjacent vertices neither of which is incident with a diagonal.

Proof. Since a house, a hole and a domino all contain a cycle of length at least 5 that has a pair of adjacent vertices neither of which is incident with a diagonal, the necessity of our result follows.

For sufficiency, suppose $G$ is HHD-free. Let $C$ be a cycle of length at least 5. Suppose $C$ contains two adjacent vertices $u$ and $v$ neither of which is incident with a diagonal of $C$. Let $u'$ and $v'$ be the neighbours of $u$ and $v$, respectively, on $C$. Then $u'v'$ is an edge of $G$; otherwise, $G$ contains a hole. If $u'$ and $v'$ have a common neighbour in $V(C)\setminus\{u,v\}$, then $G$ contains a house as induced subgraph, contrary to our assumption. Suppose thus that $u'$ and $v'$ have no common neighbours on $C$. Let $u''$ and $v''$ be the distinct neighbours of $u'$ and $v'$, respectively, in $C\setminus\{u,v\}$. If $u''v'' \in E(G)$, then $\langle\{u,v,u',u'',v''\}\rangle$ is a domino, which is not possible. So suppose $u''v'' \notin E(G)$. Then $C-\{u,v\}$ is a path of order at least 5, say $u_1,u_2,\ldots,u_k$, $k \geq 5$, where $u_1 = u'$, $u_2 = u''$, $u_{k-1} = v''$ and $u_k = v'$. Let $i$ be the smallest integer such that $v''u_i \in E(G)$. Since we are assuming that $u'$ and $v'$ have no common neighbour $i \neq 1$ and since we are also assuming that $u''v'' \notin E(G)$, $i \neq 2$. So $i \geq 3$. Now if $u'u_i \in E(G)$, then by assumption $v'u_i \notin E(G)$ and thus $\langle\{u,v,u',v',u_i\}\rangle$ is a domino. Suppose $u'u_i \notin E(G)$. Let $j$ be the largest integer $2 \leq j < i$ such that $u'uj \in E(G)$. So $u'_i u_j, u_{j+1}, \ldots, u_i, v', v''$ is a cycle of length at least 5. If $v'$ is not adjacent with any $u_j$ $(j+1 \leq l \leq i)$, then $G$ contains a hole. If $v'u_{j+1} \in E(G)$, then $\langle\{u,v,u',v',u_j,u_{j+1}\}\rangle$ induces a domino. If $v'u_{j+1} \notin E(G)$ but $v'u_i \in E(G)$ for $j+1 < l \leq i$, then $G$ again contains a hole, which is not possible. □

Lemma 1. Suppose $G$ is a graph with the property that every cycle of length at least 6 does not have adjacent vertices neither of which is incident with a diagonal. Then every vertex of $G$ has an eccentric vertex which is a contour vertex.

Proof. Suppose this is not the case. Among all vertices that do not have an eccentric vertex that is contour let $v$ be one of largest eccentricity. Let $v(e)$ be an eccentric vertex for $v$ of largest eccentricity. Since $v(e)$ is not a contour vertex, it is adjacent with some vertex $u$ such that $\text{ecc}(u) > \text{ecc}(v(e))$. By our choice of $v(e)$, $u$ cannot be an eccentric vertex for $v$. So $d_G(v,u) = d_G(v,v(e)) - 1$. Hence, $u \in I[v,v(e)]$. Let $P$ be a $v-v(e)$ geodesic that contains $u$, say $P : v^{(e)}, u, v_1, v_2, \ldots, v_k = v$.

Since $\text{ecc}(u) > \text{ecc}(v(e)) \geq \text{ecc}(v)$, by our assumption about $v$, $u$ has an eccentric vertex $u(e)$ which is a contour vertex. Let $Q : u, v^{(e)}, u_1, u_2, \ldots, u_i (=u(e))$ be a $u-u(e)$ geodesic containing $v(e)$. We now show $V(P) \cap V(Q) = \{u, v^{(e)}\}$. Suppose $u_j = v_l$ for some $i$ and $j$. Since $d_G(u,v_i) = i$ and $d_G(u,u_j) = j+1$, we have $i = j+1$. Also, since $d_G(v^{(e)},u_j) = j$ and $d_G(v^{(e)},u_i) = i+1$, we have $j = i+1$, which is not possible.

Note that $u$ and $v(e)$ both have degree 2 in the subgraph induced by $V(P) \cup V(Q)$. We now show that if $u_iv_j \in E(G)$ then $i = j$ for $1 \leq i < k$ and $1 \leq j < l$.

Suppose $u_iv_j \in E(G)$. Since $d_G(v^{(e)},v_j) = j+1$ and as $v(e), u_1, u_2, \ldots, u_i, v_j$ is a $v^{(e)}-v_j$ path of length $i+1$, it follows that $j+1 \leq i + 1$ or $j \leq i$. Also, $d_G(u,u_i) = i+1$. Since $u, v_1, v_2, \ldots, v, u_i$ is a $u-u_i$ path of length $j+1$, we have $i+1 \leq j + 1$ or $i \leq j$. Thus, $i = j$.

If $u_iv_i \in E(G)$ for $i \geq 2$, then we have a cycle of length at least 6 containing two adjacent vertices, namely $u$ and $v(e)$, neither of which is incident with a chord; contrary to hypothesis.

We may assume that $u_iu_j \notin E(G)$ for $i \geq 2$. Consider the distance from $v$ to $u(e)$. Since $\text{ecc}(u(e)) > \text{ecc}(v(e))$, it follows from our choice of $v(e)$ that $d = d(v,u(e)) < \text{ecc}(v) = k + 1$. Since $l+1 = \text{ecc}(u) > \text{ecc}(v(e)) \geq \text{ecc}(u) = k+1$, we have $l > k$. Any vertex of eccentricity 1 has an eccentric vertex which is contour, so $\text{ecc}(v) \geq 2$. If $\text{ecc}(v) = 2$, then $v, u, u(e)$ is a $v-u(e)$ geodesic and by our choice of $v(e), vu(e) \in E(G)$. Then $d(u,u(e)) \leq 2$, which is not possible since $\text{ecc}(u) = d(u,u(e)) > \text{ecc}(v) = 2$. Hence, $\text{ecc}(v) \geq 3$. So $l \geq 3$.

Let $i$ be the smallest integer such that a $v-u(e)$ geodesic contains $v_i$. So $-i+1$ is a maximum number of vertices that a $v-u(e)$ geodesic can have in common with $P$. Then $u, v_1, v_2, \ldots, v_i$ together with the $v_i-u(e)$ subpath of a $v-u(e)$ geodesic containing $v_i$ is a $u-u(e)$ path of length $i + (d-k+i)$ and thus has length at least $l+1$. Since $d < k+1$, it follows that $2i+1 > l+1$ or $i > l/2$. Thus, $i \geq 2$.

Let $\mathcal{S}$ be the collection of all $v-u(e)$ geodesics containing $v_i$. Let $j$ be the smallest integer such that $u_j$ belongs to some path in $\mathcal{S}$. Let $R$ be a $v-u(e)$ geodesic containing both $v_i$ and $u_j$. We may assume that $R$ begins with $(v)e=v_k, v_{k-1}, \ldots, v_i$ and ends with $u_j, u_{j+1}, \ldots, u_i (=u(e))$. Let $R' : (v_i)v_0, u_1, \ldots, u_j (=u_j)$ be the $v_i-u_j$ subpath of $R$. Then $R'$ is a $v_i-u_j$ geodesic. By our choice of $i$ and $j$ and since $i \geq 2$, the path $v_i, v_{i-1}, \ldots, v_1, u, v(e), u_1, u_2, \ldots, u_j$ together...
with \( R' \) produces a cycle of length at least 6. (Note \( s \geq 2 \), since \( v_iv_j \notin E(G) \) for \( i \neq j \) and for \( i, j \geq 2 \).) Since \( k - i + s + l - j = d < k + 1 \) we have \( k - i + s < k + 1 \). So \( s \leq i \). Also if equality holds, then \( l = j \).

If \( u \) (or \( v^{(e)} \)) is adjacent to \( w_p \) for \( 1 \leq p \leq s - 2 \leq i - 2 \), then \( v = v_k, v_{k-1}, \ldots, v_1, w_1, w_2, \ldots, w_p, u, v^{(e)} \) (or \( v = v_k, v_{k-1}, \ldots, v_1, w_1, w_2, \ldots, w_p, v^{(e)} \), respectively) is a \( v - v^{(e)} \) path of length at most \( k - i + i - 2 + 2 = k \), which is impossible. Similarly, if \( v^{(e)} \) is adjacent to \( w_{s-1} \) or if \( s < i \) and \( u \) is adjacent to \( w_{s-1} \), then we have a \( v - v^{(e)} \) path of length at most \( k < \text{ecc}(v) \). If \( s = i \), then \( l = j \) and \( w_3 = u^{(e)} \). If \( u \) is adjacent to \( w_{s-1} \), then \( u, w_{s-1}, w_3 \) is a \( u - u^{(e)} \) path of length \( 2 < \text{ecc}(u) \), which is impossible.

Thus, we have a cycle of length at least 6 with two adjacent vertices \( u \) and \( v^{(e)} \), neither of which is on a diagonal. This contradicts the hypothesis. \( \square \)

Suppose now that \( v = v_0 \) is any vertex in a connected graph. If \( v_0 \) is not a contour vertex, then \( v_0 \) is adjacent with a vertex \( v_1 \) such that \( \text{ecc}(v_1) > \text{ecc}(v_0) \). Moreover, if \( v_1^{(e)} \) is an eccentric vertex for \( v_1 \), then \( v_1^{(e)} \) is an eccentric vertex for \( v_0 \) and there is a \( v_1 - v_1^{(e)} \) geodesic containing \( v_0 \). If \( v_1 \) is not a contour vertex, then \( v_1 \) is adjacent with a vertex \( v_2 \) such that \( \text{ecc}(v_2) > \text{ecc}(v_1) \). Moreover, if \( v_2^{(e)} \) is an eccentric vertex for \( v_2 \), then \( v_2^{(e)} \) is an eccentric vertex for \( v_1 \) and there is a \( v_2 - v_2^{(e)} \) geodesic containing \( v_1, v_0 \). Continuing in this manner we construct a sequence \( v_0, v_1, \ldots, v_k \) of vertices such that \( \text{ecc}(v_0) < \text{ecc}(v_1) < \cdots \). This process must terminate with some vertex \( v_k \) that is necessarily a contour vertex of \( G \). Moreover, if \( v_k^{(e)} \) is an eccentric vertex for \( v_k \), then there is a \( v_k - v_k^{(e)} \) geodesic that contains the path \( v_k, v_{k-1}, \ldots, v_1, v_0 \). We call the sequence \( v_0, v_1, \ldots, v_k \) a backtrack sequence for \( v_0 \).

Observe that if \( G \) is a 3-SDH graph, then \( G \) has the property that every cycle of length at least 6 does not have a pair of adjacent vertices neither of which is incident with a diagonal. This is certainly the case if \( G \) is distance hereditary since distance hereditary graphs are precisely those graphs that have the property that every cycle of length at least 5 has a pair of crossing diagonals. If \( G \) is not distance hereditary, then this observation follows from part 1(b) of Theorem 1. With the aid of Lemma 1 the next result can easily be established.

**Theorem 2.** If \( G \) is a 3-SDH graph, then the contour of \( G \) is a geodetic set.

**Proof.** Let \( v = v_0 \) be any vertex of \( G \) and suppose \( v_0, v_1, \ldots, v_k \) is a backtrack sequence for \( v \). Then, by Lemma 1, \( v_k \) has an eccentric vertex \( v_k^{(e)} \) that is a contour vertex. From the above we know that there is a \( v_k - v_k^{(e)} \) geodesic that contains \( v \). The result now follows. \( \square \)

**Theorem 3.** If \( G \) is a HHD-free graph, then the contour of \( G \) is a geodetic set.

**Proof.** This follows as for 3-SDH graphs using the characterization of HHD-free graphs given in Proposition 1. \( \square \)

3. Geodetic and Steiner geodetic numbers in 3-SDH graphs

We show here that the Steiner geodetic number is an upper bound for the geodetic number for 3-SDH graphs.

**Theorem 4.** If \( G \) is a 3-SDH graph and \( S \subseteq V(G) \), then \( I(S) \subseteq I[S] \).

**Proof.** In [19] it was shown that if \( G \) is distance hereditary and \( S \subseteq V(G) \), then \( I(S) \subseteq I[S] \). For the remainder of the proof we assume that \( G \) is a 3-SDH graph that is not distance hereditary. If \( |S| = 2 \), the result is immediate. Suppose thus that \( |S| \geq 3 \).

Let \( v \in I(S) \). If \( v \in S \), then \( v \in I[S] \). Suppose \( v \in I(S) \setminus S \). Then there is some Steiner tree \( T \) of \( S \) that contains \( v \). Let \( H = (V(T)) \). Then \( |V(H)| - 1 = d_G(S) \). Vertex \( v \) must be a cut-vertex of \( H \); otherwise, \( H - v \) is a connected subgraph of \( G \) that contains \( S \) and has smaller order than \( T \) which is not possible.

Each component \( C \) of \( H - v \) contains at least one vertex of \( S \); otherwise, the removal of \( V(C) \) from \( H \) produces a connected subgraph of \( H \), of smaller order than \( H \), containing \( S \). This is not possible since \( H \) is a connected subgraph of \( G \) of smallest order containing \( S \). Since \( |S| \geq 3 \), there are three vertices \( x, y, z \) such that \( y, z \) do not belong to the same component as \( x \) in \( H - v \). Let \( T' \) be a Steiner tree for \( S' = \{x, y, z\} \) in \( H \). Then \( T' \) has at most three leaves and is thus either a path or homeomorphic to \( K_{1,3} \). In the latter case the paths beginning at the vertex of degree 3 and terminating
at a leaf must be geodesics, otherwise $T'$ is not a Steiner tree for $\{x, y, z\}$. Let $P$ be the $x$-$y$ path in $T'$. Since $G$ is 3-$SDH$, $|E(T')| = d_H(S') = d_G(S')$. Moreover, as $v$ is on every $x$-$y$ path in $H$, $v$ is on $P$ and hence in $T'$. The $x$-$v$ path in $T'$ is necessarily a $x$-$v$ geodesic; otherwise, $T'$ is not a Steiner tree for $S'$.

We will show that $v$ lies on a geodesic between some pair of vertices of $S'$.

Suppose $yz \in E(G)$ and that $d_{T'}(y, x) \leq d_{T'}(z, x)$. In this case, $P$ is necessarily an $x$-$y$ geodesic; otherwise, an $x$-$y$ geodesic together with $z$ and the edge $yz$ produces an $S'$-tree of smaller size than $T'$, which is not possible. So $d_G(y, z) \geq 2$.

Suppose that $P$ is not an $x$-$y$ geodesic containing $v$.

**Claim 1.** Let $Q$ be an $x$-$y$ geodesic. Then $P$ and $Q$ are internally disjoint.

**Proof of Claim 1.** Suppose $P$ and $Q$ have an internal vertex $u$ in common. Then $Q$ must be a Steiner tree for $\{x, u, y\}$ since it is an $x$-$y$ geodesic. Since $G$ is 3-$SDH$, $(V(T'))$ must contain a Steiner tree for $\{x, u, y\}$ which necessarily contains $v$ and has the same size as an $x$-$y$ geodesic. So $v$ is on an $x$-$y$ geodesic in this case. □

**Claim 2.** $(V(T'))$ contains an $x$-$y$ path $T''$ that passes through $v$ and the $x$-$v$ and $v$-$y$ paths in $T'$ are geodesics.

**Proof of Claim 2.** The subgraph $(V(P))$ contains $\{x, v, y\}$ and, since $G$ is 3-$SDH$, it contains a Steiner tree $T''$ for $\{x, v, y\}$. Since $v$ is a cut vertex of such a tree, $T''$ is an $x$-$y$ path. Thus, the $x$-$v$ and $v$-$y$ paths in $T''$ are necessarily geodesics. This completes the proof of Claim 2. □

**Case 1.** $d(x, v), d(y, v)$, and $d(z, v)$ are each at most 2.

Suppose first that two of $x, y, z$ are adjacent to $v$. If $x, y$ are both adjacent with $v$, then $x, v, y$ is a geodesic since $xy \notin E(G)$. Similarly if $x, z$ are both adjacent with $v$, then $x, v, z$ is a geodesic containing $v$. Suppose now that $x$ is not adjacent with $v$ and that $z, y$ are both adjacent with $v$. If $yz \notin E(G)$, then $y, v, z$ is a geodesic containing $v$. Suppose thus that $yz \in E(G)$. Now if $x$ and $y$ have a common neighbour $a$, then the path $z, y, a, x$ is a tree containing $\{x, y, z\}$ and having fewer edges than $T'$ which is not possible. Similarly $x$ and $z$ have no common neighbour. Thus $d_G(x, y) \geq 3$.

Thus the $x$-$v$ path of $T'$ followed by the edge $vy$ is necessarily an $x$-$y$ geodesic. We may thus assume that at most one of $x, y, z$ is adjacent to $v$.

**Case 1A.** $xv \in E(G)$.

Then, by the case we are considering, $d_G(y, v) = d_G(z, v) = d_G(x, v) = 2$. Suppose the $v$-$z$ and $v$-$y$ paths in $T'$ have only $v$ in common. If $v$ is not on an $x$-$z$ or an $x$-$y$ geodesic, then $d_G(x, z) = 2$ and $d_G(x, y) = 2$. However, then $T'$ is not a Steiner tree for $\{x, y, z\}$, since the union of an $x$-$z$ geodesic and an $x$-$y$ geodesic produces a connected graph containing $\{x, y, z\}$ but having fewer edges than $T'$.

We may assume that the $v$-$z$ and $v$-$y$ paths in $T'$ have a vertex $c$ in common, so $vc, cz, cy \in E(T')$. If $v$ is not on an $x$-$z$ or an $x$-$y$ geodesic, then $d_G(x, z) = d_G(x, y) = 2$. If some $x$-$z$ geodesic and some $x$-$y$ geodesic share a common internal vertex, then $T'$ is not a Steiner tree. We may assume that an $x$-$z$ geodesic is $x, a, z$ and an $x$-$y$ geodesic is $x, b, y$ where $a \neq b$ (see Fig. 3).

Consider the 6-cycle $x, b, y, c, z, a, x$. Since $v$ is a cut-vertex in $T'$, $x$ is not adjacent to $z, c, or y$. If $ay$ or $bz \in E(G)$, then $T'$ is not a Steiner tree. We know $zy \notin E(G)$. Therefore, the only possible chords are $ab, bc, ca$.

By Theorem 1 a 6-cycle in a 3-$SDH$ graph has either two skew diagonals or an internal triangle such as $a, b, c$ above. Thus, we may assume that all three chords are present. We have an induced subgraph as shown in Fig. 4.

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Fig. 3. Case 1A. If $v$-$z$ and $v$-$y$ paths have vertex $c$ in common.
The vertex $v$ is adjacent to $c$ and $x$ and not adjacent to $z$ and $y$. If $v$ is adjacent to $a$ or $b$, then $\langle\{a, b, c, v, z, y\}\rangle$ is isomorphic to a forbidden subgraph shown in (c) of Theorem 1; otherwise $v$ is adjacent to neither $a$ nor $b$ and $\langle\{a, b, c, v, x, y\}\rangle$ is also isomorphic to a forbidden subgraph (c) of Theorem 1 (see Fig. 5).

In any case, we have a contradiction to the fact that $G$ is 3-SDH.

Case 1B. $xv \notin E(G)$.

In this case, $d_G(x, v) = 2$ and at least one of $d_G(v, y)$ and $d_G(v, z)$ is equal to 2. Notice that $T'$ has at least 5 edges. If $v$ is not on a $x$–$y$ geodesic, then $d_G(x, y) < d_G(x, v) + d_G(v, y) \leq 4$, so $d(x, y) \leq 3$. Similarly, if $v$ is not on an $x$–$z$ geodesic, then $d(x, z) \leq 3$. If $d_G(x, y) = 2$ and $d_G(x, z) = 2$, then the union of the $x$–$y$ and $x$–$z$ geodesics forms an $\{x, y, z\}$-tree with 4 edges, and $T'$ is not a Steiner tree. We may assume that one of $y$ and $z$, say $y$, is distance 3 from $x$ and at distance 2 from $v$.

Consider the $x$–$y$ path $P$ in $T'$, say $x, a, v, b, y$ and an $x$–$y$ geodesic $Q : x, c, d, y$. By Claim 1, $P$ and $Q$ are internally disjoint. They thus produce a 7-cycle. We know $v$ is not adjacent to $x$ or $y$ and $x$ is not adjacent to $y$. By Claim 1, $a$, $v$, and $b$ are not on any $x$–$y$ geodesic, so $ab, ad, and cb \notin E(G)$. Since $d_G(x, y) = 3, ay, bx, xd$, and $cy \notin E(G)$. The only possible chords are $ac, cv, vd, and db$ (see Fig. 6). According to Theorem 1, a 7-cycle in a 3-SDH graph must have skew diagonals, so we have a contradiction.

Case II. At least one of $d_G(x, v), d_G(y, v)$, and $d_G(z, v)$ is at least 3. We may assume that either $d_G(x, v)$ or $d_G(y, v)$ is at least 3.
Let $T''$ be the $x$-$y$ path of Claim 2 and $Q$ any $x$-$y$ geodesic. Suppose $d_G(x, v) \geq 3$. If $P$ is not an $x$-$y$ geodesic, then by Claim 1, $T''$ and $Q$ are internally disjoint. Let $T'' : (x=) x_0, x_1, \ldots, x_k(=v), x_{k+1}, \ldots, x_l(=y)$. Let $Q : (x=) y_0, y_1, \ldots, y_m(=y)$. Let $C$ be the cycle formed by taking the union of $T''$ and $Q$. Since $T''$ is a Steiner tree for $\{x, v, y\}$, both the $x$-$v$ and $v$-$y$ subpaths of $T''$ are geodesics and hence have no chord. Moreover, since $v$ is a cut-vertex of $H$, no vertex of the $x$-$v$ path in $T''$ is adjacent with any vertex of the $v$-$y$ path in $T''$. Hence, $T''$ has no chords. Also, since $Q$ is an $x$-$y$ geodesic, it has no chords. For $1 \leq i \leq m-2$, there is no edge between $x_i$ and $y_j$ for $i < j \leq m - 1$; otherwise, we have a contradiction to Claim 1 or $Q$ is not an $x$-$y$ geodesic. Also, $y_j$ is not adjacent with $x_i$ for $j + 2 \leq i \leq k$; otherwise, the $x$-$v$ path in $T''$ is not a geodesic. So the only possible edges between $(x_1, x_2, \ldots, x_k)$ and $(y_1, y_2, \ldots, y_m)$ are $x_i y_j (1 \leq i \leq k)$ and $x_{j+1} y_j (1 \leq j \leq k - 1)$.

Since we assume $k \geq 3$ it follows that $l \geq 4$. Also $m \geq 3$; otherwise, $T'$ is not a Steiner tree for $\{x, y, z\}$ (since a connected graph containing $x, y$ and $z$ and having fewer vertices than $T'$ can be produced by deleting the internal vertices of the $x$-$v$ path in $T'$ and adding a $x$-$y$ geodesic).

Let $i \geq k + 1$ be the smallest integer such that $x_i y_s$ is an edge for some $s$, $1 \leq s < m$. Note that $i$ may be $l$. Among all such integers $s$ let $j$ be the smallest one. Then $x_0, x_1, \ldots, x_j, y_j, y_{j-1}, \ldots, y_0$ is a cycle of length at least 6 with out crossing diagonals. This is not possible since $G$ is 3-SDH.

**Corollary 1.** If $G$ is a 3-SDH graph, then $g(G) \leq sg(G)$.

**Proof.** This follows from Theorem 4 since every Steiner geodetic set is a geodetic set. □

4. An algorithm for finding Steiner intervals in 3-SDH graphs

Goddard [13] showed, that if a graph is $k$-SDH, then it is $t$-SDH for all $t \geq k$. The following algorithm, see [10], finds the Steiner distance of a set $S$ of vertices in a $k$-SDH graph. We use this algorithm to find the Steiner interval for $S$. We say that a set $S$ of vertices of a graph $G$ is separated in an induced subgraph $H$ of $G$ that contains $S$ if the vertices of $S$ do not belong to the same component of $H$.

**Algorithm to find the Steiner distance of a set $S$, of at least three vertices, in a 3-SDH graph.**

- label $V(G) \setminus S$ in arbitrary order $v_1, v_2, \ldots, v_m$
- $G_1 = G$
- for $i = 1$ to $m$
  - if $S$ is separated in $G_i - v_i$
    - then $G_{i+1} \leftarrow G_i$
    - else $G_{i+1} \leftarrow G_i - v_i$
- $d_G(S) = |V(G_m+1)| - 1$

**Theorem 5.** If $G$ is a 3-SDH graph and if $v_m$, in the final step of the algorithm above, does not separate $S$ in $G_m$, then $v_m \notin I(S)$.

**Proof.** Suppose, to the contrary, that $v_m$ does not separate $S$ in $G_m$ but $v_m \in I(S)$. Then $v_m$ is in a Steiner tree for $S$ in $G = G_1$.

Suppose first that $v_m$ is in a Steiner tree $T$ for $S$ in $G_m$. Since $v_m$ does not separate $S$ in $G_m$, $G_m$ must also contain some $v_j \notin V(T)$ where $j < m$. But then, in the Algorithm when $v_j$ is considered, $T \subseteq G_j$ and $v_j$ would not separate $S$ in $G_j - v_j$. Hence, $v_j$ would have been removed. We may assume that there is no Steiner tree for $S$ in $G_m$ containing $v_m$.

Let $j$ be the minimum index so that $v_m$ is in a Steiner tree for $S$ in $G_j$ but not in $G_{j+1}$. Necessarily, $G_{j+1} = G_j - v_j$. Also since $v_m \notin S$, $v_m$ is necessarily an internal vertex of every Steiner tree for $S$ in $G_j$. Define $S' = S \cup \{v_m\}$. Then $d(S') = d(S)$. We know $S$ is not separated in $G_j - v_j$, so if $S'$ is separated, that would mean that $v_m$ is in a different component than any $w \in S$. But then any $v_m - w$ path in $G_j$ goes through $v_j$; this contradicts the fact that $G_j$ contains a Steiner tree for $S$ containing $v_m$ as internal vertex. So without loss of generality, $S'$ is not separated in $G_j - v_j$. Since $G$ is 3-SDH, this means $d_G(v_m - v_j) = d_G(S)$. Since $G_j$ contains a Steiner tree for $S$ with $v_m$ in it, $d_G(S') = d_G(S)$. But then $d_G(-v_j) = d_G(S) = d_G(S) = d_G(S)$. So $G_j - v_j$ contains a Steiner tree for $S'$ in $G_j$ with size $d_G(S)$, which must be a Steiner tree for $S$. This is a contradiction. □
Thus, in order to find $I(S)$ in a 3-SDH graph, we may perform the given algorithm $m$ times, with each of the vertices of $V(G) \setminus S$ in turn as the last vertex $v_m$ in the sequence of vertices input in the algorithm. If $v_m$ does not separate $S$ in $G_m$, then it is not in $I(S)$; otherwise, it is.

5. Conclusion

The condition that every contour has an eccentric vertex that is contour is sufficient to guarantee that the contour of a graph is a geodetic set. However it is not necessary. The graph of Fig. 7 has contour vertices, namely $v_3$ and $v_7$ whose unique eccentric vertices $v_6$ and $v_4$, respectively, are not contour vertices.

This example illustrates that the techniques used in the proof of Theorems 2 and 3 cannot be applied to bipartite graphs and thus not for any class of graphs that contain the bipartite graphs such as the parity graphs and the Gallai graphs.

Note however that the graph of Fig. 7 still possesses the property that its contour vertices form a geodetic set. Not all perfect graphs possess this property as was shown in [9]. It remains an open problem to characterize graphs for which the contour is a geodetic set and to determine the subclass that possess the property that every contour vertex has an eccentric vertex that is a contour vertex.

References