In this paper, the theory of flow barriers in discontinuous dynamical systems is systematically presented as a new theory for the first time, which helps one rethink the existing theories of stability and control in dynamical systems. The concept of flow barriers in discontinuous dynamical systems is introduced, and the passability of a flow to the separation boundary with flow barriers is presented. Because the flow barriers exist on the separation boundary, the switchability of a flow to such a separation boundary is changed accordingly. The coming and leaving flow barriers in passable flows are discussed first, and the necessary and sufficient conditions for a flow to pass through the boundary with flow barrier are developed. Flow barriers for sink and source flows are also discussed. Once the sink flow is formed, the boundary flow will exist. When the boundary flow disappears from the boundary, the boundary flow barrier on the boundary may exist, which is independent of vector fields in the corresponding domains. Thus, the necessary and sufficient conditions for formations and vanishing of the boundary flow are developed. A periodically forced friction model is presented as an example for a better understanding of flow barrier existence in physical problems. The flow barrier theory presented in this paper may provide a theoretic base to further develop control theory and stability.

Keywords: Discontinuous systems; flow barriers; passable flows; sink flows; source flows; boundary flows; friction-oscillator with static friction; stick motion.
1. Introduction

Before discontinuous dynamical systems are discussed, continuous dynamical systems should be briefly reviewed. In the 17th century, Newton summarized the previous results and observations in mechanics (e.g., Kepler and Galileo) and proposed the three motion laws based on qualitative mechanics. For a better description of the motion of an object, the Lipschitz condition has been adopted. Based on such conditions, calculus was developed to provide a base for modern science. The quantitative theory of mechanics (especially smooth dynamics) has been systematically developed. The stability of dynamical systems was one of the central issues, which was discussed through the series expansion techniques.

At the end of the 19th century, Poincaré [1892] presented a qualitative, geometric method to investigate the stability of dynamical systems through differential geometry and topology instead of the traditional calculus, differential equations and variational theory. The three-body problem in celestial mechanics is one of the origins to attract much interest and to develop the fundamental theories and methodologies in dynamics. The ordinary differential equation theory in the real domain is at the core of an important role in dynamical systems. In the 20th century, one followed the Poincaré’s idea to develop and apply the qualitative theory to understand the complexity in dynamical systems. In the 20th century, one followed the Poincaré’s idea to develop and apply the qualitative theory to understand the complexity in dynamical systems. Lyapunov [1907] developed the Lyapunov direct method based on differential equations rather than the potential energy or an energy-alike quantity in general. Birkhoff [1913, 1927] pushed further the development and applications of Poincaré’s qualitative theory. The Taylor series expansion and perturbation analyses play a central role in qualitative and quantitative analyses. However, the Taylor series expansion analysis is valid in the finite domain under certain convergent conditions, and the perturbation analysis based on the small parameters, as an approximate estimate, is only acceptable for a very small domain during a short time period.
simulations, one observed complex motion characteristics which cannot be analyzed by the traditional, analytical tools. To investigate the complex motion of continuous dynamical systems, the catastrophe and bifurcation theories have further been developed to explain strange phenomena in nonlinear dynamic systems (e.g., [Arnold, 1989; Guckenheimer & Holmes, 1983]).

However, one wants to develop the expected dynamic behaviors to satisfy specific requirements. Hence, discontinuous constraints destroying the Lipschitz conditions are added to dynamic systems. The early investigation of discontinuous systems in mechanical engineering can be found in 1930’s (e.g., [Den Hartog, 1930, 1931; Den Hartog & Mikina, 1932]). For instance, smooth linear dynamical systems with periodic impacting (e.g., [Masri & Caughey, 1966; Shaw & Holmes, 1983a; Luo & Han, 1996]) have complicated dynamical behaviors which are unpredictable from the traditional dynamical theories. The Lipschitz condition cannot be satisfied for such problems. For piecewise linear systems, Levinson [1949] used a piecewise linear system to investigate the periodically excited Van der Pol equation, and found infinitely many periodic solutions which cannot be perturbed away. Further results for this piecewise model of the Van der Pol equation were presented in [Levi, 1978, 1981].

Shaw and Holmes [1983b] used mapping techniques to investigate the chaotic motion of a piecewise linear system with a single discontinuity. Natsiavas [1989] investigated the periodic motion and stability for a system with a symmetric, tri-linear spring. Nordmark [1991] introduced the grazing mapping to investigate nonperiodic motion. Kleczka et al. [1992] investigated the periodic motion and bifurcations of piecewise linear oscillator motion, and numerically observed the grazing motion. Leine and Campen [2002] investigated the discontinuous bifurcations of periodic solutions through the Floquet multipliers of periodic solutions. The analytical prediction of periodic responses of piecewise linear systems was presented (e.g., [Luo & Menon, 2004; Menon & Luo, 2005]). Normal formal mapping for piecewise smooth dynamical systems with/without sliding were discussed (e.g., [di Barnardo et al., 2001; di Barnardo et al., 2002]). In 1983, Chua introduced a circuit system for exhibiting a variety of bifurcation and chaos [Chua, 1992]. Matsumoto [1984] proved the existence of chaos in the Chua circuit system, and Zhong and Ayrom [1984] experimentally observed the chaotic phenomena in the Chua circuit system. Chua et al. [1986] presented a strict proof of chaos existence. Recently, Luo and Xue [2009] used the theory of discontinuous dynamical systems to give an analytical prediction of periodic flows in the Chua circuit.

For discontinuous dynamical systems, Filippov [1964] presented differential equations with discontinuous right-hand sides through the Coulomb friction oscillator. To determine the sliding motion along the discontinuous boundary, the differential inclusion was introduced via the set-valued analysis, and the existence and uniqueness of the solution for such a discontinuous differential equation were discussed. Since the discontinuity exists widely in engineering and control systems, one applied the Filippov’s concept to control dynamical systems. Aizerman and Pyatnitski [1974a, 1974b] extended the Filippov’s concepts to develop a generalized theory for discontinuous systems. Utkin [1978] presented sliding modes and the corresponding variable structure systems, and the theory of automatic control systems described with variable structures and sliding motions was also developed in [Utkin, 1981]. DeCarlo et al. [1988] gave a review on the development of the sliding mode control. Filippov [1988] systematically presented a geometrical theory of the differential equations with discontinuous right-hand sides, and the local singularity theory of the discontinuous boundary was discussed qualitatively. Ye et al. [1988] discussed the stability theory for hybrid systems. From geometrical points of view, Broucke et al. [2001] investigated structural stability of piecewise smooth systems. The aforementioned theories are based on the Filippov’s theory. However, the Filippov’s theory mainly focused on the existence and uniqueness of the solutions for nonsmooth dynamical systems. The differential inclusion was introduced. The local singularity of a flow to the separation boundary was not discussed. Such a differential inclusion theory with discontinuity is still difficult to use for determining the complexity of nonsmooth dynamical systems. Luo [2005a] discussed the local singularity of nonsmooth dynamical systems on connectable domains, and Luo [2005b] introduced the imaginary, sink and source flows to determine the sliding and source flows in nonsmooth dynamical systems. From such real and imaginary flows, the discontinuity and singularity in discontinuous systems can be easily described. The detailed discussion of the local singularity in discontinuous dynamical systems was given in [Luo, 2006, 2008a, 2008b]. Luo
and Gegg [2006a] used the local singularity theory to develop the force criteria for the harmonically driven linear oscillator with dry-friction. Luo and Gegg [2006b] analytically investigated periodic motions in such an oscillator. From differential geometry points of view, Luo [2008a, 2008b] introduced the G-function to measure the local singularity, and presented a theory for flow switchability in discontinuous dynamical systems.

In the above mentioned theories, the discontinuity in discontinuous dynamical systems is based on different vector fields. Once flow barriers exist on the boundary, the current theories for discontinuous dynamical systems cannot be used. Luo [2007] introduced the flow barriers in discontinuous dynamical systems, and Luo and Zwiegel [2008] used such a flow barrier theory to investigate the friction-induced oscillator with static friction. For a better understanding of flow switchability, the flow barriers and the corresponding flow passability in discontinuous systems will be systematically presented herein.

2. Discontinuous Dynamical Systems

As in [Luo, 2005a, 2006, 2008a, 2008b], consider a dynamic system consisting of $N$ subdynamic systems in a universal domain $\Omega \subset \mathbb{R}^n$. The accessible domain in phase space means that a continuous dynamical system can be defined on such a domain. The inaccessible domain in phase space means that no dynamical system can be defined on such a domain. The universal domain in phase space is divided into $N$ accessible subdomains $\Omega_\alpha$ plus the inaccessible domain $\Omega_0$. The union of all the accessible subdomains is $\bigcup_{\alpha=1}^{N} \Omega_\alpha$, and the universal domain is $\Omega = \bigcup_{\alpha=1}^{N} \Omega_\alpha \cup \Omega_0$, which can be expressed through two $n_1$-dimensional and $n_2$-dimensional subvectors $x_{n_1}$ and $x_{n_2}$ ($n_1 + n_2 = n$). $\Omega_0$ is the union of the inaccessible domains, which is the complement of the union of the accessible subdomain (i.e. $\Omega_0 = \Omega \setminus \bigcup_{\alpha=1}^{N} \Omega_\alpha$). If all the accessible domains are connected, the universal domain in phase space is called the connectable domain. If the accessible domains are separated by the separable domain, the universal domain is called the inaccessible domains, as presented in Fig. 1. To investigate the relation between the flows on two disconnected domains without any common boundary, the transport laws should be employed. Such an issue can be referred to [Luo, 2006]. The flow switchability in discontinuous dynamical system will be discussed for two connected domains with a common boundary. For example, the boundary between two open domains $\Omega_\alpha$ and $\Omega_\beta$ are $\partial \Omega_\alpha \cap \partial \Omega_\beta$, as sketched in Fig. 2. This boundary is formed by the intersection of the closed subdomains.

On the $\alpha$th open subdomain $\Omega_\alpha$, there is a $C^{r_\alpha}$-continuous system ($r_\alpha \geq 1$) in a form of

$$
\dot{x}^{(\alpha)} \equiv F^{(\alpha)}(x^{(\alpha)}, t, p_{\alpha}) \in \mathbb{R}^n, \\
x^{(\alpha)} = (x_1^{(\alpha)}, x_2^{(\alpha)}, \ldots, x_{n_\alpha}^{(\alpha)})^T \in \Omega_\alpha. 
$$

The time is $t$ and $\dot{x}^{(\alpha)} = dx^{(\alpha)}/dt$. In an accessible subdomain $\Omega_\alpha$, the vector field $F^{(\alpha)}(x^{(\alpha)}, t, p_{\alpha})$ with parameter vectors $p_\alpha = (p_\alpha^{(1)}, p_\alpha^{(2)}, \ldots, p_\alpha^{(m)})^T \in \mathbb{R}^m$ is $C^{r_\alpha}$-continuous ($r_\alpha \geq 1$) in a state vector $x$ and for time $t$; and the continuous flow in Eq. (1) $x^{(\alpha)}(t) = \Phi^{(\alpha)}(x^{(\alpha)}(t_0), t, p_{\alpha})$ with $x^{(\alpha)}(t_0) = \Phi^{(\alpha)}(x^{(\alpha)}(t_0), t_0, p_{\alpha})$ is $C^{r_\alpha+1}$-continuous for time $t$. 

![Diagram](image-url)
developed for flow continuity in [Luo, 2006]. For a boundary $\partial \Omega_j$ of a bounded domain $\Omega_j$, i.e., the normal vector of the reference surface $n(t)$ on the boundary is defined by $\eta = \frac{\mathbf{n}_m - \mathbf{n}_n}{\parallel \mathbf{n}_m - \mathbf{n}_n \parallel}$.

Consider a dynamical system in a bounded domain $\Omega_j$ with $\partial \Omega_j$ defined as $\partial \Omega_j = \partial \Omega_j \cup \Omega_j$, where the normal vector of the reference surface $\mathbf{n}(t)$ on the boundary is defined by $\eta = \frac{\mathbf{n}_m - \mathbf{n}_n}{\parallel \mathbf{n}_m - \mathbf{n}_n \parallel}$.

As in [Luo, 2006, 2008a, 2008b], discontinuous dynamical systems possess the following hypothesis.

(H1) The switching between two adjacent subsytems possesses time-continuity.

(H2) For an unbounded, accessible subdomain $\Omega_n$, there is a bounded domain $\Omega_n \subset \Omega_n$ and the corresponding vector field and its flow are bounded, i.e.

$$\|F^{(n)}\| \leq K_1(\text{const}) \quad \text{and} \quad \|\Phi^{(n)}\| \leq K_2(\text{const}) \quad \text{on} \quad D_n \quad \text{for} \quad t \in [0, \infty). \quad (2)$$

(H3) For a bounded, accessible domain $\Omega_n$, there is a bounded domain $\Omega_n \subset \Omega_n$ and the corresponding vector field is bounded, but the flow may be unbounded, i.e.

$$\|F^{(n)}\| \leq K_1(\text{const}) \quad \text{and} \quad \|\Phi^{(n)}\| < \infty \quad \text{on} \quad D_n \quad \text{for} \quad t \in [0, \infty). \quad (3)$$

Because dynamical systems on the different accessible subdomains are different, the relation between flows in the two subdomains has been developed for flow continuity in [Luo, 2006]. For a subdomain $\Omega_n$, there are many pieces of boundaries. Consider a boundary set of any two adjacent subdomains.

Definition 1. The boundary in $n$-dimensional phase space is defined as

$$S_{ij} \equiv \partial \Omega_j \cap \Pi_i \cap \Pi_j = \{ \mathbf{x} \mid \psi_{ij}(\mathbf{x}, t, \lambda) = 0, \quad \psi_{ij} \in C^r\text{-continuous (}r \geq 1) \} \subset \mathbb{R}^{n-1}. \quad (4)$$

Based on the boundary definition, one obtains $\partial \Omega_j = \partial \Omega_j$. On the separation boundary $\partial \Omega_j$, with $\psi_{ij}(\mathbf{x}, t, \lambda) = 0$, there is a dynamical system as

$$\dot{\mathbf{x}}^{(n)} = F^{(n)}(\mathbf{x}^{(n)}, t, \lambda) \quad (5)$$

where $\mathbf{x}^{(n)}(t) = (x_{ij}^{(n)}(t), \ldots, x_{mn}^{(n)}(t))^T$. The flow is given by $\mathbf{x}^{(n)}(t) = \Phi^{(n)}(\mathbf{x}^{(n)}(0), t, \lambda)$ with $\dot{\mathbf{x}}(t) = \dot{\mathbf{x}}^{(n)}(t)$ for time $t$.

3. G-Functions

From [Luo, 2008a, 2008b], consider two infinitesimal time intervals $[t_m - \varepsilon, t_m]$ and $[t_m, t_m + \varepsilon]$. There are two flows in two domains $\Omega_2(t_n) (n = i, j)$ and on the boundary $\partial \Omega_j$, and on the boundary $\partial \Omega_j$ in Eqs. (1) and (5). As in [Luo, 2006c], the vector difference between the two flows for three time instants are given by $\mathbf{x}^{(n)}(t) - \mathbf{x}^{(n)}(t)' + \lambda \mathbf{n}$ and $\mathbf{x}^{(n)}(t) - \mathbf{x}^{(n)}(t)' + \lambda \mathbf{n}$. The normal vectors of the boundary relative to the corresponding flow $\mathbf{x}^{(n)}(t)$ are expressed by $\eta_n \mathbf{n}$, the normal vectors of the boundary $\partial \Omega_j$, and the corresponding tangential vectors of the flow $\mathbf{x}^{(n)}(t)$ on the boundary are expressed by $\mathbf{t}^{(n)}$, $\mathbf{t}^{(n)}$, and $\mathbf{t}^{(n)} + \mathbf{t}^{(n)}$. The vector difference between flows in domain and on the boundary is defined by

$$d_{\varepsilon}^{(n)} = \eta_n \mathbf{n} \mathbf{x}^{(n)}(t) - \mathbf{x}^{(n)}(t)' \quad (6)$$

where the normal vector of the reference surface $\partial \Omega_j$ at point $\mathbf{x}^{(n)}(t)$ is given by $\mathbf{n} \mathbf{x}^{(n)}(t)$ and $\partial \mathbf{x}^{(n)}(t)$.

Definition 2. Consider a dynamical system in Eq. (1) in domain $\Omega_n (n = i, j)$ which has the flow $\mathbf{x}^{(n)}(t) = \Phi^{(n)}(\mathbf{x}^{(n)}(t), p, t)$ with the initial condition $(t_0, \mathbf{x}^{(n)}(0))$. On the boundary $\partial \Omega_j$, there is a boundary flow $\mathbf{x}^{(n)}(t) = \Phi^{(n)}(\mathbf{x}^{(n)}(t), \lambda, t)$ with the initial condition $(t_0, \mathbf{x}^{(n)}(0))$. For an arbitrarily small $\varepsilon > 0$, there are two time intervals $[t - \varepsilon, t]$ or $(t, t + \varepsilon)$ for flow $\mathbf{x}^{(n)}(t) (n = i, j)$. The G-functions $(G^{(n)})$ of the
flow $x_i^{(o)}$ to the boundary flow $x_i^{(0)}$ in the normal direction of the boundary $\partial \Omega_j$ are defined as

$$
G^{(o)}_{\partial \Omega_j}(x_i^{(o)} - x_i^{(0)}, p, \lambda) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} n_{\partial \Omega_j}^T \cdot \left[ (x_i^{(o)} - x_i^{(0)}) - \varepsilon \frac{\partial}{\partial x_i^{(o)}} (x_i^{(o)} - x_i^{(0)}) \right],
$$

$$
G^{(o)}_{\partial \Omega_j}(x_i^{(o)} - x_i^{(0)}, p, \lambda) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} n_{\partial \Omega_j}^T \cdot \left[ (x_i^{(0)} - x_i^{(o)}) - \varepsilon \frac{\partial}{\partial x_i^{(o)}} (x_i^{(o)} - x_i^{(0)}) \right].
$$

From Eq. (8), since $x_i^{(o)}$ and $x_i^{(0)}$ are the solutions of Eqs. (1) and (5), their derivatives exist. Further, by use of the Taylor series expansion, Eq. (8) gives

$$
G^{(o)}_{\partial \Omega_j}(x_i^{(o)} - x_i^{(0)}, p, \lambda) = D_j n_{\partial \Omega_j}^T \cdot (x_i^{(0)} - x_i^{(o)}) + n_{\partial \Omega_j}^T \cdot (x_i^{(o)} - x_i^{(0)})
$$

where the total derivative $D_j(\cdot)$ is $\partial x_i^{(o)}(\cdot) + \partial_i(\cdot)$. Using Eqs. (1) and (5), the G-function in Eq. (9) becomes

$$
G^{(o)}_{\partial \Omega_j}(x_i^{(m)}, t_m, p, \lambda) = n_{\partial \Omega_j}^T \cdot (x_i^{(0)} - x_i^{(o)}) + \nabla \varphi_j(x_i^{(0)}, t, \lambda) \cdot (x_i^{(0)} - x_i^{(o)})
$$

With Eqs. (1) and (5), Eq. (11) can be rewritten as

$$
G^{(o)}_{\partial \Omega_j}(x_i^{(m)}, t_m, p, \lambda) = n_{\partial \Omega_j}^T \cdot (x_i^{(0)} - x_i^{(o)}) \cdot [F(x_i^{(0)}, t, p, \lambda) - F^{(0)}(x_i^{(0)}, t, \lambda)]
$$

and $F^{(0)}(x_i^{(0)}, t, \lambda)$ are $C^{n-1}_{\infty}$-continuous ($r_n \geq k$) for time $t$. The flow $x_i^{(o)}$ and $x_i^{(0)}$ are $C^{n}_{\infty}$-continuous ($r_n \geq k$) for time $t$, $|x_i^{(o)} - x_i^{(0)}| < \infty$, and $|\partial x_i^{(o)} / \partial t^{n+1}| < \infty$. The k-th order G-functions of the compared flow $x_i^{(m)}$ to the boundary flow $x_i^{(0)}$ in the normal direction of the boundary surface $\partial \Omega_j$ are defined as

$$
G^{(k,\beta)}_{\partial \Omega_j}(x_i^{(m)}, t, x_i^{(0)}, p, \lambda) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[ n_{\partial \Omega_j}^T \cdot (x_i^{(m)} - x_i^{(0)}) - \varepsilon \frac{\partial}{\partial x_i^{(o)}} (x_i^{(m)} - x_i^{(0)}) \right]
$$

and $F^{(0)}(x_i^{(0)}, t, \lambda)$ are $C^{n-1}_{\infty}$-continuous ($r_n \geq k$) for time $t$. The flow $x_i^{(o)}$ and $x_i^{(0)}$ are $C^{n}_{\infty}$-continuous ($r_n \geq k$) for time $t$, $|x_i^{(o)} - x_i^{(0)}| < \infty$, and $|\partial x_i^{(o)} / \partial t^{n+1}| < \infty$. The k-th order G-functions of the compared flow $x_i^{(m)}$ to the boundary flow $x_i^{(0)}$ in the normal direction of the boundary surface $\partial \Omega_j$ are defined as

$$
G^{(k,\beta)}_{\partial \Omega_j}(x_i^{(o)} - x_i^{(0)}, p, \lambda) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left[ n_{\partial \Omega_j}^T \cdot (x_i^{(o)} - x_i^{(0)}) - \varepsilon \frac{\partial}{\partial x_i^{(o)}} (x_i^{(o)} - x_i^{(0)}) \right].
$$
Now we consider the following passage from the document:

\[ G_{\partial\Omega_j}^{(k,\alpha)}(x_j^{(0)}, t, x_i^{(\alpha)}, p_n, \lambda) = \lim_{s \to 0} \frac{1}{z^{k+1}} \left[ n_{\partial\Omega_j}^T \left( x_i^{(\alpha)} - x_j^{(0)} \right) - t n_{\partial\Omega_j}^T \left( x_i^{(\alpha)} - x_j^{(0)} \right) \right] \]

\[ - \sum_{s=0}^{k-1} G_{\partial\Omega_j}^{(s,\alpha)}(x_j^{(0)}, t, x_i^{(\alpha)}, p_n, \lambda) z^{s+1} \]  

Again, the Taylor series expansion applying to the foregoing equation yields

\[ G_{\partial\Omega_j}^{(k,\alpha)}(x_j^{(0)}, t, x_i^{(\alpha)}, p_n, \lambda) = \sum_{s=0}^{k-1} C^s_{k-1} D_0^{k-1-s} n_{\partial\Omega_j}^T \left( \frac{d^s x_i^{(\alpha)}}{dz^s} - \frac{d^s x_j^{(0)}}{dz^s} \right) \left|_{x_j^{(0)}, x_i^{(\alpha)}} \right. \]

\[ + D_0^k n_{\partial\Omega_j}^T \left( x_i^{(\alpha)} - x_j^{(0)} \right), \]

with \( C^s_{k-1} = ((k+1)!k(k-1)! \cdots (k+s-1)!) / s! \) and \( C^0_{k-1} = 1 \) and \( s! = 1 \times 2 \times \cdots \times s \). \( D_0(t) = \lambda_{x_i^{(\alpha)}}(t) k^{(\alpha)} + \lambda_{x_j}(t) \). The function \( G_{\partial\Omega_j}^{(k,\alpha)}(x_j^{(0)}, t, x_i^{(\alpha)}, p_n, \lambda) \) is the time-rate of \( G_{\partial\Omega_j}^{(k-1,\alpha)}(x_j^{(0)}, t, x_i^{(\alpha)}, p_n, \lambda) \). If the flow contacting with the boundary \( \partial\Omega_j \) at time \( t_m \) (i.e. \( x_i^{(\alpha)} = x_i^{(0)} \)) and \( n_{\partial\Omega_j}^T = n_{\partial\Omega_j}^T \), the \( k \)-th order \( G \)-function is computed by

\[ G_{\partial\Omega_j}^{(k,\alpha)}(x_m, t_m, x_i^{(\alpha)}, p_n, \lambda) = \sum_{s=0}^{k-1} C^s_{k-1} D_0^{k-1-s} n_{\partial\Omega_j}^T \left( \frac{d^s x_i^{(\alpha)}}{dz^s} - \frac{d^s x_j^{(0)}}{dz^s} \right) \left|_{x_j^{(0)}, x_i^{(\alpha)}, x_m} \right. \]

\[ = \sum_{s=0}^{k-1} C^s_{k-1} D_0^{k-1-s} n_{\partial\Omega_j}^T \left[ D_0^{k-1} F(x, t, p_n) - D_0^{k-1} F(x^{(\alpha)}, t, \lambda) \right] \left|_{x_i^{(0)}, x_m} \right. \]

For \( k = 0 \), \( G_{\partial\Omega_j}^{(0,\alpha)}(x_m, t_m, x_i^{(\alpha)}, p_n, \lambda) \) and the \( G \)-function for the boundary flow \( x_i^{(0)} \) is always zero, which is represented by \( G_{\partial\Omega_j} = 0 \).


In [Luo, 2008a, 2009], the passability of a flow to the boundary is dependent on the vector fields on both sides of the boundary. If a flow goes through the boundary \( \partial\Omega_j \) from domain \( \Omega_i \) to domain \( \Omega_j \), there are three vectors fields to form three dynamical systems

\[ \dot{x}^{(\alpha)} = F^{(\alpha)}(x^{(\alpha)}) \]

in \( \Omega_i \) (\( \alpha = i, j \)),

\[ \dot{x}^{(0)} = F^{(0)}(x^{(0)}) \]

with \( \phi_{ij} (x^{(0)}, t, \lambda) = 0 \) on \( \partial\Omega_j \).

For simplicity in discussion, the following sign function is introduced as

\[ h_n = \begin{cases} +1 & \text{for } n_{\partial\Omega_j} \to \Omega_i \\ -1 & \text{for } n_{\partial\Omega_j} \to \Omega_j \end{cases} \]

Without any flow barriers, the necessary and sufficient conditions for a flow to pass through the boundary are obtained from [Luo, 2009], i.e.

\[ h_n G_{\partial\Omega_j}^{(\alpha)}(x_m, t_m, p_n, \lambda) > 0 \text{ and } h_n G_{\partial\Omega_j}^{(\beta)}(x_m, t_m, p_n, \lambda) > 0. \]
Definition 4. For a discontinuous dynamical system in Eq. (17), there is a point \( x^{(0)}(t_m) \in \Omega_{ij} \) at time \( t_m \) between two adjacent domains \( \Omega_{ij} (\alpha = i, j) \). There is a vector field of \( F^{(\omega)}(x^{(\lambda)}, t, \pi, q^{(\lambda)}) \) for \( q^{(\lambda)} \in [q_{1}^{(\lambda)}, q_{2}^{(\lambda)}] \) \( (\rho, \gamma \in \{0, 1, j\}, \gamma \neq \gamma_0 \) if \( \rho \neq 0 \) on the boundary \( \partial \Omega_{ij} \).

The function of the vector field is defined as

\[
G^{(\omega)}(x_m, t_m, \pi, \lambda, q^{(\lambda)}) \equiv n_{\Omega_{ij}}^{\top}(x^{(0)}, t, \lambda) \cdot \left[ F^{(\omega)}(x^{(\lambda)}, t, \pi, q^{(\lambda)}) - F^{(0)}(x^{(0)}, t, \lambda) \right]_{\Omega_{ij} x^{(0)}(t_m).}
\]

The higher-order \( G \)-function of the vector field \( F^{(\omega)}(x^{(\lambda)}, t, \pi, q^{(\lambda)}) \) is defined for \( k = 1, 2, \ldots \) as

\[
G^{(k,\omega)}(x_m, t_m, \pi, \lambda, q^{(\lambda)}) = \sum_{\gamma = 1}^{k+1} \left[ D^{(\omega)}_{\lambda} F^{(\omega)}(x^{(\lambda)}, t, \pi, q^{(\lambda)}) \right]_{\Omega_{ij} x^{(0)}(t_m)}. \]

For simplicity, the following notations are adopted.

\[
G^{(k,\omega)}(x_m, t_m, \pi, \lambda, q^{(\lambda)}) \equiv G^{(k,\omega)}(x_m, t_m, \pi, \lambda),
\]

\[
G^{(k,\omega)}(x_m, t_m, \pi, q^{(\lambda)}) \equiv G^{(k,\omega)}(x_m, t_m, \pi, \lambda, q^{(\lambda)}). \]

4.1. Coming flow barriers for passable flows

In this section, the coming flow barriers in the semi-passable flow to the boundary will be discussed. The basic concepts of the flow barriers on the boundary will be introduced through a coming flow to the boundary.

Definition 5. For a discontinuous dynamical system in Eq. (17), there is a point \( x^{(0)}(t_m) \in \Omega_{ij} \) at time \( t_m \) between two adjacent domains \( \Omega_{ij} (\alpha = i, j) \). Suppose there is a vector field of \( F^{(\omega)}(x^{(\lambda)}, t, \pi, q^{(\lambda)}) \) for \( q^{(\lambda)} \in [q_{1}^{(\lambda)}, q_{2}^{(\lambda)}] \) on the boundary \( \partial \Omega_{ij} \) with

\[
h_{\alpha} G^{(\omega,\beta)}(x_m, t_m, \lambda, q^{(\alpha)}) \in [h_{\alpha} G^{(\omega,\beta)}(x_m, q_{1}^{(\alpha)}), h_{\alpha} G^{(\omega,\beta)}(x_m, q_{2}^{(\alpha)})] \subset [0, +\infty)
\]

\((\alpha, \beta \in \{i, j\} \text{ and } \alpha \neq \beta)\). The coming and leaving flows in the semi-passable flow satisfy

\[
h_{\alpha} G^{(\omega,\beta)}(x_m, t_m -) > 0 \quad \text{and} \quad h_{\alpha} G^{(\omega,\beta)}(x_m, t_m +) > 0.
\]

The vector field of \( F^{(\omega,\beta)}(x^{(0)}, t, \pi, q^{(\lambda)}) \) is called the coming flow barrier in the semi-passable flow on the \( \alpha \)-side if the following conditions are satisfied. The critical values of \( F^{(\omega,\beta)}(x^{(0)}, t, \pi, q^{(\lambda)}) \) \((\sigma = 1, 2)\) are called the lower and upper limits of the coming flow barrier on the \( \alpha \)-side.

(i) The coming flow of \( x^{(\alpha)} \) cannot be switched to the leaving flow of \( x^{(\beta)} \) if

\[
x^{(\alpha)}(t_m -) = x^{(\omega,\alpha)}(t_m, q^{(\alpha)} = x_m),
\]

\[
h_{\alpha} G^{(\omega,\beta)}(x_m, t_m -) \in (h_{\alpha} G^{(\omega,\beta)}(x_m, q_{1}^{(\alpha)}), h_{\alpha} G^{(\omega,\beta)}(x_m, q_{2}^{(\beta)})),
\]

\[
h_{\alpha} G^{(\omega,\beta)}(x_m, t_m +) > 0.
\]

(ii) The coming flow of \( x^{(\alpha)} \) cannot be switched to the leaving flow of \( x^{(\beta)} \) at the critical points of the flow barrier \((i.e. q^{(\alpha)} = q^{(\beta)}), \sigma \in \{1, 2\}) \) if

\[
x^{(\alpha)}(t_m -) = x^{(\omega,\beta)}(t_m, q^{(\alpha)} = x_m),
\]

\[
G^{(\alpha)}(x_m, t_m -) = G^{(\omega,\beta)}(x_m, q^{(\alpha)} = 0) \neq 0
\]

\(\text{for } s_{\alpha} = 0, 1, 2, \ldots, t_m - 1;\)

\[
(-1)^{s_{\alpha}} h_{\alpha} G^{(\omega,\beta)}(x^{(0)}, t_{m + s}) \cdot x^{(\omega,\beta)}(t_{m + s}, q^{(\alpha)} = 0) < 0.
\]

(iii) The coming flow of \( x^{(\alpha)} \) is switched to the leaving flow of \( x^{(\beta)} \) at the critical points of the flow barrier \((i.e. q^{(\alpha)} = q^{(\beta)}), \sigma \in \{1, 2\}) \) if

\[
x^{(\alpha)}(t_m -) = x^{(\omega,\beta)}(t_m, q^{(\alpha)} = x_m),
\]

\[
G^{(\alpha,\beta)}(x_m, t_m -) = G^{(\omega,\beta)}(x_m, q^{(\beta)} = 0) \neq 0\]

\(\text{for } s_{\alpha} = 0, 1, 2, \ldots, t_m - 1;\)
(-1)^{\varepsilon} b_{\alpha} n_{\alpha i j}(x^{(0)}(t_{m+i}))
\cdot [x^{(\alpha)}(t_{m+i}) - x^{(\alpha-\beta)}(t_{m+i}, q^{(\alpha)})] > 0.
(27)

Definition 6. For a discontinuous dynamical system in Eq. (17), there is a point $x^{(0)}(t_{m}) \equiv x_{m} \in \partial \Omega_{1}$ at time $t_{m}$ between two adjacent domains $\Omega_{\alpha}$ ($\alpha = i, j$). Suppose there is a vector field $F^{(\alpha-\beta)}(x^{(\alpha)}, t, \pi_{\alpha}, q^{(\alpha)})$ for $q^{(\alpha)} \in [q^{(\alpha)}, q^{(\alpha)+}]$ on the boundary $\partial \Omega_{1}$ with the $G$-functions

\begin{align}
G_{ij}^{(\alpha, \beta)}(x^{(\alpha)}, t_{m}, q^{(\alpha)}) &= 0 \quad \text{for} \quad s_{a} = 0, 1, \ldots, 2k_{a} - 1; \\
G_{ij}^{(2k_{a}, \alpha-\beta)}(x^{(\alpha)}, t_{m}, q^{(\alpha)}) &\in [b_{\alpha} G_{ij}^{(2k_{a}, \alpha-\beta)}, (x^{(\alpha)}, q^{(\alpha)})] \\
&\subset [0, \infty)
\end{align}
(28)

$(\alpha, \beta \in \{i, j\}$ and $\alpha \neq \beta)$. The coming and leaving flows of the $(2k_{a}, m_{\beta})$-semi-passable flow satisfy

\begin{align}
G_{ij}^{(\alpha, \beta)}(x^{(\alpha)}, t_{m-1}) &= 0 \quad \text{for} \quad s_{a} = 0, 1, \ldots, 2k_{a} - 1; \\
G_{ij}^{(\alpha, \beta)}(x^{(\alpha)}, t_{m+1}) &= 0 \quad \text{for} \quad s_{j} = 0, 1, \ldots, m_{j} - 1; \\
&b_{\alpha} G_{ij}^{(2k_{a}, \alpha, \beta)}(x^{(\alpha)}, t_{m}, q^{(\alpha)}, t_{m}) > 0 \\
&b_{\beta} G_{ij}^{(m_{\beta} \alpha, \beta)}(x^{(\alpha)}, t_{m}) > 0.
\end{align}
(29)

The vector field $F^{(\alpha-\beta)}(x^{(\alpha)}, t, \pi_{\alpha}, q^{(\alpha)}) \in [q^{(\alpha)}, q^{(\alpha)+}]$ is called the coming flow barrier in the $(2k_{a}, m_{\beta})$-semi-passable flow on the $\alpha$-side if the following conditions are satisfied. The critical values of $F^{(\alpha-\beta)}(x^{(\alpha)}, t, \pi_{\alpha}, q^{(\alpha)})$ for $q^{(\alpha)} \in [q^{(\alpha)}, q^{(\alpha)+}]$ are called the lower and upper limits of the coming flow barriers on the $\alpha$-side.

(i) The coming flow of $x^{(\alpha)}$ cannot be switched to the leaving flow of $x^{(\beta)}$ if

\begin{align}
x^{(\alpha)}(t_{m}) &= x^{(\alpha-\beta)}(t_{m}, q^{(\alpha)}) = x_{m} \\
G_{ij}^{(2k_{a}, \alpha, \beta)}(x^{(\alpha)}, t_{m}, q^{(\alpha)}) &\in [b_{\alpha} G_{ij}^{(2k_{a}, \alpha-\beta)}, (x^{(\alpha)}, q^{(\alpha)})] \\
b_{\alpha} G_{ij}^{(2k_{a}, \alpha, \beta)}(x^{(\alpha)}, q^{(\alpha)+})
\end{align}
(30)

(ii) The coming flow of $x^{(\alpha)}$ cannot be switched to the leaving flow of $x^{(\beta)}$ at the critical points of the flow barrier (i.e. $q^{(\alpha)} = q^{(\alpha)+}, \sigma \in \{1, 2\}$) if

\begin{align}
x^{(\alpha)}(t_{m}) &= x^{(\alpha-\beta)}(t_{m}, q^{(\alpha)}) = x_{m} \\
G_{ij}^{(\alpha, \beta)}(x^{(\alpha)}, t_{m-1}) &= G_{ij}^{(\alpha, \beta)}(x^{(\alpha)}, t_{m+1}) \\
&\cdot [x^{(\alpha)}(t_{m+i}) - x^{(\alpha-\beta)}(t_{m+i}, q^{(\alpha)})] < 0.
\end{align}
(31)

(iii) The coming flow of $x^{(\alpha)}$ is switched to the leaving flow of $x^{(\beta)}$ at the critical points of the flow barrier (i.e. $q^{(\alpha)} = q^{(\alpha)+}, \sigma \in \{1, 2\}$) if

\begin{align}
x^{(\alpha)}(t_{m}) &= x^{(\alpha-\beta)}(t_{m}, q^{(\alpha)}) = x_{m} \\
G_{ij}^{(\alpha, \beta)}(x^{(\alpha)}, t_{m}) &= G_{ij}^{(\alpha, \beta)}(x^{(\alpha)}, t_{m}^{(1)}) \\
&\cdot [x^{(\alpha)}(t_{m+i}) - x^{(\alpha-\beta)}(t_{m+i}, q^{(\alpha)})] > 0.
\end{align}
(32)

To explain the above concept, the $G$-functions for the coming flow barriers on the $\alpha$-side of the boundary are presented in Fig. 3. The $G$-function of the leaving flow of $x^{(\beta)}$, relative to the coming flow barrier on the $\alpha$-side, is denoted by dashed curves. The thick line on the boundary $\partial \Omega_{1}$ represents the $G$-function of the coming flow barrier. For $\mathbf{n}_{\alpha i j} \rightarrow \Omega_{\beta}$, one has $b_{\alpha} = +1$. If $G_{ij}^{(\alpha, \beta)} > 0$ and $G_{ij}^{(\alpha, \beta)} > 0$, without the flow barrier, the coming flow can be switched to the leaving flow from domain $\Omega_{\alpha}$ to $\Omega_{\beta}$. However, if there is a coming flow barrier with lower and upper limits $G_{ij}^{(\alpha, \beta)} \in [q^{(\alpha)}, q^{(\alpha)+}]$, the coming flow cannot be switched to the leaving flow in domain $\Omega_{\beta}$. When a coming flow of $G_{ij}^{(\alpha, \beta)} > 0$ arrives at the boundary $\partial \Omega_{1}$ with a flow barrier, such a coming flow cannot be switched from domain $\Omega_{\alpha}$ to $\Omega_{\beta}$ only if $G_{ij}^{(\alpha, \beta)} \notin [G_{ij}^{(\alpha, \beta)}, q^{(\alpha)}, \epsilon G_{ij}^{(\alpha, \beta)}, q^{(\alpha)+}]$. For this case, the flow barrier is sketched in Fig. 3(a). In fact, the $G$-function for the lower limit of the coming flow barrier can be less than zero (i.e. $G_{ij}^{(\alpha, \beta)}(q_{1}^{(\alpha)}) < 0$). However, for $G_{ij}^{(\alpha, \beta)} < 0$,
the coming flow of $x^{(a)}$ becomes a source flow, which will never pass through the boundary. Thus, the lower limit of the coming flow barrier of $G_{\partial \Omega_{ij}}^{(2k_{\alpha},m_{\beta})}(q^{(a)}_{1}) < 0$ is not important for $\mathbf{n}_{\partial \Omega_{ij}} \to \Omega_\beta$. Therefore, the lower limit of the coming flow barrier with $G_{\partial \Omega_{ij}}^{(2k_{\alpha},m_{\beta})}(q^{(a)}_{1}) = 0$ can be considered as the lower limit. However, the $G$-function of the upper coming flow barrier can approach to infinity (i.e. $G_{\partial \Omega_{ij}}^{(2k_{\alpha},m_{\beta})}(q^{(a)}_{2}) \to +\infty$). Similarly, for $\mathbf{n}_{\partial \Omega_{ij}} \to \Omega_\alpha$, the coming flow barrier on the $\alpha$-side can be similarly discussed, as presented in Fig. 3(b). The $G$-functions of the lower and upper limits of the coming flow barrier can be zero and negative infinity, respectively. The lower limit of $G_{\partial \Omega_{ij}}^{(2k_{\alpha},m_{\beta})}(q^{(a)}_{1}) > 0$ is not significant for $\mathbf{n}_{\partial \Omega_{ij}} \to \Omega_\alpha$. Of course, to block the flow from both sides of the boundary, one can simply define the coming flow barrier with negative and positive limits of the $G$-functions. In addition, the $G$-function of the coming flow barrier varies with the location of the boundary. On some subsets of the boundary, no flow barriers exist. The coming flow barrier on such a boundary is a partial coming flow barrier. If the coming flow barrier exists on the entire boundary,
the flow barrier is a full flow barrier on the boundary. The strict definitions are given as follows.

**Definition 7.** For a discontinuous dynamical system in Eq. (17), there is a point $x^{(0)}(t_m) \equiv x_m \in \partial \Omega_{ij}$ at time $t_m$ between two adjacent domains $\Omega_\alpha (\alpha = i, j)$. Suppose a coming flow barrier of $F^{(\alpha \succ \beta)}(x^{(\alpha)}, t, \pi, q^{(\alpha)})$ exists on the $\alpha$-side in the $(2k_\alpha : m_\beta)$-semi-passable flow for $q^{(\alpha)} \in [q^{(\alpha)}_1, q^{(\alpha)}_2]$, $\alpha \in \{i, j\}$ and $\alpha \neq \beta$.

(i) The coming flow barrier in the $(2k_\alpha : m_\beta)$-semi-passable flow is partial on the $\alpha$-side if $x_m \in \partial \Omega_{ij}$. 

(ii) The coming flow barrier in the $(2k_\alpha : m_\beta)$-semi-passable flow is full on the $\alpha$-side if $x_m \in S = \partial \Omega_{ij}$.

For $k_\alpha = m_\beta = 0$, the above definitions are suitable for the fundamental flow barriers. The partial and full coming flow barriers on the $\alpha$-side of the boundary $\partial \Omega_{ij}$ are sketched in Fig. 4. The coming flow barriers only exist on subsets of the boundary (i.e. $S \subset \partial \Omega_{ij}$). On the other subsets $(\partial \Omega_{ij} \setminus S)$, the coming flow barriers do not exist. Once a coming flow in domain $\Omega_\alpha$ arrives to such subsets, the coming flow in the semi-passable flow can be switched to the leaving flow on the $\beta$-side.

---

**Fig 4.** The coming flow barrier on the $\alpha$-side in the $(2k_\alpha : m_\beta)$-semi-passable flow: (a) partial flow barrier and (b) full flow barrier. The dark surface is for the flow barrier on $\partial \Omega_{ij}$. The red solid and dashed curves with arrows are $G$-functions of flows on $\alpha$ and $\beta$-domains. The blue curves are semi-passable flows. The white surface represents “no flow barrier” ($k_\alpha, m_\beta \in \{0, 1, 2, \ldots\}$).
If the coming flow barrier exists on \( S = \partial \Omega_{ij} \), such a flow barrier is a full flow barrier. Any coming flow barriers possess the lower and upper limits. As discussed before, the lowest limit of the coming flow barrier is \( G(2k, \alpha \succ \beta)\partial \Omega_{ij}(q_{1}) = 0 \). If \( G(2k, \alpha \succ \beta)\partial \Omega_{ij}(q_{2}) \) is finite, the coming flow barrier with the lowest limit is a flow barrier with the upper limit. If the \( G\)-function of the upper limit of the coming flow barrier is infinity but \( G(2k, \alpha \succ \beta)\partial \Omega_{ij}(q_{1}) \neq 0 \), the coming flow barrier is a flow barrier with a lower limit. If such a flow barrier exists on \( S \subseteq \partial \Omega_{ij} \), the partial and full coming flow barriers with an upper or lower limit are sketched in Figs. 5 and 6, respectively. If \( G(2k, \alpha \succ \beta)\partial \Omega_{ij}(q_{1}) = 0 \) and \( G(2k, \alpha \succ \beta)\partial \Omega_{ij}(q_{2}) \to +\infty \) for \( x_{m} \in \partial \Omega_{ij} \), the coming flow of \( x^{(\alpha)} \) at the boundary point cannot pass over the boundary. The coming flow barrier at this point is an absolute flow barrier. Of course, if the entire boundary possesses such absolute flow barriers, the coming flow barrier is a flow barrier wall. If there are many partial coming flow barriers on the boundary, a coming flow barrier fence can be formed. If the \( G\)-function of

![Image](image_url)

Fig. 5. The coming flow barrier on the \( \alpha\)-side with upper limits in the \((2k, m_{\beta})\)-semi-passable flow: (a) partial flow barrier and (b) full flow barrier. The dark surface is the flow barrier surface on \( \partial \Omega_{ij} \). The red solid and dashed curves with arrows are \( G\)-functions of flows on \( \alpha \) and \( \beta \)-domains. The blue curves are semi-passable flows. The white surface represents “no flow barrier” \((k_{\alpha}, m_{\beta} \in \{0, 1, 2, \ldots\})\).
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The infinity coming flow barrier on the $\alpha$-side in the $(2k_{\alpha}:m_{\beta})$-semi-passable flow: (a) partial flow barrier and (b) full flow barrier. The dark surface is the flow barrier surface on $\partial \Omega_{ij}$. The red solid and dashed curves with arrows are $G$-functions of flows on $\alpha$ and $\beta$-domains. The blue curves are semi-passable flows. The white surface represents “no flow barrier” ($k_{\alpha}, m_{\beta} \in \{0, 1, 2, ...\}$).

The flow barrier is not defined on $S \subset \partial \Omega_{ij}$ with $q^{(\alpha)} \in (q^{(\alpha)}_1, q^{(\alpha)}_2)$, the window of the flow barrier can be formed where no flow barriers on such a portion exist. The definitions are given as follows.

**Definition 8.** For a discontinuous dynamical system in Eq. (17), there is a point $x^{(0)}(t_m) \equiv x_m \in \partial \Omega_{ij}$ at time $t_m$ between two adjacent domains $\Omega_{ij} (\alpha = i, j)$. Suppose a coming flow barrier of $F^{(\alpha)\beta}(x^{(\alpha)}; t, \pi_{\alpha}, q^{(\alpha)})$ exists on the $\alpha$-side in the $(2k_{\alpha}:m_{\beta})$-semi-passable flow for $q^{(\alpha)} \in [q^{(\alpha)}_1, q^{(\alpha)}_2]$ ($\alpha, \beta \in \{i, j\}$ and $\alpha \neq \beta$).

(i) The coming flow barrier in the $(2k_{\alpha}:m_{\beta})$-semi-passable flow is with an upper limit for $x_m \in S \subset \partial \Omega_{ij}$ if

$$\begin{align*}
&b_{\alpha} G^{(2k_{\alpha}:m_{\beta})}_{\partial \Omega_{ij}}(x_m, q^{(\alpha)}_1) = 0, \\
&b_{\alpha} G^{(2k_{\alpha}:m_{\beta})}_{\partial \Omega_{ij}}(x_m, q^{(\alpha)}_2) \neq \infty.
\end{align*}$$

(ii) The coming flow barrier in the $(2k_{\alpha}:m_{\beta})$-semi-passable flow is with an upper limit for $x_m \in S \subset \partial \Omega_{ij}$ if

$$\begin{align*}
&b_{\alpha} G^{(2k_{\alpha}:m_{\beta})}_{\partial \Omega_{ij}}(x_m, q^{(\alpha)}_1) = 0, \\
&b_{\alpha} G^{(2k_{\alpha}:m_{\beta})}_{\partial \Omega_{ij}}(x_m, q^{(\alpha)}_2) \neq \infty.
\end{align*}$$

Fig. 6. The infinity coming flow barrier on the $\alpha$-side in the $(2k_{\alpha}:m_{\beta})$-semi-passable flow: (a) partial flow barrier and (b) full flow barrier. The dark surface is the flow barrier surface on $\partial \Omega_{ij}$. The red solid and dashed curves with arrows are $G$-functions of flows on $\alpha$ and $\beta$-domains. The blue curves are semi-passable flows. The white surface represents “no flow barrier” ($k_{\alpha}, m_{\beta} \in \{0, 1, 2, ...\}$).
(iii) The coming flow barrier in the $(2k_m: m_3)$-semi-passable flow is with a lower bound for $x_m \in S \subseteq \partial \Omega_j$ if
\[
\begin{align*}
 h_0 G^{(2k_m, \alpha, \beta)}(x_m, q^{(\alpha)}) &\neq 0 \quad \text{and} \\
 h_0 G^{(2k_m, \alpha, \beta)}(x_m, q^{(\beta)}) &\rightarrow \infty.
\end{align*}
\] (34)

(iii) The coming flow barrier in the $(2k_m: m_3)$-semi-passable flow is absolute for $x_m \in S \subseteq \partial \Omega_j$ if
\[
\begin{align*}
 h_0 G^{(2k_m, \alpha, \beta)}(x_m, q^{(\alpha)}) &\neq 0 \quad \text{and} \\
 h_0 G^{(2k_m, \alpha, \beta)}(x_m, q^{(\beta)}) &\rightarrow \infty.
\end{align*}
\] (35)

(iv) The coming flow barrier in the $(2k_m: m_3)$-semi-passable flow is a flow barrier wall on the $\alpha$-side if the absolute flow barrier exists at $x_m \in S = \partial \Omega_j$.

(v) The coming flow barrier in the $(2k_m: m_3)$-semi-passable flow is a flow barrier fence on the $\alpha$-side if the flow barriers exist on $S_{\alpha} \subseteq \partial \Omega_j$ and no flow barriers on $S_{\beta} \subseteq \partial \Omega_j$ ($k_1, k_2 \in \{1, 2, \ldots \}$) for $S_{\alpha} \cap S_{\beta} = \emptyset$.

**Definition 9.** For a discontinuous dynamical system in Eq. (17), there is a point $x^{(0)}(t_m) \equiv x_m \in \partial \Omega_j$ at time $t_m$ between two adjacent domains $\Omega_k$ ($\alpha = i, j$). The coming and leaving flows in the $(2k_m: m_3)$-semi-passable flow satisfy the conditions in Eq. (29). Many flow barriers $F^{(\alpha, \beta)}(x^{(\alpha)}, t, \pi_\alpha, q^{(\alpha)})$ exist on the $\alpha$-side for $x_m \in S \subseteq \partial \Omega_j$ and $q^{(\alpha)} \in (q^{(\alpha)}_{2n-1}, q^{(\alpha)}_{2n})$ ($n = 1, 2, \ldots$) with $\sigma = 2n - 1$:
\[
\begin{align*}
 h_0 G^{(2k_m, \alpha, \beta)}(x_m, q^{(\alpha)}) &= 0 \\
 h_0 G^{(2k_m, \alpha, \beta)}(x_m, q^{(\alpha)}) &\in [h_0 G^{(2k_m, \alpha, \beta)}(x_m, q^{(\alpha)}_{2n-1}), h_0 G^{(2k_m, \alpha, \beta)}(x_m, q^{(\alpha)}_{2n})] \\
 &\subseteq [0, \infty),
\end{align*}
\] (36)

For $q^{(\alpha)} \in (q^{(\alpha)}_{2n-1}, q^{(\alpha)}_{2n})$ ($n = 2, 3, \ldots$), no flow barriers exist on $S \subseteq \partial \Omega_j$. Thus, the coming flow of $x^{(\alpha)}$ can be switched to the leaving flow of $x^{(\beta)}$ for
\[
\begin{align*}
 h_0 G^{(2k_m, \alpha, \beta)}(x_m, q^{(\alpha)}) &\equiv h_0 G^{(2k_m, \alpha, \beta)}(x_m, q^{(\beta)}_{2n-2}) \\
 h_0 G^{(2k_m, \alpha, \beta)}(x_m, q^{(\alpha)}_{2n-1}) &\subseteq [0, \infty).
\end{align*}
\] (37)

The $G$-function intervals for all $x_m \in S \subseteq \partial \Omega_j$ with $q^{(\alpha)} \in (q^{(\alpha)}_{2n-2}, q^{(\alpha)}_{2n})$ are called the window of the coming flow barrier on the $\alpha$-side in the $(2k_m: m_3)$-semi-passable flow.

**Definition 10.** For a discontinuous dynamical system in Eq. (17), there is a point $x^{(0)}(t_m) \equiv x_m \in \partial \Omega_j$ at time $t_m$ between two adjacent domains $\Omega_k$ ($\alpha = i, j$). Suppose there is a coming flow barrier of $F^{(\alpha, \beta)}(x^{(\alpha)}, t, \pi_\alpha, q^{(\alpha)})$ on the $\alpha$-side in the $(2k_m: m_3)$-semi-passable flow for $q^{(\alpha)} \in [q^{(\alpha)}_{2n-1}, q^{(\alpha)}_{2n})$ ($\alpha, \beta \in \{i, j\}$ and $\alpha \neq \beta$).

(i) The coming flow barrier in the $(2k_m: m_3)$-semi-passable flow is permanent on the $\alpha$-side if the flow barrier is independent of time $t \in [0, \infty)$.
(ii) The coming flow barrier in the $(2k_m: m_3)$-semi-passable flow is instantaneous on the $\alpha$-side if the flow barrier is continuously dependent on time $t \in [0, \infty)$.
(iii) The coming flow barrier in the $(2k_m: m_3)$-semi-passable flow is intermittent on the $\alpha$-side if the flow barrier exists for time $t \in [t_1, t_{k+1}]$ with $k \in \mathbb{Z}$.

On the window area, the flow can be switched. Similarly, the permanent and instantaneous windows of the flow barrier on the boundary can be discussed. Further, the concept of the door for the flow barrier wall can be introduced.

**Definition 11.** For a discontinuous dynamical system in Eq. (17), there is a point $x^{(0)}(t_m) \equiv x_m \in \partial \Omega_j$ at time $t_m$ between two adjacent domains $\Omega_k$ ($\alpha = i, j$). Suppose a coming flow barrier of $F^{(\alpha, \beta)}(x^{(\alpha)}, t, \pi_\alpha, q^{(\alpha)})$ exists on the $\alpha$-side for $x_m \in S \subseteq \partial \Omega_j$ and $q^{(\alpha)} \in (q^{(\alpha)}_{2n-1}, q^{(\alpha)}_{2n})$ ($n = 1, 2, \ldots$) and there is a window of the coming flow barrier for $S \subseteq \partial \Omega_j$ and $q^{(\alpha)} \in (q^{(\alpha)}_{2n-2}, q^{(\alpha)}_{2n-1})$ in the $(2k_m: m_3)$-semi-passable flow.

(i) The flow barrier window in the $(2k_m: m_3)$-semi-passable flow is permanent on the $\alpha$-side if the window is independent of time $t \in [0, +\infty)$.
(ii) The flow barrier window in the $(2k_m: m_3)$-semi-passable flow is instantaneous on the
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(iii) The flow barrier window in the $\{k; m\}$-semi-passable flow is intermittent on the $\alpha$-side if the window exists for time $t \in [t_k, t_{k+1}]$ with $k \in \mathbb{Z}$.
(iv) The flow barrier window in the $\{k; m\}$-semi-passable flow is intermittent and static on the $\alpha$-side if the window is independent of time $t \in [t_k, t_{k+1}]$ with $k \in \mathbb{Z}$.

Definition 12. For a discontinuous dynamical system in Eq. (17), there is a point $x^{(0)}(t_m) \equiv x_m \in \partial \Omega_{ij}$ at time $t_m$ between two adjacent domains $\Omega_\alpha$ ($\alpha = i, j$). Suppose a coming flow barrier wall $F^{(\alpha \rightarrow \beta)}(x^{(0)}, t, \pi_m, q^{(0)})$ on $S = \partial \Omega_{ij}$ exists in the $\{k; m\}$-semi-passable flow for $q^{(0)} \in [0, \infty)$ and there is an intermittent, static window of the coming flow barrier wall on $S \subset \partial \Omega_{ij}$ for $q^{(0)} \in [q_1^{(0)}, q_2^{(0)}]$ and $t \in [t_k, t_{k+1}]$ with $k \in \mathbb{Z}$.

(i) The window of the flow barrier in the $\{k; m\}$-semi-passable flow is called a door of the flow barrier wall on the $\alpha$-side if the window and flow barriers exist alternatively.
(ii) The door of the coming flow barrier wall in the $\{k; m\}$-semi-passable flow is open on the $\alpha$-side for time $t \in [t_k, t_{k+1}]$ with $k \in \mathbb{Z}$ if the window exists.
(iii) The door of the coming flow barrier wall in the $\{k; m\}$-semi-passable flow is closed on the $\alpha$-side for time $t \in [t_k, t_{k+1}+2]$ with $k \in \mathbb{Z}$ if the window exists.
(iv) The door of the coming flow barrier wall in the $\{k; m\}$-semi-passable flow is permanently open on the $\alpha$-side if the window exists for time $t \in [t_k, \infty)$.
(v) The door of the coming flow barrier wall is permanently closed on the $\alpha$-side if the flow barrier exists for time $t \in [t_k+1, \infty)$.

From Definition 9, the window of the flow barrier is sketched in Fig. 7. On the window area, the flow can be switched. For the permanent windows of the flow barrier, the flow may or may not be switched at the boundary for the next moment. In addition, the door of the absolute and permanent, coming flow barrier in the $\{k; m\}$-semi-passable flow on the $\alpha$-side of the boundary $\partial \Omega_{ij}$ is sketched in Fig. 8.

The door is a path on the boundary with the flow barrier wall. If the door is open, the flow can be switched from one domain into another domain via the boundary. If the door is closed, such a door looks like a wall of the flow barrier. The flow cannot be switched from one domain into another domain via the boundary. In Fig. 8(a), it is shown that the door of the flow barrier is open. The coming flow of $x^{(0)}$ can be switched into the leaving flow of $x^{(0)}$. However, in Fig. 8(b), the door of the flow barrier is closed. During a certain time period, the flow cannot pass through this door.

For a passable flow on the boundary $\partial \Omega_{ij}$, if the flow barrier exists on such a boundary with a subset $S \subset \partial \Omega_{ij}$, the flow cannot pass through the boundary under the following conditions

$$h_\alpha G^{(2k; \alpha \rightarrow \beta)}(x_m, t_m) \in \{h_\alpha G^{(2k; \alpha \rightarrow \beta)}(x_m, q_1^{(0)}), h_\alpha G^{(2k; \alpha \rightarrow \beta)}(x_m, q_2^{(0)})\}.$$  

For $x_m \in S \subset \partial \Omega_{ij}$. For this case, the dynamical system on the $\alpha$-side of the boundary will be constrained by the boundary $\partial \Omega_{ij}$, i.e.

$$\dot{x}^{(0)} = F^{(\alpha \rightarrow \beta)}(x^{(0)}, t, \pi_m) \text{ in } \Omega_\alpha (\alpha = i, j),$$

with $\varphi_t(x^{(0)}, t, \lambda) = 0$ on $\partial \Omega_{ij}$. (39)

The vector field of the dynamical system on the $\alpha$-side of the boundary is given by the vector field on the domain of $\Omega_\alpha$. Because the coming flow barrier exists on the $\alpha$-side of the boundary $\partial \Omega_{ij}$, the coming flow will be along the $\alpha$-side of the boundary until the condition in Eq. (38) cannot be satisfied. If the tangential components of vector fields in Eq. (39) to the boundary are zero, the coming flow will stay on the specific point of the boundary until the flow barrier can be passed over. The coming flow at this point is called the “standing flow” on the $\alpha$-side of the boundary. For this case, the transport laws may exist, which transport to another vector field in the same domain or another accessible domain or on the other boundary.

Theorem 1. For a discontinuous dynamical system in Eq. (17), there is a point $x^{(0)}(t_m) \equiv x_m \in \partial \Omega_{ij}$ at time $t_m$ between two adjacent domains $\Omega_\alpha$ ($\alpha = i, j$). Suppose a coming flow barrier $F^{(\alpha \rightarrow \beta)}(x^{(0)}, t, \pi_m, q^{(0)})$ for $q^{(0)} \in [q_1^{(0)}, q_2^{(0)}]$ in the semi-passable flow exists on the boundary $\partial \Omega_{ij}$ for
Fig. 7. The coming barrier windows on the α-side of ∂Ω_{ij} in the (2k_α,m_β)-semi-passable flow: (a) partial flow barrier and (b) full flow barrier. The dark surface is the flow barrier surface. The red solid and dashed curves with arrows are G-functions of flows on α and β-domains. The blue curves are semi-passable flows. The white surface represents “no flow barrier” (x_\alpha, m_\beta \in \{0, 1, 2, \ldots \}).

x_m \in S \subseteq \partial \Omega_{ij}

\begin{align*}
&h_{\alpha} G_{\partial \Omega_{ij}}^{(\alpha) (\beta)} (x_m, q^{(\alpha)}) \in [h_{\alpha} G_{\partial \Omega_{ij}}^{(\alpha) (\beta)} (x_m, q_1^{(\alpha)}), \]
&\quad [h_{\alpha} G_{\partial \Omega_{ij}}^{(\alpha) (\beta)} (x_m, q_2^{(\alpha)})] \\
&\subset [0, \infty].
\end{align*}

(i) The coming flow in the semi-passable flow cannot pass through the flow barrier on the α-side of the boundary ∂Ω_{ij} at q^{(\alpha)} \in (q_1^{(\alpha)}, q_2^{(\alpha)}) if and only if

\begin{align*}
&h_{\alpha} G_{\partial \Omega_{ij}}^{(\alpha)} (x_m, t_m -) \in [h_{\alpha} G_{\partial \Omega_{ij}}^{(\alpha)} (x_m, q_1^{(\alpha)}), \\
&\quad h_{\alpha} G_{\partial \Omega_{ij}}^{(\alpha)} (x_m, q_2^{(\alpha)})].
\end{align*}

(ii) The coming flow in the semi-passable flow cannot pass over the flow barrier on the α-side of the boundary at q^{(\alpha)} = \tilde{q}_\sigma^{(\alpha)} (\sigma \in \{1, 2\}) if and
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Fig. 8. The door of the flow barrier wall on the $\alpha$-side: (a) opened door (b) closed door. The dark surface is the flow barrier surface. The red solid and dashed curves with arrows are $G$-functions of flows on $\alpha$ and $\beta$-domains. The blue curves are semi-passable flows. The white surface represents "no flow barrier" ($k, m, n \in \{0, 1, 2, \ldots\}$).

\begin{align}
\text{only if } & \quad \mathcal{h}_\alpha G_{ij}^{(s_\alpha, n)}(x_m, t_{m-}) \\
& = \mathcal{h}_\alpha G_{ij}^{(s_\alpha, n, s_\alpha - \beta)}(x_m, q^{(\alpha)}_m) \
& \quad \text{for } s_\alpha = 0, 1, \ldots, l_\alpha - 1; \nonumber \\
\text{and } & \quad (-1)^{s_\alpha} \mathcal{h}_\alpha G_{ij}^{(l_\alpha, n)}(x_m, t_{m-}) \\
& - G_{ij}^{(l_\alpha, n, s_\alpha - \beta)}(x_m, q^{(\alpha)}_m) < 0. \quad (43)
\end{align}

\begin{align}
\text{the boundary at } & \quad q^{(\alpha)}_m = q^{(\alpha)}_m(\sigma \in \{1, 2\}) \text{ if and only if} \\
& \quad \mathcal{h}_\alpha G_{ij}^{(s_\alpha, n)}(x_m, t_{m-}) \\
& = \mathcal{h}_\alpha G_{ij}^{(s_\alpha, n, s_\alpha - \beta)}(x_m, q^{(\alpha)}_m(\sigma)) \in (0, \infty) \
& \quad \text{for } s_\alpha = 0, 1, \ldots, l_\alpha - 1; \nonumber \\
\text{and } & \quad (-1)^{s_\alpha} \mathcal{h}_\alpha G_{ij}^{(l_\alpha, n)}(x_m, t_{m-}) \\
& - G_{ij}^{(l_\alpha, n, s_\alpha - \beta)}(x_m, q^{(\alpha)}_m(\sigma)) > 0. \quad (44)
\end{align}

(iii) The coming flow in the semi-passable flow passes over the flow barrier on the $\alpha$-side of
(i) From Definition 5, the condition in Eq. (40) is obtained, vice versa.

(ii) An auxiliary flow determined by the flow barrier is introduced as a fictitious flow $x^{(\alpha)}(t)$. Since $x^{(\alpha)}(t_{m \pm}) = x(0)(t_{m \pm})$ and $x^{(\alpha)}(t_{m \pm}) = x(0)(t_{m \pm})$, the $G$-function definition gives

$$n_{G_{\alpha_{i+j}}}^{\prime}(0)(t_{m \pm}) = \sum_{s_{i}=0}^{l_{i}-1} G_{\alpha_{i+j}}^{(\alpha_{i})} \cdot (x_{s_{i}}, t_{m \pm}, q^{(\alpha_{i})}_{s_{i}}) + o(\epsilon_{l_{i}+1}),$$

for the upper limit of the coming flow barrier (i.e. $\sigma = 2$), and

$$n_{G_{\alpha_{i+j}}}^{\prime}(0)(t_{m \pm}) = \sum_{s_{i}=0}^{l_{i}-1} G_{\alpha_{i+j}}^{(\alpha_{i})} \cdot (x_{s_{i}}, t_{m \pm}, q^{(\alpha_{i})}_{s_{i}}) + o(\epsilon_{l_{i}+1}),$$

for the lower limit of the coming flow barrier (i.e. $\sigma = 1$). With Eq. (18), if a flow cannot pass over the semi-passable flow barrier, one obtains the conditions in Eq. (43), vice versa.

(iii) Similar to the proof of (ii), the definition of a coming flow passing through the flow barrier gives

$$n_{G_{\alpha_{i+j}}}^{\prime}(0)(t_{m \pm}) = \sum_{s_{i}=0}^{l_{i}-1} G_{\alpha_{i+j}}^{(\alpha_{i})} \cdot (x_{s_{i}}, t_{m \pm}, q^{(\alpha_{i})}_{s_{i}}) + o(\epsilon_{l_{i}+1}),$$

for the upper limit of the coming flow barrier (i.e. $\sigma = 2$), and

$$n_{G_{\alpha_{i+j}}}^{\prime}(0)(t_{m \pm}) = \sum_{s_{i}=0}^{l_{i}-1} G_{\alpha_{i+j}}^{(\alpha_{i})} \cdot (x_{s_{i}}, t_{m \pm}, q^{(\alpha_{i})}_{s_{i}}) + o(\epsilon_{l_{i}+1}),$$

for the lower limit of the coming flow barrier (i.e. $\sigma = 1$). If the flow passes over the
The coming and leaving flows in the semi-passable flow barrier, one obtains the conditions in Eq. (44), and vice versa.

Theorem 2. For a discontinuous dynamical system in Eq. (17), there is a point $x(0,t_m) \equiv x_m \in \partial\Omega$ at time $t_m$ between two adjacent domains $\Omega_n, \Omega_m (n = i,j)$. Suppose a coming flow barrier $\mathbf{F}^{(\alpha)}(\mathbf{x}(0,t),\mathbf{x}_n,\mathbf{q}(0))$ in the $(2k_a:m_3)$-passable flow exists on the boundary $\partial\Omega_i$ for $q(0) \in [q_1^{(\alpha)}, q_2^{(\alpha)}]$ for $x_m \in S \subseteq \partial\Omega_i$,

$$C^{(\alpha)}(x_m, q(0)) = 0 \quad \text{for } s_\alpha = 0, 1, \ldots, 2k_a - 1;$$

$$h_\alpha G^{(2k_a^{(\alpha)},q(0))}(x_m, q(0)) \in [h_\alpha G^{(2k_a^{(\alpha)},q(0))}(x_m, q_1^{(\alpha)}), h_\alpha G^{(2k_a^{(\alpha)},q(0))}(x_m, q_2^{(\alpha)})] \subset [0,\infty).$$

The coming and leaving flows in the $(2k_a:m_3)$-semi-passable flow satisfies

$$C^{(\alpha)}(x_m, i_{m-}) = 0 \quad \text{for } s_\alpha = 0, 1, \ldots, 2k_a - 1;$$

$$C^{(\alpha)}(x_m, i_{m+}) = 0 \quad \text{for } s_\alpha = 0, 1, \ldots, m_3 - 1;$$

$$h_\alpha G^{(2k_a^{(\alpha)},q(0))}(x_m, i_{m-}) > 0 \quad \text{and} \quad h_\alpha G^{(2k_a^{(\alpha)},q(0))}(x_m, i_{m+}) > 0.$$ (45)

(i) The coming flow in the $(2k_a:m_3)$-semi-passable flow cannot pass through the flow barrier on the $\alpha$-side at $q^{(\alpha)} \in (q_1^{(\alpha)}, q_2^{(\alpha)})$ if and only if

$$h_\alpha G^{(2k_a^{(\alpha)},q(0))}(x_m, l_{m-}) = h_\alpha G^{(2k_a^{(\alpha)},q(0))}(x_m, q_1^{(\alpha)}),$$

$$h_\alpha G^{(2k_a^{(\alpha)},q(0))}(x_m, l_{m+}) = h_\alpha G^{(2k_a^{(\alpha)},q(0))}(x_m, q_2^{(\alpha)}).$$ (46)

(ii) The coming flow in the $(2k_a:m_3)$-semi-passable flow cannot pass over the flow barrier on the $\alpha$-side at $q^{(\alpha)} = q_\alpha^{(\beta)} (\sigma \in \{1,2\})$ if and only if

$$h_\alpha G^{(\alpha)}(x_m, l_{m-}) = h_\alpha G^{(\alpha)}(x_m, q_\alpha^{(\beta)}),$$

$$h_\alpha G^{(\alpha)}(x_m, l_{m+}) = h_\alpha G^{(\alpha)}(x_m, q_2^{(\beta)}).$$

Proof

(i) From Definition 3, the condition in Eq. (30) is obtained vice versa.

(ii) An auxiliary flow determined by the flow barrier is introduced as a fictitious flow $x(\alpha^{(\alpha)}(t))$. Since $x(\alpha^{(\alpha)}(t_{m-})) = x(0)(t_{m-})$ and $x(\alpha^{(\alpha)}(t_{m+})) = x(0)(t_{m+})$, the $G$-function definition gives

$$h_\alpha G^{(\alpha)}(x(0)(t_{m+}))(x(0)(t_{m-}), q_\alpha^{(\beta)}) = [x(\alpha^{(\alpha)}(t_{m-}), q_\alpha^{(\beta)}) - x(0)(t_{m+}, q_\alpha^{(\beta)})].$$

$$= \sum_{s_\alpha=0}^{l_{m-}} G^{(\alpha)}(x_m, l_{m+}, q_\alpha^{(\beta)}) \cdot e^{s_\alpha+1}$$

$$+ \sum_{s_\alpha=2k_a}^{l_{m-}+1} G^{(\alpha)}(x_m, l_{m+}, q_\alpha^{(\beta)}) \cdot e^{s_\alpha+1},$$

$$h_\alpha G^{(\alpha)}(x(0)(t_{m+}))(x(0)(t_{m-}), q_\alpha^{(\beta)}) = [x(\alpha^{(\alpha)}(t_{m-}), q_\alpha^{(\beta)}) - x(0)(t_{m+}, q_\alpha^{(\beta)})].$$

$$= \sum_{s_\alpha=0}^{2k_a} G^{(\alpha)}(x_m, l_{m+}, q_\alpha^{(\beta)}) \cdot e^{s_\alpha+1}$$

$$+ \sum_{s_\alpha=2k_a}^{l_{m-}+1} G^{(\alpha)}(x_m, l_{m+}, q_\alpha^{(\beta)}) \cdot e^{s_\alpha+1}.$$
Because of the upper limit of the flow barrier (i.e. \( \sigma < 1 \)), with Eq. (48), if a coming flow cannot pass over the flow barrier at the critical point of the flow barrier in the semi-passable flow, one obtains the conditions in Eq. (48), vice versa.

(iii) In a similar fashion as in (ii), the definitions for the coming flow of \( \mathbf{x}^{(\alpha)}(t) \) passing over the coming flow barrier lead to

\[
\mathbf{n}^T_{\partial \Omega_{ij}}(\mathbf{x}^{(0)}(t_{m+})) \cdot [\mathbf{x}^{(\alpha)}(t_{m+}) - \mathbf{x}^{(\alpha-\beta)}(t_{m+})] = \begin{cases} 0 & \text{for } \mathbf{n}_{\partial \Omega_{ij}} \rightarrow \Omega_j, \\ \neq 0 & \text{for } \mathbf{n}_{\partial \Omega_{ij}} \rightarrow \Omega_\alpha \end{cases}
\]

for the upper limit of the flow barrier (i.e. \( \sigma = 2 \)), and

\[
\mathbf{n}^T_{\partial \Omega_{ij}}(\mathbf{x}^{(0)}(t_{m+})) \cdot [\mathbf{x}^{(\alpha)}(t_{m+}) - \mathbf{x}^{(\alpha-\beta)}(t_{m+})] = \begin{cases} 0 & \text{for } \mathbf{n}_{\partial \Omega_{ij}} \rightarrow \Omega_j, \\ \neq 0 & \text{for } \mathbf{n}_{\partial \Omega_{ij}} \rightarrow \Omega_\alpha \end{cases}
\]

for the lower limit of the flow barrier (i.e. \( \sigma = 1 \)). With Eq. (18), if a coming flow cannot pass over the flow barrier on the \( \alpha \)-side at the critical point of the flow barrier in the semi-passable flow, one obtains the conditions in Eq. (48), vice versa.

4.2. Leaving flow barriers for passable flows

For a leaving flow of the semi-passable passable flow to the boundary, as in the coming flow, there is a leaving flow barrier on the boundary.

Definition 13. For a discontinuous dynamical system in Eq. (17), there is a point \( \mathbf{x}^{(0)}(t_m) \equiv x_m \in \partial \Omega_{ij} \) at time \( t_m \) between two adjacent domains \( \Omega_\alpha (\alpha = i,j) \). There is a vector field...
flows in the semi-passable flow satisfy
\[ \partial_x f_{\text{passable flow if the following conditions are satisfied.}} \]

(ii) The leaving flow of \( \alpha, \beta \) at the critical points of the flow barrier (i.e. \( x_i, j \) for \( \sigma = 1, 2 \)) are called the lower and upper limits of the leaving flow barrier on the \( \beta \)-side.

(i) The leaving flow of \( x^{(\beta)} \) cannot enter the domain \( \Omega_1 \) at the point of \( q^{(\beta)} \in (q_1, q_2) \) if
\[ x^{(\beta)}(t_{m+}) = x^{(\alpha^{(\beta)})}(t_{m+}; q_2^{(\beta)}) = x_m, \]
\[ h_{\alpha}G^{(\alpha^{(\beta)})}_{\Omega_1}(t_m, \Omega_1) > 0 \text{ and } h_{\alpha}G^{(\alpha^{(\beta)})}_{\Omega_1}(t_m, \Omega_1) > 0. \]

The vector field of \( F^{(\alpha^{(\beta)})}(x^{(\beta)}, t, \pi, q^{(\beta)}) \) is called the leaving flow barrier on the \( \beta \)-side in the semi-passable flow if the following conditions are satisfied. The critical values of \( F^{(\alpha^{(\beta)})}(x^{(\beta)}, t, \pi, q^{(\beta)}) \) for \( \sigma = 1, 2 \) are called the lower and upper limits of the leaving flow barrier on the \( \beta \)-side.

Definition 14. For a discontinuous dynamical system in Eq. (17), there is a point \( x^{(\beta)}(t_m) \equiv x_m \in \partial \Omega_1 \) at time \( t_m \) between two adjacent domains \( \Omega_{\alpha} \) (\( \alpha = i, j \)). There is a vector field of \( F^{(\alpha^{(\beta)})}(x^{(\beta)}, t, \pi, \sigma^{(\beta)}) \) for \( \sigma^{(\beta)} \in [q_1^{(\beta)}, q_2^{(\beta)}] \) on the boundary \( \partial \Omega_1 \) with the \( G \)-functions
\[ G^{(\alpha^{(\beta)})}_{\partial \Omega_1}(x_m, \sigma^{(\beta)}) = 0 \quad \text{for } s_{\beta} = 0, 1, \ldots, m_{\beta} - 1; \]
\[ h_{\alpha}G^{(\alpha^{(\beta)})}_{\partial \Omega_1}(x_m, \sigma^{(\beta)}) > 0 \text{ and } h_{\alpha}G^{(\alpha^{(\beta)})}_{\partial \Omega_1}(x_m, \sigma^{(\beta)}) > 0. \]

(iii) The leaving flow of \( x^{(\beta)} \) enters the domain \( \Omega_\beta \) at the critical points of the flow barrier (i.e. \( q^{(\beta)} = q_{\beta}^{(\beta)}, \sigma \in [1, 2] \) if
\[ x^{(\beta)}(t_{m+}) = x^{(\alpha^{(\beta)})}(t_{m+}; q_{\beta}^{(\beta)}) = x_m, \]
\[ h_{\alpha}G^{(\alpha^{(\beta)})}_{\partial \Omega_1}(t_m, \Omega_1) = h_{\alpha}G^{(\alpha^{(\beta)})}_{\partial \Omega_1}(x_m, q_{\beta}^{(\beta)}) \neq 0 \]
for \( s_{\beta} = 0, 1, 2, \ldots, l_{\beta} - 1; \)
\[ (-1)^{i}h_{\alpha}G^{(\alpha^{(\beta)})}_{\partial \Omega_1}(x_m(t_{m+})) = x^{(\beta)}(t_{m+}) \quad \text{and } \quad h_{\alpha}G^{(\alpha^{(\beta)})}_{\partial \Omega_1}(x_m, \Omega_1) > 0. \]

The vector field of \( F^{(\alpha^{(\beta)})}(q^{(\beta)}, t, \pi, q^{(\beta)}) \) is called the leaving flow barrier on the \( \beta \)-side in the (2\( k_{\alpha} \); \( m_{\beta} \))-semi-passable flow if the following conditions are satisfied. The critical values of \( F^{(\alpha^{(\beta)})}(q^{(\beta)}, t, \pi, \sigma_{\beta}^{(\beta)}) \) (\( \sigma = 1, 2 \)) are called the lower and upper limits of the leaving flow barrier on the \( \beta \)-side.

(i) The leaving flow of \( x^{(\beta)} \) cannot enter the domain \( \Omega_\beta \) at \( q^{(\beta)} \in (q_1^{(\beta)}, q_2^{(\beta)}) \),
\[ x^{(\beta)}(t_{m+}) = x^{(\alpha^{(\beta)})}(t_{m+}; q_{\beta}^{(\beta)}) = x_m, \]
\[ h_{\alpha}G^{(\alpha^{(\beta)})}_{\partial \Omega_1}(t_m, \Omega_1) = h_{\alpha}G^{(\alpha^{(\beta)})}_{\partial \Omega_1}(x_m, q_{\beta}^{(\beta)}) \neq 0 \]
for \( s_{\beta} = 0, 1, 2, \ldots, l_{\beta} - 1; \)
\[ (-1)^{i}h_{\alpha}G^{(\alpha^{(\beta)})}_{\partial \Omega_1}(x_m(t_{m+})) = x^{(\beta)}(t_{m+}) \quad \text{and } \quad h_{\alpha}G^{(\alpha^{(\beta)})}_{\partial \Omega_1}(x_m, \Omega_1) > 0. \]
\[
G^{(m_{1}, \beta)}_{(\partial \Omega_{1}, \xi, t_{m-})}(x_{m}, t_{m-}) \in \left( h_{0} G^{(m_{2}, \alpha, \sigma_{0}, \beta)}_{(\partial \Omega_{1}, \xi, t_{m-})}(x_{m}, q_{1}^{(\beta)}), h_{0} G^{(m_{2}, \alpha, \sigma_{1}, \beta)}_{(\partial \Omega_{1}, \xi, t_{m-})}(x_{m}, q_{2}^{(\beta)}) \right).
\]

(ii) The leaving flow of \( x^{(\beta)} \) cannot enter the domain \( \Omega_{\beta} \) at the critical points of the flow barrier (i.e. \( q^{(\beta)} = q_{\sigma_{\beta}}^{(\beta)}, \sigma \in \{1, 2\} \)) if
\[
x^{(\beta)}(t_{m-}) = x^{(\alpha, \sigma_{\beta})}(t_{m-}, q_{\sigma_{\beta}}^{(\beta)}) = x_{m},
\]

\[
h_{0} G^{(m_{2}, \alpha, \sigma_{0}, \beta)}_{(\partial \Omega_{1}, \xi, t_{m-})} (x_{m}, t_{m-})
\]

\[
= h_{0} G^{(m_{2}, \alpha, \sigma_{0}, \beta)}_{(\partial \Omega_{1}, \xi, t_{m-})} (x_{m}, q_{2}^{(\beta)}) \neq 0
\]

for \( s_{\beta} = m_{\beta}, m_{\beta} + 1, \ldots, \beta - 1; \)
\[
-1)^{\alpha} h_{m} \partial_{n} \partial_{n_{\beta}} (x^{(\alpha)}(t_{m+}))
\]

\[
\cdot [x^{(\beta)}(t_{m+}) - x^{(\alpha, \sigma_{\beta})}(t_{m+}, q_{\sigma_{\beta}}^{(\beta)})] < 0.
\]

(iii) The leaving flow of \( x^{(\beta)} \) enters the domain \( \Omega_{\beta} \) at the critical points of the flow barrier (i.e. \( q^{(\beta)} = q_{\sigma_{\beta}}^{(\beta)}, \sigma \in \{1, 2\} \)) if
\[
x^{(\beta)}(t_{m-}) = x^{(\alpha, \sigma_{\beta})}(t_{m-}, q_{\sigma_{\beta}}^{(\beta)}) = x_{m},
\]

\[
h_{0} G^{(m_{2}, \alpha, \sigma_{0}, \beta)}_{(\partial \Omega_{1}, \xi, t_{m-})} (x_{m}, t_{m-})
\]

\[
= h_{0} G^{(m_{2}, \alpha, \sigma_{0}, \beta)}_{(\partial \Omega_{1}, \xi, t_{m-})} (x_{m}, q_{2}^{(\beta)}) \neq 0
\]

for \( s_{\beta} = m_{\beta}, m_{\beta} + 1, \ldots, \beta - 1; \)
\[
-1)^{\alpha} h_{m} \partial_{n} \partial_{n_{\beta}} (x^{(\alpha)}(t_{m+}))
\]

\[
\cdot [x^{(\beta)}(t_{m+}) - x^{(\alpha, \sigma_{\beta})}(t_{m+}, q_{\sigma_{\beta}}^{(\beta)})] > 0.
\]

To explain flow barriers on the \( \beta \)-side of the boundary, the corresponding \( G \)-functions are presented in Fig. 9. The \( G \)-function of the \( \beta \)-flow relative to the leaving flow barrier is denoted by the dashed curve. The thick line on the boundary represents the \( G \)-function of the flow barrier. For \( m_{\beta} \rightarrow -1 \), one has \( h_{m} = +1 \). Suppose there is a leaving flow barrier on the boundary \( \partial \Omega_{1} \), with lower and upper limits of \( G^{(m_{2}, \alpha, \sigma_{0}, \beta)}_{(\partial \Omega_{1}, \xi, t_{m-})} (q_{1}^{(\beta)}) > 0 \) and \( G^{(m_{2}, \alpha, \sigma_{1}, \beta)}_{(\partial \Omega_{1}, \xi, t_{m-})} (q_{2}^{(\beta)}) > 0 \). The leaving flow cannot leave the boundary \( \partial \Omega_{1} \) for \( G^{(m_{2}, \alpha, \sigma_{0}, \beta)}_{(\partial \Omega_{1}, \xi, t_{m-})} \) \( \in \left[ G^{(m_{2}, \alpha, \sigma_{0}, \beta)}_{(\partial \Omega_{1}, \xi, t_{m-})} (q_{1}^{(\beta)}), G^{(m_{2}, \alpha, \sigma_{1}, \beta)}_{(\partial \Omega_{1}, \xi, t_{m-})} (q_{2}^{(\beta)}) \right] \). Because no flow barriers exist on the \( \alpha \)-side of the boundary, the \( \alpha \)-flow with \( G^{(m_{1}, \beta)}_{(\partial \Omega_{1}, \xi, t_{m-})} > 0 \) can arrive to the \( \beta \)-side of the boundary \( \partial \Omega_{1} \). Only if \( G^{(m_{1}, \beta)}_{(\partial \Omega_{1}, \xi, t_{m-})} \notin \left[ G^{(m_{2}, \alpha, \sigma_{0}, \beta)}_{(\partial \Omega_{1}, \xi, t_{m-})} (q_{1}^{(\beta)}), G^{(m_{2}, \alpha, \sigma_{1}, \beta)}_{(\partial \Omega_{1}, \xi, t_{m-})} (q_{2}^{(\beta)}) \right] \), the leaving flow can leave the boundary, as sketched in Fig. 9(a). Similarly, for \( m_{\alpha} \rightarrow -1 \), the corresponding \( G \)-functions of the leaving flow are presented in Fig. 9(b). Similarly, the partial and full flow barriers on the \( \beta \)-side can be defined as in Sec. 4.1.

Definition 15. For a discontinuous dynamical system in Eq. (17), there is a point \( x^{(\beta)}(t_{m}) \equiv x_{m} \in \partial \Omega_{1} \) at time \( t_{m} \) between two adjacent domains \( \Omega_{\alpha} (x = i, j) \). Suppose there is a leaving flow barrier of \( x^{(\alpha, \beta)}(t_{m}, x_{m}, q_{\beta}) \) on the \( \beta \)-side in the \( (2k_{m}, m_{\beta}) \)-semi-passable flow for \( q^{(\beta)} \in \left[ q_{1}^{(\beta)}, q_{2}^{(\beta)} \right] \) on the boundary \( \partial \Omega_{1} (\alpha = i, j) \) and \( \alpha \neq \beta \).

(i) The leaving flow barrier in the \( (2k_{m}, m_{\beta}) \)-semi-passable flow is partial on the \( \beta \)-side if \( x_{m} \in S \subset \partial \Omega_{1} \).

(ii) The leaving flow barrier in the \( (2k_{m}, m_{\beta}) \)-semi-passable flow is full on the \( \beta \)-side if \( x_{m} \in S \subset \partial \Omega_{1} \).

The partial and full leaving flow barriers on the \( \beta \)-side of the boundary \( \partial \Omega_{1} \) are sketched in Fig. 10. The partial flow barriers only exist on subsets of the boundary (i.e. \( S \subset \partial \Omega_{1} \)). On the other subsets \( (\partial \Omega_{1}/S) \), the leaving flow barriers do not exist, and the leaving flow from such subsets of the boundary can get into the domain \( \Omega_{\beta} \), as shown in Fig. 10(a). If the leaving flow barrier exists on \( S = \partial \Omega_{1} \), such a flow barrier is called the full leaving flow barrier, which is presented in Fig. 10(b). The flow barriers possess the lower and upper limits of the flow barrier on the boundary. The other discussions about the leaving flow barriers on the \( \beta \)-side can be similarly discussed as for the coming flow on the \( \alpha \)-side, such as the infinity flow barrier, flow barrier fences, flow barrier widows and flow barrier doors of the leaving flow on the \( \beta \)-side for the semi-passable flow.

For a semi-passable flow on the boundary \( \partial \Omega_{1} \), if the leaving flow barrier exists on the \( \beta \)-side of such a boundary with a subset \( S \subset \partial \Omega_{1} \), the flow still cannot pass through the boundary under the following condition
\[
G^{(m_{2}, \alpha, \sigma_{0}, \beta)}_{(\partial \Omega_{1}, \xi, t_{m-})} (x_{m}, t_{m-}) \in \left( h_{0} G^{(m_{2}, \alpha, \sigma_{0}, \beta)}_{(\partial \Omega_{1}, \xi, t_{m-})} (x_{m}, q_{1}^{(\beta)}), h_{0} G^{(m_{2}, \alpha, \sigma_{1}, \beta)}_{(\partial \Omega_{1}, \xi, t_{m-})} (x_{m}, q_{2}^{(\beta)}) \right).
\]
for $\mathbf{x}_m \in S \subseteq \partial \Omega_{ij}$. For this case, the dynamical system along the boundary in the $\beta$-domain will be constrained by the boundary, i.e.

$$\dot{\mathbf{x}}^{(\beta)} = \mathbf{F}^{(\beta)}(\mathbf{x}^{(\beta)}, t, \mathbf{p}_\beta) \quad \text{in} \quad \Omega_\beta \{ \beta \in \{i, j\} \},$$

with $\varphi_j(x^{(\beta)}, t, \lambda) = 0$ on $\partial \Omega_{ij}$.

(61)

The vector field is given by the vector field on the domain of $\Omega_\beta$. Because the flow barrier exists on the $\beta$-side of the boundary $\partial \Omega_{ij}$, the flow will be along the boundary on the $\beta$-side until the condition in Eq. (60) cannot be satisfied. If the tangential component of the vector field in Eq. (61) on the boundary is zero, the system will stay on the specific point of the boundary until the normal vector field overcomes the flow barrier. This flow can be called the “standing flow” on the $\beta$-side of the boundary $\partial \Omega_{ij}$.

**Theorem 3.** For a discontinuous dynamical system in Eq. (17), there is a point $\mathbf{x}^{(0)}(t_m) \equiv \mathbf{x}_m \in \partial \Omega_{ij}$ at time $t_m$ between two adjacent domains $\Omega_\alpha$ ($\alpha = i, j$). For $\mathbf{x}_m \in S \subseteq \partial \Omega_{ij}$, there is a semi-passable flow barrier $\mathbf{F}^{(\alpha)}(\mathbf{x}^{(\beta)}, t, \pi_\beta, q^{(\beta)})$ for $q^{(\beta)} \in [q^{(\beta)}_1, q^{(\beta)}_2]$ on the boundary $\partial \Omega_{ij}$ with

$$h_{\alpha, \beta} \mathcal{G}^{(\alpha, \beta)}(\mathbf{x}_m, q^{(\beta)})$$

$$\in [h_{\alpha, \beta} \mathcal{G}^{(\alpha, \beta)}(\mathbf{x}_m, q^{(\beta)}_1), h_{\alpha, \beta} \mathcal{G}^{(\alpha, \beta)}(\mathbf{x}_m, q^{(\beta)}_2)] \subset [0, \infty)$$

(62)
(\alpha, \beta \in \{i, j\} \text{ and } \alpha \neq \beta). \text{ The semi-passable flow satisfies}
\begin{align*}
h_0 G_{\partial \Omega}^{(\alpha)}(\mathbf{x}_m, t_m -) > 0 \quad \text{and} \\
h_0 G_{\partial \Omega}^{(\beta)}(\mathbf{x}_m, t_m +) > 0. \quad (63)
\end{align*}

(i) \text{ The leaving flow in the semi-passable flow cannot pass through the flow barrier on the } \beta \text{-side at } q^{(\beta)} = \varphi_{\beta}(\sigma \in \{1, 2\}) \text{ if and only if}
\begin{align*}
h_0 G_{\partial \Omega}^{(\alpha)}(\mathbf{x}_m, t_m -) \in (h_0 G_{\partial \Omega}^{(\alpha \rightarrow \beta)}(\mathbf{x}_m, q^{(\beta)}), h_0 G_{\partial \Omega}^{(\alpha \rightarrow \beta)}(\mathbf{x}_m, q^{(\beta)})).
\end{align*}

(ii) \text{ The leaving flow in the semi-passable flow cannot enter the domain } \Omega_\beta \text{ on the } \beta \text{-side at } q^{(\beta)} = \varphi_{\beta}(\sigma \in \{1, 2\}) \text{ if and only if}
\begin{align*}
h_0 G_{\partial \Omega}^{(\alpha \rightarrow \beta)}(\mathbf{x}_m, t_m +) = \pm h_0 G_{\partial \Omega}^{(\alpha \rightarrow \beta)}(\mathbf{x}_m, q^{(\beta)}) \in (0, \infty) \\
\text{for } s_3 = 0, 1, \ldots, l_3 - 1; \\
- G_{\partial \Omega}^{(\alpha \rightarrow \beta)}(\mathbf{x}_m, q^{(\beta)}) < 0.
\end{align*}

(64)
(iii) The leaving flow in the semi-passable flow enters the domain $\Omega_j$ on the $\beta$-side at $q^{(\beta)} = q^{(\beta)}(\alpha \in [1, 2])$ if and only if

$$h_\delta G^{(m,\alpha,\beta)}_{\partial\Omega_j}(x_m, t_{m+}) = h_\delta G^{(m,\alpha,\beta)}_{\partial\Omega_j}(x_m, q^{(\beta)} ) \in (0, \infty) \text{ for } s_j = 0, 1, \ldots, t_j - 1;$$

(67)

$$-G^{(m,\alpha,\beta)}_{\partial\Omega_j}(x_m, q^{(\beta)} ) < 0.$$  

\textbf{Proof.} The proof of this theorem is similar to the proof of Theorem 1. \hfill \blacksquare

\textbf{Theorem 4.} For a discontinuous dynamical system in Eq. (17), there is a point $x^{(\alpha)}(t_m) \equiv x_m \in \partial\Omega_j$ at time $t_m$ between two adjacent domains $\Omega_j (\alpha = i, j)$. Suppose a $(2k_\alpha ; m_\alpha )$-semi-passable flow barrier $F^{(\alpha,\beta)}(x^{(\alpha)}, t, \pi_\alpha q^{(\beta)})$ exists on the boundary $\partial\Omega_j$ for $q^{(\beta)} \in [q^{(1)}, q^{(2)}]$ and $x_m \in S \subseteq \partial\Omega_j$ with

$$G^{(m,\alpha,\beta)}_{\partial\Omega_j}(x_m, q^{(\beta)} ) = 0 \text{ for } s_j = 0, 1, \ldots, m_j - 1;$$

$$h_\delta G^{(m,\alpha,\beta)}_{\partial\Omega_j}(x_m, t_{m+}) \in [h_\delta G^{(m,\alpha,\beta)}_{\partial\Omega_j}(x_m, q^{(1)}), h_\delta G^{(m,\alpha,\beta)}_{\partial\Omega_j}(x_m, q^{(2)}]) \subset [0, \infty)$$

$$\alpha, \beta \in \{i, j\} \text{ and } \alpha \neq \beta.$$  

The coming and leaving flows in the $(2k_\alpha ; m_\alpha )$-semi-passable flow satisfy

$$G^{(m,\alpha,\beta)}_{\partial\Omega_j}(x_m, t_{m-}, p_x, \lambda) = 0 \text{ for } s_m = 0, 1, \ldots, 2k_\alpha - 1;$$

$$G^{(m,\alpha,\beta)}_{\partial\Omega_j}(x_m, t_{m+}, p_x, \lambda) = 0 \text{ for } s_m = 0, 1, \ldots, m_j - 1;$$

$$h_\delta G^{(m,\alpha,\beta)}_{\partial\Omega_j}(x_m, t_{m-}) > 0 \text{ and }$$

$$h_\delta G^{(m,\alpha,\beta)}_{\partial\Omega_j}(x_m, t_{m+}) > 0.$$  

\textbf{Proof.} The proof of this theorem is the same as the proof of Theorem 2. \hfill \blacksquare

4.3. \textbf{Passable flows with both flow barriers}

In the previous two sections, the coming and leaving flow barriers of the semi-passable flow were considered separately. In fact, the coming and leaving flow barriers for the semi-passable flow can exist together. The switchability of the semi-passable flow with two flow barriers becomes more complex, which will be discussed as follows.

\textbf{Definition 16.} For a discontinuous dynamical system in Eq. (17), there is a point $x^{(\alpha)}(t_m) \equiv x_m \in \partial\Omega_j$ at time $t_m$ between two adjacent domains $\Omega_j (\alpha = i, j)$. Suppose there is a flow barrier of $F^{(\alpha,\beta)}(x^{(\alpha)}, t, \pi_\alpha q^{(\beta)})$ at $q^{(\alpha)} \in [q^{(1)}, q^{(2)}]$ for the coming flow of $x^{(\alpha)}$ on the $\alpha$-side of boundary $\partial\Omega_j$.
with
\[G^{(\alpha,\beta)}(x_m, q^{(\alpha)}) = [h_aG^{(\alpha,\beta)}(x_m, q^{(\alpha)}), h_bG^{(\alpha,\beta)}(x_m, q^{(\alpha)})] \subset [0, +\infty), \quad (72)\]

and also there is a flow barrier \(F^{(\alpha,\beta)}(x^{(\beta)}), t, \pi, q^{(\beta)}\) at \(q^{(\beta)} \in [q^{(\beta)}_1, q^{(\beta)}_2]\) for the leaving flow of \(x^{(\beta)}\) on the \(\beta\)-side of boundary \(\partial \Omega_j\) with
\[G^{(\alpha,\beta)}(x_m, q^{(\beta)}) = [h_aG^{(\alpha,\beta)}(x_m, q^{(\beta)}_1), h_bG^{(\alpha,\beta)}(x_m, q^{(\beta)}_2)] \subset [0, +\infty) \quad (73)\]

\((\alpha, \beta \in \{i, j\} \text{ and } \alpha \neq \beta)\). The coming and leaving flows in the semi-passable flow satisfy
\[h_aG^{(\alpha)}(x_m, t_{m-}) > 0 \quad \text{and} \quad h_bG^{(\beta)}(x_m, t_{m+}) > 0. \quad (74)\]

(i) The coming flow of \(x^{(\alpha)}\) cannot be switched to the leaving flow of \(x^{(\beta)}\) to form a semi-passable flow at the boundary if
\[
\begin{align*}
&\text{either} \\
&h_aG^{(\alpha)}(x_m, t_{m-}) \in \{h_aG^{(\alpha,\beta)}(x_m, q^{(\alpha)}), h_bG^{(\alpha,\beta)}(x_m, q^{(\alpha)})\}, \quad (75) \\
\text{or} \\
&h_bG^{(\beta)}(x_m, t_{m+}) \in \{h_aG^{(\alpha,\beta)}(x_m, q^{(\beta)}), h_bG^{(\alpha,\beta)}(x_m, q^{(\beta)})\}. \quad (76)
\end{align*}
\]

(ii) The coming flow of \(x^{(\alpha)}\) cannot be switched to the leaving flow of \(x^{(\beta)}\) to form a semi-passable flow at the boundary if \(\sigma_a \in \{1, 2\} \text{ and } \sigma_b \in \{1, 2\}\)
\[
\begin{align*}
&\text{either} \\
&G^{(\alpha,\alpha)}(x_m, t_{m-1}) = G^{(\alpha,\alpha)}(x_m, t_{m-1}, q^{(\alpha)}_{\sigma_a}) \neq 0 \quad \text{for } s_a = 0, 1, 2, \ldots, l_a - 1; \quad (77) \\
&\quad \quad \quad \quad \quad \quad (-1)^{\sigma_a}h_bG^{(\beta)}(x^{(\beta)}(t_{m+})): [x^{(\alpha)}(t_{m+}) - x^{(\alpha,\beta)}(t_{m+}, q^{(\beta)}_{\sigma_b})] < 0, \\
\text{or} \\
&G^{(\alpha,\beta)}(x_m, t_{m+}) = G^{(\alpha,\beta)}(x_m, q^{(\alpha)}_{\sigma_b}) \neq 0 \quad \text{for } s_b = 0, 1, 2, \ldots, l_b - 1; \quad (78) \\
&\quad \quad \quad \quad \quad \quad (-1)^{\sigma_b}h_aG^{(\alpha)}(x^{(\alpha)}(t_{m+})): [x^{(\beta)}(t_{m+}) - x^{(\alpha,\beta)}(t_{m+}, q^{(\alpha)}_{\sigma_a})] < 0.
\end{align*}
\]

(iii) The coming flow of \(x^{(\alpha)}\) is switched to the leaving flow of \(x^{(\beta)}\) to form a semi-passable flow at the boundary if \(\sigma_a \in \{1, 2\} \text{ and } \sigma_b \in \{1, 2\}\)
\[
\begin{align*}
&\text{both} \\
&G^{(\alpha,\alpha)}(x_{m}, t_{m-1}) = G^{(\alpha,\alpha)}(x_m, q^{(\alpha)}_{\sigma_a}) \neq 0 \quad \text{for } s_a = 0, 1, 2, \ldots, l_a - 1; \quad (79) \\
&\quad \quad \quad \quad \quad \quad (-1)^{\sigma_a}h_bG^{(\beta)}(x^{(\beta)}(t_{m+})): [x^{(\alpha)}(t_{m+}) - x^{(\alpha,\beta)}(t_{m+}, q^{(\beta)}_{\sigma_b})] > 0, \\
\text{and} \\
&G^{(\alpha,\beta)}(x_m, t_{m+}) = G^{(\alpha,\beta)}(x_m, q^{(\beta)}_{\sigma_b}) \neq 0 \quad \text{for } s_b = 0, 1, 2, \ldots, l_b - 1; \\
&\quad \quad \quad \quad \quad \quad (-1)^{\sigma_b}h_aG^{(\alpha)}(x^{(\alpha)}(t_{m+})): [x^{(\beta)}(t_{m+}) - x^{(\alpha,\beta)}(t_{m+}, q^{(\alpha)}_{\sigma_a})] > 0. \quad (80)
\end{align*}
\]

Definition 17. For a discontinuous dynamical system in Eq. (17), there is a point \(x^{(\beta)}(t_m) \equiv x_m \in \partial \Omega_j\) at time \(t_m\) between two adjacent domains \(\Omega_a \ (a = i, j)\). Suppose there is a flow barrier \(F^{(\alpha,\beta)}(x^{(\alpha)}, t, \pi, q^{(\beta)})\)
at \( q^{(a)} \in \{q_1^{(r)}, q_2^{(r)} \} \) for the incoming flow of \( x^{(a)} \) on the \( \alpha \)-side of boundary \( \partial \Omega_{ij} \) with

\[
G^{(s, \alpha \rightarrow \beta)}_{\partial \Omega_{ij}}(x_m, q^{(a)}) = 0 \quad \text{for} \quad s_a = 0, 1, \ldots, 2k_a - 1; \\
G^{(2k_a, \alpha \rightarrow \beta)}_{\partial \Omega_{ij}}(x_m, q^{(a)}) \in \{ h_a G^{(2k_a, \alpha \rightarrow \beta)}_{\partial \Omega_{ij}}(x_m, q_1^{(a)}), h_a G^{(2k_a, \alpha \rightarrow \beta)}_{\partial \Omega_{ij}}(x_m, q_2^{(a)}) \} \subset [0, +\infty),
\]

and also there is a flow barrier \( F^{(\alpha \rightarrow \beta)}(x^{(b)}, t, \pi, q^{(b)}) \) at \( q^{(b)} \in \{ q_1^{(r)}, q_2^{(r)} \} \) for the leaving flow of \( x^{(b)} \) on the boundary of \( \beta \)-side of boundary \( \partial \Omega_{ij} \) with

\[
G^{(m, \beta \rightarrow \alpha)}_{\partial \Omega_{ij}}(x_m, q^{(b)}) = 0 \quad \text{for} \quad s_b = 0, 1, \ldots, m_b - 1; \\
G^{(m, \beta \rightarrow \alpha)}_{\partial \Omega_{ij}}(x_m, q^{(b)}) \in \{ h_a G^{(m, \beta \rightarrow \alpha)}_{\partial \Omega_{ij}}(x_m, q_1^{(b)}), h_a G^{(m, \beta \rightarrow \alpha)}_{\partial \Omega_{ij}}(x_m, q_2^{(b)}) \} \subset [0, +\infty)
\]

\((\alpha, \beta) \in \{i, j\} \) and \( \alpha \neq \beta \). The coming and leaving flows in the \((2k_a : m_b)\)-semi-passable flow satisfies

\[
G^{(s, \alpha \rightarrow \beta)}_{\partial \Omega_{ij}}(x_m, t_m = 0) = 0 \quad \text{for} \quad s_a = 0, 1, \ldots, 2k_a - 1; \\
G^{(s, \alpha \rightarrow \beta)}_{\partial \Omega_{ij}}(x_m, t_m = 0) = 0 \quad \text{for} \quad s_b = 0, 1, \ldots, m_b - 1; \\
h_a G^{(m, \beta \rightarrow \alpha)}_{\partial \Omega_{ij}}(x_m, t_m = 0) > 0 \quad \text{and} \quad h_a G^{(m, \beta \rightarrow \alpha)}_{\partial \Omega_{ij}}(x_m, t_m = 0) > 0.
\]

(i) The coming flow of \( x^{(a)} \) cannot be switched to the leaving flow of \( x^{(b)} \) to form the \((2k_a : m_b)\)-semi-passable flow at the boundary if

either

\[
h_a G^{(s, \alpha \rightarrow \beta)}_{\partial \Omega_{ij}}(x_m, t_m = 0) \in \{ h_a G^{(2k_a, \alpha \rightarrow \beta)}_{\partial \Omega_{ij}}(x_m, q_1^{(a)}), h_a G^{(2k_a, \alpha \rightarrow \beta)}_{\partial \Omega_{ij}}(x_m, q_2^{(a)}) \},
\]

or

\[
h_a G^{(m, \beta \rightarrow \alpha)}_{\partial \Omega_{ij}}(x_m, t_m = 0) \in \{ h_a G^{(m, \beta \rightarrow \alpha)}_{\partial \Omega_{ij}}(x_m, q_1^{(b)}), h_a G^{(m, \beta \rightarrow \alpha)}_{\partial \Omega_{ij}}(x_m, q_2^{(b)}) \}.
\]

(ii) The coming flow of \( x^{(a)} \) cannot be switched to the leaving flow of \( x^{(b)} \) to form a \((2k_a : m_b)\)-semi-passable flow at the boundary if \( s_a \in \{1, 2\} \) and \( s_b \in \{1, 2\} \)

\[
x^{(a)}(t_m = 0) \in x^{(a \rightarrow \beta)}(t_m = 0), \\
g^{(s, \alpha \rightarrow \beta)}_{\partial \Omega_{ij}}(x_m, t_m = 1) \neq 0 \quad \text{for} \quad s_a = 2k_a, 2k_a + 1, \ldots, t_a - 1; \\
(\ref{eq:hs}) h_a G^{(s, \alpha \rightarrow \beta)}_{\partial \Omega_{ij}}(x_m, t_m = 1) \cdot x^{(a)}(t_m = 1) - x^{(a \rightarrow \beta)}(t_m = q_1^{(a)}); \\
x^{(b)}(t_m = 1) \neq x^{(a \rightarrow \beta)}(t_m = 1), \\
g^{(s, \beta \rightarrow \alpha)}_{\partial \Omega_{ij}}(x_m, t_m = 1) = G^{(s, \beta \rightarrow \alpha)}_{\partial \Omega_{ij}}(x_m, t_m = 1) ; \\
(\ref{eq:hs}) h_a G^{(s, \beta \rightarrow \alpha)}_{\partial \Omega_{ij}}(x_m, t_m = 1) \cdot x^{(b)}(t_m = 1) - x^{(a \rightarrow \beta)}(t_m = q_2^{(a)}); \\

\]

(iii) The coming flow of \( x^{(a)} \) is switched to the leaving flow of \( x^{(b)} \) to form a \((2k_a : m_b)\)-semi-passable flow at the boundary if \( s_a \in \{1, 2\} \) and \( s_b \in \{1, 2\} \)

\[
x^{(a)}(t_m = 0) \in x^{(a \rightarrow \beta)}(t_m = 0), \\
g^{(s, \alpha \rightarrow \beta)}_{\partial \Omega_{ij}}(x_m, t_m = 1) \neq 0 \quad \text{for} \quad s_a = 2k_a, 2k_a + 1, \ldots, t_a - 1; \\
(\ref{eq:hs}) h_a G^{(s, \alpha \rightarrow \beta)}_{\partial \Omega_{ij}}(x_m, t_m = 1) \cdot x^{(a)}(t_m = 1) - x^{(a \rightarrow \beta)}(t_m = q_1^{(a)}); \\
x^{(b)}(t_m = 1) \neq x^{(a \rightarrow \beta)}(t_m = 1), \\
g^{(s, \beta \rightarrow \alpha)}_{\partial \Omega_{ij}}(x_m, t_m = 1) = G^{(s, \beta \rightarrow \alpha)}_{\partial \Omega_{ij}}(x_m, t_m = 1) ; \\
(\ref{eq:hs}) h_a G^{(s, \beta \rightarrow \alpha)}_{\partial \Omega_{ij}}(x_m, t_m = 1) \cdot x^{(b)}(t_m = 1) - x^{(a \rightarrow \beta)}(t_m = q_2^{(a)});
To explain the above definition of the coming and leaving flow barriers in the semi-passable flow, the flow barriers on both sides of the boundary $\partial \Omega_{ij}$ are presented through the $G$-functions in Fig. 11. The red curves represent the $G$-function of the flows pertaining to the flow barriers in each domain $\Omega_\alpha$. The thick lines denote the $G$-functions of the flow barriers on both sides of the boundary. The shaded area is zoomed in for the boundary flow of $x^{(0)}$. To more intuitively illustrate the semi-passable flow with the flow barriers for the coming and leaving flows on the boundary. The $(2k_\alpha:m_\beta)$-semi-passable flow with the partial and full flow barriers on both sides of the boundary $\partial \Omega_{ij}$ are sketched in Fig. 12. For $x_m \in S \subset \partial \Omega_{ij}$, the two different colored surfaces represent the flow barriers at the $\alpha$ and $\beta$-side boundaries of the boundary $\partial \Omega_{ij}$. The two flow barriers can be the same, but the coming and leaving flows to the boundary are depicted by the solid and dashed curves.

**Theorem 5.** For a discontinuous dynamical system in Eq. (17), there is a point $x^{(0)}(t_m) \equiv x_m \in \partial \Omega_{ij}$ at time $t_m$ between two adjacent domains $\Omega_\alpha$ ($\alpha = i, j$). Suppose there is a flow barrier

![Diagram](image-url)
On Flow Barriers and Switchability in Discontinuous Dynamical Systems

Fig. 12. The coming and leaving flow barriers on both sides of the boundary $\partial \Omega_{ij}$ in the $(2k_{\alpha}, m_{\beta})$-semi-passable flow: (a) partial flow barrier and (b) full flow barrier. The dark and blue shaded surfaces are for two flow barriers on $\partial \Omega_{ij}$. The red curves are the $G$-functions relative to the flow barrier. The blue curves are semi-passable flows ($m_{\alpha}, m_{\beta} \in \{0, 1, 2, \ldots\}$).

The coming and leaving flows in the semi-passable flow satisfy

$$\hat{b}_\alpha G_{\partial \Omega_{ij}}^{(\alpha)}(x_{m}, t_{m-}) > 0 \quad \text{and} \quad \hat{b}_\beta G_{\partial \Omega_{ij}}^{(\beta)}(x_{m}, t_{m+}) > 0.$$
The coming flow of $x^{(s)}$ cannot be switched to the leaving flow of $x^{(s)}$ to form a semi-passable flow at the boundary if and only if

$$\begin{align*}
\text{either} & \quad h_n G_{\delta t}^{(s)}(x_m, t_m) \in \{ h_n G_{\delta t}^{(s)}, G_\sigma(x_m, q_1^{(s)}), h_n G_{\delta t}^{(s)}(x_m, q_2^{(s)}) \}, \\
\text{or} & \quad h_n G_{\delta t}^{(s)}(x_m, t_m) \in \{ h_n G_{\delta t}^{(s)}, G_\sigma(x_m, q_1^{(s)}), h_n G_{\delta t}^{(s)}(x_m, q_2^{(s)}) \}.
\end{align*}$$

(iii) The coming flow of $x^{(s)}$ is switched to the leaving flow of $x^{(s)}$ to form a semi-passable flow on the boundary if and only if for $\sigma_n \in \{1, 2\}$ and $\sigma_j \in \{1, 2\}$

$$\begin{align*}
\text{either} & \quad h_n G_{\delta t}^{(s)}(x_m, t_m) = h_n G_{\delta t}^{(s, \alpha, \beta)}(x_m, q_1^{(s)}), \\
& \quad (-1)^m h_n G_{\delta t}^{(s, \alpha, \beta)}(x_m, t_m) - G_{\delta t}^{(s, \alpha, \beta)}(x_m, q_1^{(s)}) < 0; \\
\text{or} & \quad h_n G_{\delta t}^{(s, \alpha, \beta)}(x_m, t_m) = h_n G_{\delta t}^{(s, \alpha, \beta)}(x_m, q_2^{(s)}), \\
& \quad (-1)^m h_n G_{\delta t}^{(s, \alpha, \beta)}(x_m, t_m) - G_{\delta t}^{(s, \alpha, \beta)}(x_m, q_2^{(s)}) < 0.
\end{align*}$$

Proof. Similar to the proof of Theorem 1, this theorem can be proved. □

Theorem 6. For a discontinuous dynamical system in Eq. (17), there is a point $x^{(s)}(t_m) \equiv x_m \in \partial \Omega_\alpha$ at time $t_m$ between two adjacent domains $\Omega_\alpha \ (\alpha = 1, 2)$. Suppose there is a flow barrier $F^{(s, \alpha, \beta)}(x^{(s)}, t, \pi, \alpha, \beta)$ at $q^{(s)} \in \{ q_1^{(s)}, q_2^{(s)} \}$ for the incoming flow of $x^{(s)}$ on the $\alpha$-side of boundary $\partial \Omega_\alpha$ with

$$G_{\delta t}^{(s, \alpha, \beta)}(x_m, q_1^{(s)}) = 0 \quad \text{for} \quad s_\alpha = 0, 1, \ldots, k_\alpha - 1;$$

$$G_{\delta t}^{(2s, \alpha, \beta)}(x_m, q_1^{(s)}) \in \{ h_n G_{\delta t}^{(2s, \alpha, \beta)}(x_m, q_1^{(s)}), h_n G_{\delta t}^{(2s, \alpha, \beta)}(x_m, q_2^{(s)}) \} \subset [0, +\infty),$$

and also there is a flow barrier $F^{(s, \alpha, \beta)}(x^{(s)}, t, \pi, \alpha, \beta)$ at $q^{(s)} \in \{ q_1^{(s)}, q_2^{(s)} \}$ for the $(m, \beta)$-order leaving flow of $x^{(s)}$ on the $\beta$-side of boundary $\partial \Omega_\beta$ with

$$G_{\delta t}^{(s, \alpha, \beta)}(x_m, q_2^{(s)}) = 0 \quad \text{for} \quad s_\beta = 0, 1, \ldots, m_\beta - 1;$$

$$G_{\delta t}^{(m, \alpha, \beta)}(x_m, q_2^{(s)}) \in \{ h_n G_{\delta t}^{(m, \alpha, \beta)}(x_m, q_1^{(s)}), h_n G_{\delta t}^{(m, \alpha, \beta)}(x_m, q_2^{(s)}) \} \subset [0, +\infty).$$

Proof. Similar to the proof of Theorem 1, this theorem can be proved. □
(α, β ∈ {i, j} and α ≠ β). The coming and leaving flows in the \((2k_α : m_β)\)-semi-passable flow satisfy

\[
G^{(s, k)}(x_m, l_m) = 0 \quad \text{for} \quad s = 0, 1, \ldots, 2k_α - 1,
\]

\[
G^{(s, k)}(x_m, l_m) = 0 \quad \text{for} \quad s = 0, 1, \ldots, m_β - 1,
\]

\[
h_αG^{(s, k)}(x_m, l_m) > 0, \quad \text{and} \quad h_αG^{(s, k)}(x_m, l_m) > 0.
\]

(i) The coming flow of \(x^{(α)}\) cannot be switched to the leaving flow of \(x^{(β)}\) to form the \((2k_α : m_β)\)-semi-passable flow if

\[
h_αG^{(s, k)}(x_m, l_m) \in \{h_αG^{(s, k)}(x_m, q_1^{(α)}), h_αG^{(s, k)}(x_m, q_2^{(α)})\}, \quad (103)
\]

or

\[
h_αG^{(s, k)}(x_m, l_m) \in \{h_αG^{(s, k)}(x_m, q_1^{(β)}), h_αG^{(s, k)}(x_m, q_2^{(β)})\}. \quad (104)
\]

(ii) The coming flow of \(x^{(α)}\) cannot be switched to the leaving flow of \(x^{(β)}\) to form the \((2k_α : m_β)\)-semi-passable flow at the boundary if and only if for \(σ_α ∈ \{1, 2\}\) and \(σ_β ∈ \{1, 2\}\)

\[
G^{(s, k)}(x_m, l_m) = G^{(s, k)}(x_m, q_1^{(β)}) \neq 0 \quad \text{for} \quad s = 2k_α, 2k_α + 1, \ldots, l_α - 1;
\]

\[
-1)^{s-2k_α}h_αG^{(s, k)}(x_m, l_m) + G^{(s, k)}(x_m, q_1^{(β)}) < 0. \quad (106)
\]

or

\[
G^{(s, k)}(x_m, l_m) = G^{(s, k)}(x_m, q_2^{(β)}) \neq 0 \quad \text{for} \quad s = m_β, m_β + 1, \ldots, l_β - 1;
\]

\[
-1)^{s-m_β}h_αG^{(s, k)}(x_m, l_m) + G^{(s, k)}(x_m, q_2^{(β)}) < 0. \quad (107)
\]

(iii) The coming flow of \(x^{(α)}\) is switched to the leaving flow \(x^{(β)}\) to form the \((2k_α : m_β)\)-semi-passable flow at the boundary if and only if for \(σ_α ∈ \{1, 2\}\) and \(σ_β ∈ \{1, 2\}\)

\[
G^{(s, k)}(x_m, l_m) = G^{(s, k)}(x_m, q_1^{(β)}) \neq 0 \quad \text{for} \quad s = 2k_α, 2k_α + 1, \ldots, l_α - 1;
\]

\[
-1)^{s-2k_α}h_αG^{(s, k)}(x_m, l_m) - G^{(s, k)}(x_m, q_1^{(β)}) > 0. \quad (108)
\]

and

\[
G^{(s, k)}(x_m, l_m) = G^{(s, k)}(x_m, q_2^{(β)}) \neq 0 \quad \text{for} \quad s = m_β, m_β + 1, \ldots, l_β - 1;
\]

\[
-1)^{s-m_β}h_αG^{(s, k)}(x_m, l_m) - G^{(s, k)}(x_m, q_2^{(β)}) > 0. \quad (109)
\]

(iv) The coming flow of \(x^{(α)}\) is switched to the leaving flow of \(x^{(β)}\) to form the \((2k_α : m_β)\)-semi-passable flow if

\[
G^{(s, k)}(x_m, l_m) \notin \{h_αG^{(s, k)}(x_m, q_1^{(α)}), h_αG^{(s, k)}(x_m, q_2^{(α)})\}. \quad (110)
\]

and

\[
G^{(s, k)}(x_m, l_m) \notin \{h_αG^{(s, k)}(x_m, q_1^{(β)}), h_αG^{(s, k)}(x_m, q_2^{(β)})\}. \quad (111)
\]

Proof. As in the proof of Theorem 2, this theorem can be proved. □

The foregoing two theorems presented that the coming and leaving flows on the boundary can pass over the corresponding flow barriers to form a semi-passable flow at the boundary. Theorems 5 and 6 give all the possible conditions for the coming flow to be switched into the leaving flow. The conditions for the leaving flow to pass over the corresponding flow barriers are presented in Fig. 13. The leaving flow at the critical point passes over the flow barrier and enters the domain \(Ω_β\). Similarly, the other cases can be illustrated for a better understanding of the flow barriers in the semi-passable flow to the separation boundary.
5. Flow Barriers of Sink Flows

Without any flow barriers, from [Luo, 2008a, 2008b], the necessary and sufficient conditions of a sink flow to move along the boundary in discontinuous dynamical systems are

\[
\begin{align*}
\mathbb{g}_{\alpha}(x_{m}, t, \pi_{\alpha}, q_{\alpha}) & > 0 \quad \text{and} \\
\mathbb{g}_{\beta}(x_{m}, t, \pi_{\beta}, q_{\beta}) & < 0.
\end{align*}
\] (112)

To investigate the sink flow property to the boundary with flow barriers, the sink flow barriers on the boundary will be discussed in this section.

Definition 18. For a discontinuous dynamical system in Eq. (17), there is a point \( x^{(0)}(t_{m}) \equiv x_{m} \in \partial \Omega_{ij} \) at time \( t_{m} \) between two adjacent domains \( \Omega_{i} \) (\( i \neq j \)). Suppose there is a vector field \( F^{(\alpha = 0)}(x^{(0)}, t, \pi_{\alpha}, q^{(0)}) \) for \( q^{(0)} \in [q_{1}^{(0)}, q_{2}^{(0)}] \) on the boundary \( \partial \Omega_{ij} \) with

\[
\begin{align*}
\mathbb{h}_{\alpha}G_{\alpha}(x_{m}, q^{(0)}) & > 0 \\
\mathbb{h}_{\beta}G_{\beta}(x_{m}, q^{(0)}) & < 0.
\end{align*}
\] (113)

Fig. 13. The flow passing over the leaving flow barriers on the boundary \( \partial \Omega_{ij} \): (a) partial flow barrier and (b) full flow barrier. The dark and blue surfaces are for the flow barriers on the \( \alpha \)-side and the \( \beta \)-side of the boundary \( \partial \Omega_{ij} \), respectively. The dark and blue shaded surfaces are for two flow barriers on \( \partial \Omega_{ij} \). The red curves are the \( G \)-functions relative to the flow barrier. The dark blue curves are semi-passable flows (\( m_{\alpha}, m_{\beta} \in \{0, 1, 2, \ldots \} \)).
The two possible coming flows in the sink flow satisfy
\[ h_\alpha G_\alpha G_\beta (x_m, t_m) > 0 \quad \text{and} \quad h_\beta G_\beta G_\alpha (x_m, t_m) < 0 \]  \hspace{1cm} (114)
\( \alpha, \beta \in \{1, 2\} \) and \( \alpha \neq \beta \). The vector field of \( F_{\alpha \rightarrow \beta} (x, t) \) is called the coming flow barrier in the sink flow on the \( \alpha \)-side of the boundary if the following conditions are satisfied. The critical values of \( F_{\alpha \rightarrow \beta} (x, t) \) at \( \pi \), \( q_\pi \) \( (\sigma = 1, 2) \) on the boundary \( \partial D_j \) are called the lower and upper limits of the coming flow barriers on the \( \alpha \)-side.

(i) The coming flow of \( x^{(\sigma)} \) cannot be switched to the boundary flow of \( x^{(0)} \) if
\[ x^{(\sigma)} (t_m) = x^{(\sigma \rightarrow 0)} (t_m, q^{(\sigma)}) = x_m, \]
\[ h_\alpha G_\alpha G_\beta (x_m, t_m) > 0 \quad \text{and} \quad h_\beta G_\beta G_\alpha (x_m, t_m) < 0 \]  \hspace{1cm} (115)
\( \alpha, \beta \in \{1, 2\} \) and \( \alpha \neq \beta \). The vector field of \( F_{\alpha \rightarrow \beta} (x, t) \) is called the coming flow barrier in the sink flow on the \( \alpha \)-side of the boundary if the following conditions are satisfied. The critical values of \( F_{\alpha \rightarrow \beta} (x, t) \) at \( \pi \), \( q_\pi \) \( (\sigma = 1, 2) \) on the boundary \( \partial D_j \) are called the lower and upper limits of the coming flow barriers on the \( \alpha \)-side.

(ii) The coming flow of \( x^{(\sigma)} \) cannot be switched to the boundary flow of \( x^{(0)} \) at the critical points of the flow barrier (i.e. \( q^{(\sigma)} = q^{(0)} \), \( \sigma \in \{1, 2\} \)) if
\[ x^{(\sigma)} (t_m) = x^{(\sigma \rightarrow 0)} (t_m, q^{(\sigma)}) = x_m, \]
\[ G_{\alpha j}^{(\sigma)} (x_m, t_m) = 0 \]  \hspace{1cm} (116)
\( \alpha, \beta \in \{1, 2\} \) and \( \alpha \neq \beta \). The vector field of \( F_{\alpha \rightarrow \beta} (x, t) \) is called the coming flow barrier in the sink flow on the \( \alpha \)-side of the boundary if the following conditions are satisfied. The critical values of \( F_{\alpha \rightarrow \beta} (x, t) \) at \( \pi \), \( q_\pi \) \( (\sigma = 1, 2) \) on the boundary \( \partial D_j \) are called the lower and upper limits of the coming flow barriers on the \( \alpha \)-side.

(iii) The coming flow of \( x^{(\sigma)} \) is switched to the boundary flow of \( x^{(0)} \) at the critical points of the flow barrier (i.e. \( q^{(\sigma)} = q_\sigma \), \( \sigma \in \{1, 2\} \)) if
\[ x^{(\sigma)} (t_m) = x^{(\sigma \rightarrow 0)} (t_m, q^{(\sigma)}) = x_m, \]
\[ G_{\alpha j}^{(\sigma)} (x_m, t_m) = 0 \]  \hspace{1cm} (117)
\( \alpha, \beta \in \{1, 2\} \) and \( \alpha \neq \beta \). The vector field of \( F_{\alpha \rightarrow \beta} (x, t) \) is called the coming flow barrier in the sink flow on the \( \alpha \)-side of the boundary if the following conditions are satisfied. The critical values of \( F_{\alpha \rightarrow \beta} (x, t) \) at \( \pi \), \( q_\pi \) \( (\sigma = 1, 2) \) on the boundary \( \partial D_j \) are called the lower and upper limits of the coming flow barriers on the \( \alpha \)-side.

\[ \text{Definition 19.} \quad \text{For a discontinuous dynamical system in Eq. (17), there is a point } x^{(0)} (t_m) \equiv x_m \in \partial D_j \text{ at time } t_m \text{ between two adjacent domains } \Omega_k (a = i, j). \text{ There is a vector field } F_{\alpha \rightarrow \beta} (x, t, \pi, q^{(\sigma)}) \text{ for } q^{(\sigma)} \in [q^{(1)}, q^{(2)}] \text{ on the boundary } \partial D_j \text{ with the } G\text{-functions} \]
\[ G_{\alpha j}^{(\sigma)} (x_m, q^{(\sigma)}) = 0 \]  \hspace{1cm} (118)
\( \sigma = 0, 1, \ldots, 2k_m - 1 \); \( \alpha, \beta \in \{1, 2\} \) and \( \alpha \neq \beta \). The vector field of \( F_{\alpha \rightarrow \beta} (x, t) \) is called the coming flow barrier in the sink flow on the \( \alpha \)-side of the boundary if the following conditions are satisfied. The critical values of \( F_{\alpha \rightarrow \beta} (x, t) \) at \( \pi \), \( q_\pi \) \( (\sigma = 1, 2) \) on the boundary \( \partial D_j \) are called the lower and upper limits of the coming flow barriers on the \( \alpha \)-side.
(iii) The coming flow of $\mathbf{x}^{(\alpha)}$ is switched to the boundary flow of $\mathbf{x}^{(0)}$ at the critical points of the flow barrier (i.e. $q^{(\alpha)} = q^{(\sigma)}$, $\sigma \in \{1, 2\}$) if

$$x^{(\alpha)}(t_m - \epsilon) = x^{(\alpha-\sigma)}(t_m \pm \epsilon, q^{(\sigma)}) = x_m,$$

$$G_{\partial \Omega_{ij}}^{(\alpha, \alpha)}(x_m, t_m).$$

for $s_\alpha = 2k_\alpha, 2k_\alpha + 1, \ldots, l_\alpha - 1$;

To explain the coming flow barrier in the sink flow on the boundary, the sink flow barriers on both sides of the boundary $\partial \Omega_{ij}$ are presented through the $G$-functions in Fig. 14. The red curves represent the $G$-function of the flows pertaining to the flow barriers in each domain $\Omega_\alpha (\alpha = i, j)$. The thick lines denote the $G$-functions of the flow barriers on both sides of the boundary. The gray curves are the $G$-functions without the flow barriers, and the solid thin lines are used to connect the corresponding $G$-functions. In order to show the flow barriers,

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig14.png}
\caption{G-functions for flow barriers in the $(2k_\alpha, 2k_\beta)$-sink flow on the boundary: (a) $n_{\partial \Omega_{ij}} \rightarrow \Omega_\beta$ and (b) $n_{\partial \Omega_{ij}} \rightarrow \Omega_\alpha$. The red curves are the $G$-functions relative to the flow barrier. The thick line is the $G$-function of the flow barriers at both $\alpha$ and $\beta$-side boundaries of the boundary $\partial \Omega_{ij}$, $G_{\partial \Omega_{ij}}^{(\alpha, \alpha-\sigma)}(q^{(\sigma)})$ and $G_{\partial \Omega_{ij}}^{(\alpha, \alpha-\sigma)}(q^{(\sigma)})$ are for lower and upper barrier limits ($k_\alpha \in \{0, 1, 2, \ldots\}, \alpha = i, j$) and similarly for the $\beta$-side.}
\end{figure}
consider the $G$-function on the $\alpha$-side of the boundary $\partial\Omega_{ij}$ as a reference. So the $G$-function on the $\beta$-side of the boundary $\partial\Omega_{ij}$ is presented through $-G_{\partial\Omega_{ij}}^{(\beta)}$ because the sink flow requires the signs of the $G$-functions of the flow on both sides of the boundary should be opposite. The shaded area is zoomed in for the boundary flow of $x^{(0)}$.

**Definition 20.** For a discontinuous dynamical system in Eq. (17), there is a point $x^{(0)}(t_m) \equiv x_m \in \partial\Omega_{ij}$ at time $t_m$ between two adjacent domains $\Omega_\alpha$ ($\alpha = i, j$). Suppose there is a coming flow barrier of $F^{(\alpha \rightarrow 0)}(x^{(0)}(t_m), \pi, q)$ for $q^{(\alpha)} \in \left[ q^{(\alpha)}_1, q^{(\alpha)}_2 \right]$ in the $(2k_\alpha : 2k_\beta)$-sink flow on the $\alpha$-side of the boundary $\partial\Omega_{ij}$ ($k_\alpha, k_\beta = 0, 1, 2, \ldots$).

(i) The coming flow barrier in the $(2k_\alpha : 2k_\beta)$-sink flow is partial on the $\alpha$-side if $x_m \in S \subset \partial\Omega_{ij}$.

(ii) The coming flow barrier in the $(2k_\alpha : 2k_\beta)$-sink flow is full on the $\alpha$-side if $x_m \in S = \partial\Omega_{ij}$.

In a similar fashion, the partial and full $(2k_\alpha : 2k_\beta)$-sink flow barriers on both sides of the boundary $\partial\Omega_{ij}$ are sketched in Fig. 15 for $x_m \in S \subseteq \partial\Omega_{ij}$ through the two different colored surfaces at the $\alpha$ and $\beta$-sides of the boundary $\partial\Omega_{ij}$. To clearly show the flow barriers, the $G$-function on the $\alpha$-side of the boundary $\partial\Omega_{ij}$ is still considered as a reference, and the $G$-function on the $\beta$-side of the boundary $\partial\Omega_{ij}$ is presented by $-G_{\partial\Omega_{ij}}^{(\beta)}$. The infinity $(2k_\alpha : 2k_\beta)$-sink flow barriers are presented in Fig. 16.

Fig. 15. The $G$-functions of the flow barriers in the $(2k_\alpha : 2k_\beta)$-sink flow on the boundary $\partial\Omega_{ij}$: (a) partial flow barrier and (b) full flow barrier. The red curves are the $G$-functions relative to the flow barrier. The dark and blue surfaces are the flow barrier surfaces. The hatched area is for the zoomed boundary. The dark blue curves are coming flows ($k_\alpha, k_\beta \in \{0, 1, 2, \ldots\}$).
Fig. 16. The $G$-functions for the infinity $(2k_\alpha:2k_\beta)$-sink flow barrier with lower boundary on $\partial \Omega_{ij}$: (a) partial flow barrier and (b) full flow barrier. The red curves are the $G$-functions relative to the flow barrier. The dark and blue surfaces are the flow barrier surfaces. The hatched area is for the zoomed boundary. The dark blue curves are coming flows ($k_\alpha,k_\beta \in \{0,1,2,\ldots\}$).

Definition 21. For a discontinuous dynamical system in Eq. (17), there is a point $x^{(0)}(t_m) \equiv x_m \in \partial \Omega_{ij}$ at time $t_m$ between two adjacent domains $\Omega_{\alpha}$ ($\alpha = i,j$). There is a coming barrier of $F^{(\sigma_{\alpha}>0)}(x^{(\alpha)}, t, \pi_{\alpha}, q^{(\alpha)})$ for $q^{(\alpha)} \in \{q_1^{(\alpha)}, q_2^{(\alpha)}\}$ on the $\alpha$-side of the boundary ($k_{\alpha}, k_\beta \in \{0,1,2,\ldots\}$) in the $(2k_\alpha:2k_\beta)$-sink flow.

(i) The coming flow barrier in the $(2k_\alpha:2k_\beta)$-sink flow is with an upper limit if for $x_m \in S \subseteq \partial \Omega_{ij}$

$$h_\alpha G^{(2k_\alpha,\alpha>0)}(x_m, q_1^{(\alpha)}) = 0 \quad \text{and} \quad h_\alpha G^{(2k_\alpha,\alpha>0)}(x_m, q_2^{(\alpha)}) \neq \infty. \quad (123)$$

(ii) The coming flow barrier in the $(2k_\alpha:2k_\beta)$-sink flow is with a lower limit if for $x_m \in S \subseteq \partial \Omega_{ij}$

$$h_\alpha G^{(2k_\alpha,\sigma_{\alpha}>0)}(x_m, q_1^{(\alpha)}) \neq 0 \quad \text{and} \quad h_\alpha G^{(2k_\alpha,\sigma_{\alpha}>0)}(x_m, q_2^{(\alpha)}) \rightarrow +\infty. \quad (124)$$

(iii) The coming flow barrier in the $(2k_\alpha:2k_\beta)$-sink flow is absolute if for $x_m \in S \subseteq \partial \Omega_{ij}$

$$h_\alpha G^{(2k_\alpha,\alpha>0)}(x_m, q_1^{(\alpha)}) = 0 \quad \text{and} \quad h_\alpha G^{(2k_\alpha,\alpha>0)}(x_m, q_2^{(\alpha)}) \rightarrow +\infty. \quad (125)$$
(v) The coming flow barrier in the \((2k_{0} : 2k_{2})\)-sink flow is a \textit{flow barrier wall} on the \(\alpha\)-side if the absolute flow barrier exists on \(x_m \in \partial \Omega_l\).

(ii) The coming flow barrier in the \((2k_{0} : 2k_{2})\)-sink flow is \textit{a flow barrier fence} on the \(\alpha\)-side if the flow barriers exist on \(S_{i_{0}} \subset \partial \Omega_j\) and no flow barriers on \(S_{i_{0}} \subset \partial \Omega_j\) \((k_{1}, k_{2} \in \{1, 2, \ldots \})\) for \(S_{i_{0}} \cap S_{i_{0}} = \emptyset\).

**Definition 22.** For a discontinuous dynamical system in Eq. (17), there is a point \(x^{(0)}(t_m) \equiv x_m \in \partial \Omega_j\) at time \(t_m\) between two adjacent domains \(\Omega_{i_{0}}\) \((\alpha = i, j)\). The two possible coming flows in the sink flow satisfy Eq. (112). Suppose there is a coming flow barrier of \(F^{(\alpha = 0)}(x^{(n)}(t), t, \pi_{n}, q^{(n)})\) on the \(\alpha\)-side in the \((2k_{0} : 2k_{2})\)-sink flow \((k_{0}, k_{2} \in \{0, 1, 2, \ldots \})\) for \(x_m \in S \subset \partial \Omega_j\) and \(q^{(n)} \in [q_{2n-1}^{(n)}, q_{2n}^{(n)}] \) \((n = 1, 2, \ldots)\) with

\[
\begin{array}{l}
\partial \Omega_{i_{0}}(x_m, q_{m}^{(n)}) = 0 \\
\text{for } n = 0, 1, \ldots, 2k_{0} - 1;
\end{array}
\]

\[
k_{0}G_{\partial \Omega_{i_{0}}}^{(2k_{0}, \alpha = 0)}(x_m, q^{(n)}) \\
\in \{k_{0}G_{\partial \Omega_{i_{0}}}^{(2k_{0}, \alpha = 0)}(x_m, q_{2n}^{(n)}),
\]

\[
\in \{k_{0}G_{\partial \Omega_{i_{0}}}^{(2k_{0}, \alpha = 0)}(x_m, q_{2n+1}^{(n)}), \}
\]

(iii) The coming flow barrier in the \((2k_{0} : 2k_{2})\)-sink flow is \textit{instantaneous} on the \(\alpha\)-side if the flow barrier is \textit{continuously dependent} on time \(t \in [0, \infty)\).

(iv) The coming flow barrier in the \((2k_{0} : 2k_{2})\)-sink flow is \textit{intermittent and static} on the \(\alpha\)-side if the flow barrier is \textit{independent of time} \(t \in [t_{k}, t_{k+1}]\) with \(k \in \mathbb{Z}\).

Similarly, the permanent and instantaneous windows of the \((2k_{0} : 2k_{2})\)-sink flow barrier on the boundary can be discussed. Further, the concept of the door for flow barrier wall on the boundary can be defined.

**Definition 24.** For a discontinuous dynamical system in Eq. (17), there is a point \(x^{(0)}(t_m) \equiv x_m \in \partial \Omega_l\) at time \(t_m\) between two adjacent domains \(\Omega_{i_{0}}\) \((\alpha = i, j)\). There is a coming flow barrier of \(F^{(\alpha = 0)}(x^{(n)}(t), t, \pi_{n}, q^{(n)})\) for \(q^{(n)} \in [q_{2n-1}^{(n)}, q_{2n}^{(n)}]\) on the \(\alpha\)-side of the boundary \((k_{0}, k_{2} \in \{0, 1, 2, \ldots \})\). Suppose there is a window of the flow barrier \(S \subset \partial \Omega_l\) and \(q^{(n)} \in [q_{2n}^{(n)}, q_{2n+1}^{(n)}]\).

(i) The window of the flow barrier in the \((2k_{0} : 2k_{2})\)-sink flow is \textit{permanent} on the \(\alpha\)-side if the window is \textit{independent of time} \(t \in [0, +\infty)\).

(ii) The window of the flow barrier in the \((2k_{0} : 2k_{2})\)-sink flow is \textit{instantaneous} on the \(\alpha\)-side if the window is \textit{continuously dependent} on time \(t \in [0, +\infty)\).

(iii) The window of the flow barrier in the \((2k_{0} : 2k_{2})\)-sink flow is \textit{intermittent} on the \(\alpha\)-side if the window exists for time \(t \in [t_{k}, t_{k+1}]\) with \(k \in \mathbb{Z}\).

(iv) The window of the flow barrier in the \((2k_{0} : 2k_{2})\)-sink flow is \textit{intermittent and static} on the \(\alpha\)-side if the window is \textit{independent of time} \(t \in [t_{k}, t_{k+1}]\) with \(k \in \mathbb{Z}\).

**Definition 25.** For a discontinuous dynamical system in Eq. (17), there is a point \(x^{(0)}(t_m) \equiv x_m \in \partial \Omega_l\) at time \(t_m\) between two adjacent domains \(\Omega_{i_{0}}\) \((\alpha = i, j)\). There is a coming flow barrier of \(F^{(\alpha = 0)}(x^{(n)}(t), t, \pi_{n}, q^{(n)})\) for \(q^{(n)} \in [q_{2n-1}^{(n)}, q_{2n}^{(n)}]\) on the \(\alpha\)-side of the boundary \((k_{0}, k_{2} \in \{0, 1, 2, \ldots \})\) in the \((2k_{0} : 2k_{2})\)-sink flow. Suppose there is a \textit{flow barrier wall} on the \(\alpha\)-side for \(S = \partial \Omega_j\) and there is an \textit{intermittent}, static window of the coming flow \((2k_{0} : 2k_{2})\)-sink flow.
barrier on $S \subset \partial \Omega_{ij}$ for $q^{(α)} \in [q^{(1)}_{ij}, q^{(2)}_{ij}]$ and $t \in [t_k, t_{k+1}]$ with $k \in \mathbb{Z}$.

(i) The window of the flow barrier in the $(2k_{\alpha} ; 2k_{\beta})$-sink flow is termed a door of the flow barrier wall on the α-side if the window and flow barriers exist alternatively.

(ii) The door of the coming flow barrier in the $(2k_{\alpha} ; 2k_{\beta})$-sink flow is open on the α-side if the window exists for time $t \in [t_k, t_{k+1}]$ with $k \in \mathbb{Z}$.

(iii) The door of the coming flow barrier in the $(2k_{\alpha} ; 2k_{\beta})$-sink flow is closed on the α-side if the flow barrier exists for time $t \in [t_k, t_{k+1}]$ with $k \in \mathbb{Z}$.

(iv) The door of the coming flow barrier in the $(2k_{\alpha} ; 2k_{\beta})$-sink flow is permanently open on the α-side if the window exists for $t \in [t_k, \infty)$. 

(v) The door of the coming flow barrier in the $(2k_{\alpha} ; 2k_{\beta})$-sink flow is permanently closed on the α-side if the flow barrier exists for $t \in [t_k + 1, \infty)$.

From the previous definition, the window of the coming flow barrier in the sink flow is sketched in Fig. 17. On the window area, the flow of $x^{(α)}$ can...
Fig. 18. The door of the absolute $(2k_\alpha : 2k_\beta)$-sink flow barrier on $\partial \Omega_{ij}$: (a) opened door and (b) closed door. The red curves are the $G$-functions relative to the flow barrier. The dark and blue surfaces are the flow barrier surfaces. The hatched area is for the zoomed boundary. The dark blue curves are coming flows ($k_\alpha, k_\beta \in \{0, 1, 2, \ldots \}$).

be switched to the boundary flow of $x^{(0)}$. The door for the flow barrier wall in the $(2k_\alpha : 2k_\beta)$-sink flow on the boundary $\partial \Omega_{ij}$ is sketched in Fig. 18. In Fig. 18(a), the door of the flow barrier is open, and the coming flow of $x^{(\alpha)}$ (or $x^{(0)}$-flow) can be switched into the boundary flow of $x^{(0)}$. However, in Fig. 18(b), the door of the sink flow barrier is closed. None of the flows can be switched into the boundary flow of $x^{(0)}$ on the $\alpha$-side through this door.

For a sink flow on the boundary $\partial \Omega_{ij}$, if the sink flow barrier exists on the $\alpha$-side of such a boundary with a subset $S \subseteq \partial \Omega_{ij}$, the sink flow cannot be formed on the boundary under the following conditions for $x_m \in S \subseteq \partial \Omega_{ij}$:

\[ h_{\alpha} G_{(2k_\alpha : \alpha)}^{(2k_\alpha : \alpha)}(x_m) \in [h_{\alpha} G_{(2k_\alpha : \alpha)}^{(2k_\alpha : \alpha)}((x_m, q_1^{(\alpha)}), h_{\alpha} G_{(2k_\alpha : \alpha)}^{(2k_\alpha : \alpha)}((x_m, q_2^{(\alpha)})]. \quad (128) \]

The dynamical system will be constrained on the boundary, given by Eq. (39) because the flow barrier exists on the $\alpha$-side of the boundary $\partial \Omega_{ij}$, and the flow will be along the boundary until the condition
The two possible coming flows in the sink flow on
\( \Omega \), i.e.,
\[ \begin{align*}
\text{(i)} & \quad \alpha \in (\partial^0 \Omega, t, \pi, \alpha q_1^{(\alpha)}) \text{ at } q_1^{(\alpha)} = \{q_1, q_2^{(\alpha)}\} \text{ on the } \\
\text{(ii)} & \quad \alpha \in (\partial^0 \Omega, t, \pi, \alpha q_1^{(\alpha)}) \text{ at } q_1^{(\alpha)} = \{q_1, q_2^{(\alpha)}\} \text{ on the } \\
\end{align*} \]

The coming flow of Eq. (17) has a flow barrier
\[ \mathbf{F}^{(\alpha)}(\mathbf{x}(t), t, \mathbf{\pi}, \alpha q_1^{(\alpha)}) \text{ at } q_1^{(\alpha)} = \{q_1, q_2^{(\alpha)}\} \text{ on the } \\
\alpha \text{-side of the boundary } \partial \Omega \text{ with} \]
\[ \begin{align*}
\mathbf{G}(\mathbf{x}_m, \alpha q_1^{(\alpha)}) & \in [\mathbf{G}(\mathbf{x}_m, \alpha q_1^{(\alpha)}), \mathbf{G}(\mathbf{x}_m, \alpha q_2^{(\alpha)})] \\
\text{in } [0, \infty). \tag{129} \\
\end{align*} \]
The two possible coming flows in the sink flow on the
boundary satisfy
\[ \begin{align*}
h_u(\mathbf{G}(\mathbf{x}_m, \alpha q_1^{(\alpha)})) & > 0 \text{ and} \\
h_u(\mathbf{G}(\mathbf{x}_m, \alpha q_1^{(\alpha)})) & > 0. \tag{130} \\
\end{align*} \]

(i) The coming flow of \( \mathbf{x}(t) \) cannot be switched to the boundary flow of \( \mathbf{x}(t) \) to form a sink flow if and only if
\[ \begin{align*}
h_u(\mathbf{G}(\mathbf{x}_m, \alpha q_1^{(\alpha)})) & \in [h_u(\mathbf{G}(\mathbf{x}_m, \alpha q_1^{(\alpha)})), h_u(\mathbf{G}(\mathbf{x}_m, \alpha q_2^{(\alpha)}))] \\
\in [0, \infty). \tag{131} \\
\end{align*} \]

(ii) The coming flow of \( \mathbf{x}(t) \) cannot be switched to the boundary flow of \( \mathbf{x}(t) \) to form a sink flow at \( q_1^{(\alpha)} = q_2^{(\alpha)} \) if and only if
\[ \begin{align*}
h_u(\mathbf{G}(\mathbf{x}_m, \alpha q_1^{(\alpha)})) & \in [h_u(\mathbf{G}(\mathbf{x}_m, \alpha q_1^{(\alpha)})), h_u(\mathbf{G}(\mathbf{x}_m, \alpha q_2^{(\alpha)}))] \\
\in [0, \infty). \tag{132} \\
\end{align*} \]

(iii) The coming flow of \( \mathbf{x}(t) \) is switched to the boundary flow of \( \mathbf{x}(t) \) to form a sink flow at \( q_1^{(\alpha)} = q_2^{(\alpha)} \) if and only if
\[ \begin{align*}
h_u(\mathbf{G}(\mathbf{x}_m, \alpha q_1^{(\alpha)})) & \in (0, \infty) \\
\text{for } \alpha = 0, 1, \ldots, l_{m_{-}} - 1; \tag{133} \\
\end{align*} \]

\[ (-1)^p h_u(\mathbf{G}(\mathbf{x}_m, \alpha q_1^{(\alpha)})) = 0. \tag{134} \]

Proof. The proof of this theorem is similar to
Theorem 1. \( \Box \)

Theorem 8. For a discontinuous dynamical system
in Eq. (17), there is a point \( \mathbf{x}(t) = \mathbf{x}_m \in \partial \Omega \text{ at } t_{m_{-}} \text{ between two adjacent domains } \Omega_n (\alpha = i, j) \).

(i) A coming flow of \( \mathbf{x}(t) \) cannot be switched to the boundary flow of \( \mathbf{x}(t) \) to form a \( (2k_0 : 2k_1) \)-sink flow if and only if
\[ \begin{align*}
h_u(\mathbf{G}(\mathbf{x}_m, \alpha q_1^{(\alpha)})) & \in [h_u(\mathbf{G}(\mathbf{x}_m, \alpha q_1^{(\alpha)})), h_u(\mathbf{G}(\mathbf{x}_m, \alpha q_2^{(\alpha)}))] \\
\in [0, \infty). \tag{135} \\
\end{align*} \]

(ii) The coming flow of \( \mathbf{x}(t) \) cannot be switched to the boundary flow of \( \mathbf{x}(t) \) to form a \( (2k_0 : 2k_1) \)-sink flow at \( q_1^{(\alpha)} = q_2^{(\alpha)} \) if and
only if
\[ h_\alpha G^{(\alpha)}_{\partial I_\alpha}(x_m, t_m-) = h_\alpha G^{(\alpha)}_{\partial I_\alpha}(x_m, q^{(\alpha)}_m) \in (0, \infty) \]
for \( s_a = 2k_a, 2k_a + 1, \ldots, l_a - 1; \)
\[ (-1)^i h_\alpha G^{(\alpha)}_{\partial I_\alpha}(x_m, t_m-) \]
\[ - G^{(\alpha)}_{\partial I_\alpha}(x_m, q^{(\alpha)}_m) < 0. \]

\[ (137) \]

(iii) The coming flow of \( x^{(\alpha)} \) is switched to the boundary flow of \( x^{(\alpha)} \) to form a \((2k_a, 2k_a)\)-sink flow on the \( \alpha \)-side at \( q^{(\alpha)}_m \) \((\sigma \in \{1, 2\})\) if and only if
\[ h_\alpha G^{(\alpha)}_{\partial I_\alpha}(x_m, t_m-) = h_\alpha G^{(\alpha)}_{\partial I_\alpha}(x_m, q^{(\alpha)}_m) \in (0, \infty) \]
for \( s_a = 2k_a, 2k_a + 1, \ldots, l_a - 1; \)
\[ (-1)^i h_\alpha G^{(\alpha)}_{\partial I_\alpha}(x_m, t_m-) \]
\[ - G^{(\alpha)}_{\partial I_\alpha}(x_m, q^{(\alpha)}_m) > 0. \]

\[ (138) \]

Proof. The proof of this theorem is similar to the proof of Theorem 2. \( \blacksquare \)


Without any flow barriers, from [Luo, 2008a; Luo, 2008b], the necessary and sufficient conditions for a source flow leaving the boundary are
\[ h_\alpha G^{(\alpha)}_{\partial I_\alpha}(x_m, t_m+, q^{(\alpha)}_m) < 0 \]
\[ h_\alpha G^{(\beta)}_{\partial I_\beta}(x_m, t_m+, q^{(\beta)}_m) > 0. \]

For the sink flow, three is a boundary flow of \( x^{(\alpha)} \) on the boundary, governed by
\[ \dot{x}^{(\alpha)} = F^{(\alpha)}(x^{(\alpha)}, t, \lambda), \]
with \( \varphi^{(\alpha)}_j(x^{(\alpha)}, t, \lambda) = 0 \) on \( \partial I_\alpha \).

\[ (140) \]

The \( G \)-function for the boundary flow of \( x^{(\alpha)} \) is already zero, i.e.
\[ G^{(\alpha)}_{\partial I_\alpha}(x_m, t_m) = 0 \] on \( \partial I_\alpha \).

\[ (141) \]

6.1. Boundary flow barriers in source flows

To avoid the boundary flow leaving the boundary and entering the \( \alpha \)-domain, the boundary flow barrier should be considered, and the corresponding discussions will be given as follows.

Definition 26. For a discontinuous dynamical system in Eq. (17), there is a point \( x^{(\alpha)}(t_m) \equiv x_m \in \partial I_\alpha \) at time \( t_m \) between two adjacent domains \( \Omega \) \((\alpha = i, j)\). Suppose there is a vector field \( F^{(\alpha)}(x^{(\alpha)}, t, \pi_n, q^{(\alpha)}) \) for \( q \in \{q_1, q_2\} \) on the boundary \( \partial I_\alpha \) with
\[ 0 \in [h_\alpha G^{(\alpha)}_{\partial I_\alpha}(x_m, q^{(\alpha)}_1), h_\alpha G^{(\alpha)}_{\partial I_\alpha}(x_m, q^{(\alpha)}_2)] \subset \mathbb{R}. \]

\[ (142) \]

The two possible leaving flows in the source flow satisfy
\[ h_\alpha G^{(\alpha)}_{\partial I_\alpha}(x_m, t_m+) < 0 \]
\[ h_\alpha G^{(\beta)}_{\partial I_\beta}(x_m, t_m+) > 0 \]
\( (\alpha, \beta \in \{1, 2\} \) and \( \alpha \neq \beta \). The vector field of \( F^{(\alpha)}(x^{(\alpha)}, t, \pi_n, q^{(\alpha)}) \) is called the flow barrier of the boundary flow in the source flow on the \( \alpha \)-side if the following conditions are satisfied. The two critical values of \( F^{(\alpha)}(x^{(\alpha)}, t, \pi_n, q^{(\alpha)}) \) for \( \alpha = 1, 2 \) are called the lower and upper limits of the boundary flow barriers on the \( \alpha \)-side.

(i) The boundary flow of \( x^{(\alpha)} \) cannot be switched to the leaving flow of \( x^{(\alpha)} \) if
\[ \dot{x}^{(\alpha)}(t_m) = x^{(\alpha)}(t_m+, q^{(\alpha)}_m) = x_m \]
for \( \sigma = 1, 2; \)
\[ h_\alpha G^{(\alpha)}_{\partial I_\alpha}(x_m, q^{(\alpha)}_m) > 0 \] and \( h_\alpha G^{(\alpha)}_{\partial I_\alpha}(x_m, q^{(\alpha)}_m) < 0. \)

\[ (144) \]

(ii) The boundary flow of \( x^{(\alpha)} \) cannot be switched to the leaving flow of \( x^{(\alpha)} \) at the critical points of the flow barrier \( i.e. q^{(\alpha)} = q^{(\alpha)}_\sigma, \sigma \in \{1, 2\} \) if
\[ \dot{x}^{(\alpha)}(t_m) = x^{(\alpha)}(t_m+, q^{(\alpha)}_m) = x_m, \]
\[ h_\alpha G^{(\alpha)}_{\partial I_\alpha}(x_m, q^{(\alpha)}_m) < 0 \] and \( h_\alpha G^{(\alpha)}_{\partial I_\alpha}(x_m, q^{(\alpha)}_m) = 0 \) for \( s_a = 0, 1, 2, \ldots, l_a - 1; \)
The boundary flow of $x^{(0)}(t)$ can be switched to the leaving flow of $x^{(n)}$ at the critical points of the flow barrier (i.e., $q^{(n)} = q_s^{(n)}$, $\sigma \in \{1, 2\}$) if

$$x^{(0)}(t_m) = x^{(0,0,\ldots,0)}(t_m, q_s^{(n)}) = x_m$$
for $\sigma = 1, 2$;

$$h_{\alpha}G^{(0,0,\ldots,0)}_{ij}(x_m, q_2^{(n)}) < 0 \quad \text{and} \quad h_{\alpha}G^{(0,0,\ldots,0)}_{ij}(x_m, q_1^{(n)}) = 0 \quad \text{for } s_m = 0, 1, 2, \ldots, l_m - 1; \quad (146)$$

$$h_{\alpha}n^T_{ij}(x^{(0)}(t_m+)) \cdot [x^{(0,0,\ldots,0)}(t_m+; q_2^{(n)}) - x^{(0)}(t_m+)] < 0. \quad (145)$$

(iii) The boundary flow of $x^{(0)}$ can be switched to the leaving flow of $x^{(n)}$ at the critical points of the flow barrier (i.e., $q^{(n)} = q_s^{(n)}$, $\sigma \in \{1, 2\}$) if

$$x^{(0)}(t_m) = x^{(0,0,\ldots,0)}(t_m, q_s^{(n)}) = x_m$$
for $\sigma = 1, 2$;

$$h_{\alpha}G^{(0,0,\ldots,0)}_{ij}(x_m, q_2^{(n)}) < 0 \quad \text{and} \quad h_{\alpha}G^{(0,0,\ldots,0)}_{ij}(x_m, q_1^{(n)}) = 0 \quad \text{for } s_m = 0, 1, 2, \ldots, l_m - 1; \quad (146)$$

$$h_{\alpha}n^T_{ij}(x^{(0)}(t_m+)) \cdot [x^{(0,0,\ldots,0)}(t_m+; q_1^{(n)}) - x^{(0)}(t_m+)] < 0. \quad (145)$$

Definition 27. For a discontinuous dynamical system in Eq. (17), there is a point $x^{(0)}(t_m) \equiv x_m \in \partial\Omega_{ij}$ at time $t_m$ between two adjacent domains $\Omega_{ij}$ ($i = 1, j$. There is a vector field $F^{(0,0,\ldots,0)}(x^{(n)}, t, \pi_n, q^{(n)})$ for $q^{(n)} \in [q_1^{(n)}, q_2^{(n)}]$ on the boundary $\partial\Omega_{ij}$ with its $G$-function

$$G^{(0,0,\ldots,0)}_{ij}(x_m, q^{(n)}) = 0$$

for $s_m = 0, 1, \ldots, m_n - 1$;

$$G^{(0,0,\ldots,0)}_{ij}(x_m, q^{(n)}) < 0 \quad \text{for } s_m = m_n$$

for $\sigma = 0, 1, \ldots, m_n - 1$; \quad (147)

$$G^{(0,0,\ldots,0)}_{ij}(x_m, q^{(n)}) > 0 \quad \text{for } s_m = m_n, m_n + 1, \ldots, m_{\beta} - 1$$

The two leaving flows in the $(m_n, m_{\beta})$-source flow satisfy

(i) The boundary flow of $x^{(0)}$ cannot be switched to the $(m_n)$-th-order leaving flow of $x^{(n)}$ if

$$x^{(0)}(t_m) = x^{(0,0,\ldots,0)}(t_m; q_s^{(n)}) = x_m$$
for $\sigma = 1, 2$;

$$h_{\alpha}G^{(0,0,\ldots,0)}_{ij}(x_m, q_2^{(n)}) < 0 \quad \text{and} \quad h_{\alpha}G^{(0,0,\ldots,0)}_{ij}(x_m, q_1^{(n)}) > 0. \quad (149)$$

(ii) The boundary flow of $x^{(0)}$ cannot be switched to the $(m_n)$-th-order leaving flow of $x^{(n)}$ if

$$x^{(0)}(t_m) = x^{(0,0,\ldots,0)}(t_m; q_s^{(n)}) = x_m$$
for $\sigma = 1, 2$;

$$h_{\alpha}G^{(0,0,\ldots,0)}_{ij}(x_m, q_2^{(n)}) < 0 \quad \text{and} \quad h_{\alpha}G^{(0,0,\ldots,0)}_{ij}(x_m, q_1^{(n)}) = 0 \quad \text{for } s_m = m_n, m_n + 1, \ldots, l_m - 1;$$

$$h_{\alpha}n^T_{ij}(x^{(0)}(t_m+)) \cdot [x^{(0,0,\ldots,0)}(t_m+; q_2^{(n)}) - x^{(0)}(t_m+)] > 0. \quad (150)$$

(iii) The boundary flow of $x^{(0)}$ can be switched to the $(m_n)$-th-order leaving flow of $x^{(n)}$ if

$$x^{(0)}(t_m) = x^{(0,0,\ldots,0)}(t_m; q_s^{(n)}) = x_m$$
for $\sigma = 1, 2$;

$$h_{\alpha}G^{(0,0,\ldots,0)}_{ij}(x_m, q_2^{(n)}) < 0 \quad \text{and} \quad h_{\alpha}G^{(0,0,\ldots,0)}_{ij}(x_m, q_1^{(n)}) = 0 \quad \text{for } s_m = m_n, m_n + 1, \ldots, l_m - 1;$$

$$h_{\alpha}n^T_{ij}(x^{(0)}(t_m+)) \cdot [x^{(0,0,\ldots,0)}(t_m+; q_1^{(n)}) - x^{(0)}(t_m+)] < 0. \quad (151)$$
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Figure 19. $G$-functions for the boundary flow barriers in the $(m_\alpha, m_\beta)$-source flow: (a) $n_{\partial \Omega_{ij}} \rightarrow \Omega_\beta$ and (b) $n_{\partial \Omega_{ij}} \rightarrow \Omega_\alpha$.

The red dashed curves are the $G$-functions relative to the flow barrier. The thick line is the $G$-function of the flow barriers at both $\alpha$ and $\beta$-sides of the boundary $\partial \Omega_{ij}$. $G_{\Omega_{ij}}(m_\alpha, m_\beta)(q_{ij}^{(\alpha)})$ and $G_{\Omega_{ij}}(m_\alpha, m_\beta)(q_{ij}^{(\beta)})$ are for lower and upper barrier limits ($m_\alpha \in \{0, 1, 2, \ldots\}$, $\alpha = i, j$) and similarly for the $\beta$-side.

To explain the boundary flow barrier, the $G$-functions of the flow barriers on both sides of the boundary $\partial \Omega_{ij}$ are presented in Fig. 19. The red dashed curves represent the $G$-function of the flows pertaining to the boundary flow barriers in each domain $\Omega_\alpha$. The thick lines denote the $G$-functions of the flow barriers on both sides of the boundary. To show the boundary flow barriers, consider the $G$-function on the $\beta$-side of the boundary $\partial \Omega_{ij}$ as a reference. The shade area is also zoomed for the boundary flow of $x_0$. Because the $G$-function for the boundary flow is zero (i.e. $G_{\partial \Omega_{ij}}^{(m_\alpha, m_\beta)}(0) = 0$), the lower and upper limits of the boundary flow barriers should be negative and positive, respectively. Such flow barriers are independent of the flow of $x^{(n)}$ ($\alpha \in \{i, j\}$). However, once the flow barriers disappear, the boundary flow of $x^{(n)}$ will be switched to the flow $x^{(m)}$ controlled by $G_{\partial \Omega_{ij}}^{(m_\alpha, m_\beta)}$. For $n_{\partial \Omega_{ij}} \rightarrow \Omega_\beta$, one obtains $\tilde{h}_0 = 1$. The $G$-function $G_{\partial \Omega_{ij}}^{(m_\alpha, m_\beta)}$ for the leaving flow of $x^{(n)}$ should be negative (i.e. $G_{\partial \Omega_{ij}}^{(m_\alpha, m_\beta)} < 0$) as similar to Eq. (148). If the $G$-function is positive (i.e. $G_{\partial \Omega_{ij}}^{(m_\alpha, m_\beta)} > 0$), the leaving flow will become a coming flow in domain $\Omega_\alpha$. The leaving flow of $x^{(n)}$ on the $\alpha$-side cannot be formed. In a similar fashion, the $G$-function $G_{\partial \Omega_{ij}}^{(m_\alpha, m_\beta)}$ for the leaving flow of $x^{(n)}$ should be positive (i.e. $G_{\partial \Omega_{ij}}^{(m_\alpha, m_\beta)} > 0$) from Eq. (148) to form the source flow on the $\beta$-side. Otherwise, the leaving flow in
domain $\Omega_j$ cannot be achieved. Such characteristics of the boundary flow barrier are sketched in Fig. 19(a). For $\mathbf{n}_{\partial \Omega_j} \rightarrow \mathbf{n}_x$, one obtains $h_{\beta} = -1$. The $G$-function for the leaving flow of $q^{(0)}_x$ should be positive ($G^{(m, 0, 0)}_{ij} > 0$), and the $G$-function for the leaving flow of $x^{(0)}$ becomes negative ($G^{(m, 0, 0)}_{ij} < 0$) in order to form a source flow starting from the boundary. For this case, the $G$-functions of the flow barriers are sketched in Fig. 19(b).

Definition 28. For a discontinuous dynamical system in Eq. (17), there is a point $x^{(0)}(t_m) \equiv x_m \in \partial \Omega_i$ at time $t_m$ between two adjacent domains $\Omega_i$ ($i = i, j$). Suppose there is a boundary flow barrier of $F^{(0, 0)}(x^{(0)}_i, t, \pi^{(0)}_i, q^{(0)}_i)$ in the $(m, m_i)$-source flow for $q^{(0)}_i \in [q^{(0)}_i, q^{(0)}_j]$ on the $\alpha$-side of the boundary $\partial \Omega_j$ ($\alpha \neq \beta, m, m_i \in \Omega_j$).

(i) The boundary flow barrier in the $(m, m_i)$-source flow is on the $\alpha$-side if $x_m \in S \subset \partial \Omega_j$.

(ii) The boundary flow barrier in the $(m, m_i)$-source flow is on the $\alpha$-side if $x_m \in S = \partial \Omega_j$.

Definition 29. For a discontinuous dynamical system in Eq. (17), there is a point $x^{(0)}(t_m) \equiv x_m \in \partial \Omega_i$ at time $t_m$ between two adjacent domains $\Omega_i$ ($i = i, j$). Suppose there is a boundary flow barrier of $F^{(0, 0)}(x^{(0)}_i, t, \pi^{(0)}_i, q^{(0)}_i)$ in the $(m, m_i)$-source flow for $q^{(0)}_i \in [q^{(0)}_i, q^{(0)}_j]$ on the $\alpha$-side of the boundary $\partial \Omega_j$ ($\alpha \neq \beta, m, m_i \in \Omega_j$).

(i) The boundary flow barrier in the $(m, m_i)$-source flow is on the $\alpha$-side if $x_m \in S \subset \partial \Omega_j$.

(ii) The boundary flow barrier in the $(m, m_i)$-source flow is on the $\alpha$-side if $x_m \in S = \partial \Omega_j$.

x_m \in S \subset \partial \Omega_j

\begin{align}
G^{(m, 0, 0)}_{ij}(x_m, q^{(0)}_i) &\rightarrow +\infty \quad \text{and} \\
h_{\beta}G^{(m, 0, 0)}_{ij}(x_m, q^{(0)}_i) &< 0. \\
\end{align}

\begin{align}
G^{(m, 0, 0)}_{ij}(x_m, q^{(0)}_i) &> 0 \quad \text{and} \\
h_{\beta}G^{(m, 0, 0)}_{ij}(x_m, q^{(0)}_i) &\rightarrow -\infty. \\
\end{align}

Definition 30. For a discontinuous dynamical system in Eq. (17), there is a point $x^{(0)}(t_m) \equiv x_m \in \partial \Omega_i$ at time $t_m$ between two adjacent domains $\Omega_i$ ($i = i, j$). Suppose there is a boundary flow barrier of $F^{(0, 0)}(x^{(0)}_i, t, \pi^{(0)}_i, q^{(0)}_i)$ in the $(m, m_i)$-source flow for $q^{(0)}_i \in [q^{(0)}_i, q^{(0)}_j]$ on the $\alpha$-side of the boundary $\partial \Omega_j$.

(iii) The boundary flow barrier in the $(m, m_i)$-source flow is on the $\alpha$-side if $x_m \in S = \partial \Omega_j$.

\begin{align}
G^{(m, 0, 0)}_{ij}(x_m, q^{(0)}_i) &\rightarrow +\infty \quad \text{and} \\
h_{\beta}G^{(m, 0, 0)}_{ij}(x_m, q^{(0)}_i) &< 0. \\
\end{align}

(iv) The boundary flow barrier in the $(m, m_i)$-source flow is on the $\alpha$-side if $x_m \in S = \partial \Omega_j$.

The infinity boundary flow barriers in the $(m, m_i)$-source flow are sketched in Fig. 21.

$(i)$ The boundary flow barrier in the $(m, m_i)$-source flow is on the $\alpha$-side if $x_m \in S \subset \partial \Omega_j$.

$(ii)$ The boundary flow barrier in the $(m, m_i)$-source flow is on the $\alpha$-side if $x_m \in S = \partial \Omega_j$.

\begin{align}
G^{(m, 0, 0)}_{ij}(x_m, q^{(0)}_i) &\rightarrow +\infty \quad \text{and} \\
h_{\beta}G^{(m, 0, 0)}_{ij}(x_m, q^{(0)}_i) &< 0. \\
\end{align}

$\partial \Omega_j$.

For $q^{(0)}_i \in [q^{(0)}_{i-1}, q^{(0)}_{i+1})$, no flow barriers are defined at $x_m \in S \subset \partial \Omega_j$. Thus, the boundary flow of $x^{(0)}$ can be switched to the flow of $q^{(0)}$ for $0 \in \{h_{\beta}G^{(m, 0, 0)}_{ij}(x_m, q^{(0)}_i) \}

\begin{align}
G^{(m, 0, 0)}_{ij}(x_m, q^{(0)}_i) &\rightarrow +\infty \quad \text{and} \\
h_{\beta}G^{(m, 0, 0)}_{ij}(x_m, q^{(0)}_i) &< 0. \\
\end{align}

$\partial \Omega_j$.

For $q^{(0)}_i \in [q^{(0)}_{i-1}, q^{(0)}_{i+1})$, no flow barriers are defined at $x_m \in S \subset \partial \Omega_j$. Thus, the boundary flow of $x^{(0)}$ can be switched to the flow of $q^{(0)}$ for $0 \in \{h_{\beta}G^{(m, 0, 0)}_{ij}(x_m, q^{(0)}_i) \}

\begin{align}
G^{(m, 0, 0)}_{ij}(x_m, q^{(0)}_i) &\rightarrow +\infty \quad \text{and} \\
h_{\beta}G^{(m, 0, 0)}_{ij}(x_m, q^{(0)}_i) &< 0. \\
\end{align}

$\partial \Omega_j$.
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The G-function intervals for all \( x_m \in S \subseteq \partial \Omega_{ij} \) with \( q^{(\alpha)} \in \left[ q^{(1,\alpha)}, q^{(2,\alpha)} \right] \) are called the window of the boundary flow barrier in the \((m_{\alpha}, m_{\beta})\)-source flow on the \( \alpha \)-side of the boundary.

**Definition 31.** For a discontinuous dynamical system in Eq. (17), there is a point \( x_m(t_m) \equiv x_m \in \partial \Omega_{ij} \) at time \( t_m \) between two adjacent domains \( \Omega_{\alpha}(\alpha = i, j) \). The two possible leaving flows in the \((m_{\alpha}, m_{\beta})\)-source flow satisfy Eq. (148). Suppose there is a boundary flow barrier of \( F^{(0,0)}(\mathbf{x}^{(\alpha)}, t, \mathbf{r}_{\alpha}, q^{(\alpha)}) \) for \( q^{(\alpha)} \in \left[ q^{(1,\alpha)}, q^{(2,\alpha)} \right] \) on the \( \alpha \)-side of the boundary \((m_{\alpha}, m_{\beta}) \in \{0, 1, 2, \ldots \}\) in the \((m_{\alpha}, m_{\beta})\)-source flow.

(i) The boundary flow barrier in the \((m_{\alpha}, m_{\beta})\)-source flow is **permanent** on the \( \alpha \)-side if the flow barrier is independent of time \( t \in [0, \infty) \).

(ii) The boundary flow barrier in the \((m_{\alpha}, m_{\beta})\)-source flow is **instantaneous** on the \( \partial \Omega_{ij} \)-side if the flow barrier is continuously dependent on time \( t \in [0, \infty) \).

(iii) The boundary flow barrier in the \((m_{\alpha}, m_{\beta})\)-source flow is **intermittent** on the \( \alpha \)-side if the flow barrier exists for time \( t \in \left[ t_k, t_{k+1} \right] \) with \( k \in \mathbb{Z} \).

(iv) The boundary flow barrier in the \((m_{\alpha}, m_{\beta})\)-source flow is **intermittent and static** on the...
Fig. 21. The infinity boundary flow barrier with lower boundary on \( \partial \Omega \): (a) partial flow barrier and (b) full flow barrier. The red curves are the \( G \)-functions relative to the flow barrier. The dark and blue surfaces are the flow barrier surfaces. The hatched area is for the zoomed boundary. The dark blue curves are coming flows \((m_\alpha, m_\beta) \in \{0, 1, 2, \ldots \}\).

\( \alpha \)-side if the flow barrier is independent of time \( t \in [t_k, t_{k+1}] \) with \( k \in \mathbb{Z} \).

Similarly, the permanent and instantaneous windows of the source flow barrier on the boundary can be discussed. Further, the door for the flow barrier wall on the boundary can be discussed.

**Definition 32.** For a discontinuous dynamical system in Eq. (17), there is a point \( x^{(0)}(t_m) \equiv x_m \in \partial \Omega_j \) at time \( t_m \) between two adjacent domains \( \Omega_\alpha \) (\( \alpha = i, j \)). The two leaving flows in the \((m_\alpha, m_\beta)\)-source flow satisfy Eq. (148). There is a boundary flow barrier of \( \Psi^{(0)(m_\alpha)}(x^{(0)}, t, \pi_\alpha, q^{(\alpha)}) \) for \( q^{(\alpha)} \in [q^{(\alpha)}_{2n-1}, q^{(\alpha)}_{2n}] \) in the \((m_\alpha, m_\beta)\)-source flow on the \( \alpha \)-side of the boundary \((m_\alpha, m_\beta) \in \{0, 1, 2, \ldots \}, n = 1, 2, \ldots \). Suppose there is a window of the flow barrier for \( S \subset \partial \Omega_j \) and \( q^{(\alpha)} \in [q^{(\alpha)}_{2n-1}, q^{(\alpha)}_{2n}] \).

(i) The window of boundary flow barrier in the \((m_\alpha, m_\beta)\)-source flow is **permanent** on the \( \alpha \)-side if the window is independent of time for \( t \in [0, +\infty) \).
(ii) The window of flow barrier in the \((m_a, m_j)\)-source flow is instantaneous in the \(\alpha\)-side if the window is continuously dependent on time \(t \in [0, +\infty)\).

(iii) The window of boundary flow barrier in the \((m_a, m_j)\)-source flow is intermittent on the \(\alpha\)-side if the window exists for time \(t \in [t_k, t_{k+1}]\) with \(k \in \mathbb{Z}\).

(iv) The window of boundary flow barrier in the \((m_a, m_j)\)-source flow is intermittent and static on the \(\alpha\)-side if the window is independent of time \(t \in [t_k, t_{k+1}]\) with \(k \in \mathbb{Z}\).

Definition 33. For a discontinuous dynamical system in Eq. (17), there is a point \(x^{(0)}(t_m) \equiv x_m \in \partial \Omega_i\) at time \(t_m\) between two adjacent domains \(\Omega_i\) (\(\alpha = i,j\)). The two possible leaving flows in the \((m_a, m_j)\)-source flow satisfy Eq. (148). There is a boundary flow barrier of \(F^{(0=0)}([\alpha](t, \pi_a, q^{(0)})\) for \(q^{(0)} \in [\pi_a, \tilde{q}]\) in the \((m_a, m_j)\)-source flow on the \(\alpha\)-side of the boundary \((m_a, m_j) \in \{0, 1, 2, \ldots\}\). Suppose a boundary flow barrier wall exists on the \(\alpha\)-side of the entire boundary \(\partial \Omega_i\), and there is an intermittent and static window of the boundary flow barrier on \(S \subseteq \partial \Omega_i\) for \(q^{(0)} \in [\pi_a, \tilde{q}]\) and \(t \in [t_k, t_{k+1}]\) with \(k \in \mathbb{Z}\).

(i) The window of the boundary flow barrier in the \((m_a, m_j)\)-source flow is termed the door of the boundary flow barrier wall on the \(\alpha\)-side of the boundary \(\partial \Omega_i\) if the window and the flow barrier exists alternatively.

(ii) The door of the boundary flow barrier in the \((m_a, m_j)\)-source flow is permanently open on the \(\alpha\)-side if the window exists for time \(t \in [t_k, t_{k+1}]\) with \(k \in \mathbb{Z}\).

(iii) The door of the boundary flow barrier in the \((m_a, m_j)\)-source flow is closed on the \(\alpha\)-side if the flow barrier exists for time \(t \in [t_k, t_{k+1}]\) with \(k \in \mathbb{Z}\).

(iv) The door of the boundary flow barrier in the \((m_a, m_j)\)-source flow is permanently closed on the \(\alpha\)-side if the window exists for time \(t \in [t_k, t_{k+1}]\) with \(k \in \mathbb{Z}\).

(v) The door of the boundary flow barrier in the \((m_a, m_j)\)-source flow is permanently closed on the \(\alpha\)-side if the flow barrier exists for time \(t \in [t_k, t_{k+1}]\).

From the previous definition, the window of the boundary flow barrier in the \((m_a, m_j)\)-source flow is sketched in Fig. 22. On the window area, the source flow should satisfy the conditions in [Luo, 2006, 2008a]. The door for the boundary flow barrier wall on the boundary \(\partial \Omega_i\) is sketched in Fig. 23. In Fig. 23(a), the door of the boundary flow barrier wall is open, which implies the boundary flow of \(x^{(0)}\) can be switched to one of the \(x^{(0)}\)-flow. However, in Fig. 23(b), the door of the source flow barrier is closed, and the boundary flow of \(x^{(0)}\) cannot be switched into domain \(\Omega_i\) from the \(\alpha\)-side of the boundary.

For a source flow on the boundary \(\partial \Omega_i\), if the source flow barrier exists on the \(\alpha\)-side of such a boundary with a subset \(S \subseteq \partial \Omega_i\), the boundary flow of \(x^{(0)}\) cannot leave the boundary from the \(\alpha\)-side (\(\alpha = i,j\)) for \(x_m \in S \subseteq \partial \Omega_i\)

\[
0 \in \left[ \left( h_{0}G^{(0=0)}_{\partial \Omega_i}(x_m, q^{(0)}),\right) h_{0}G^{(0=0)}_{\partial \Omega_i}(x_m, q^{(0)}) \right].
\tag{157}
\]

If there are two boundary flow barriers on both sides of the boundary to satisfy Eq. (157), the dynamical system will be constrained on the boundary, governed by Eq. (140).

Theorem 9. For a discontinuous dynamical system in Eq. (47), there is a point \(x^{(0)}(t_m) \equiv x_m \in \partial \Omega_i\) at time \(t_m\) between two adjacent domains \(\Omega_i\) (\(\alpha = i,j\)). For \(x_m \in S \subseteq \partial \Omega_i\), there is a source flow barrier \(F^{(0=0)}([\alpha](t, \pi_a, q^{(0)})\) for \(q^{(0)} \in [\pi_a, \tilde{q}]\) on the \(\alpha\)-side of the boundary \(\partial \Omega_i\) with \(0 \in \left[ \left( h_{0}G^{(0=0)}_{\partial \Omega_i}(x_m, q^{(0)}),\right) h_{0}G^{(0=0)}_{\partial \Omega_i}(x_m, q^{(0)}) \right) \subset \mathbb{R}\). (158)

The two leaving flows in the source flow satisfy

\[
h_{0}G^{(0)}_{\partial \Omega_i}(x_m, t_{m+1}) > 0 \quad \text{and} \quad h_{0}G^{(0)}_{\partial \Omega_i}(x_m, t_{m+1}) > 0. \tag{159}
\]

(i) A boundary flow of \(x^{(0)}\) cannot be switched into the leaving flow of \(x^{(0)}\) in a source flow on the \(\alpha\)-side if and only if

\[
h_{0}G^{(0=0)}_{\partial \Omega_i}(x_m, q^{(0)}_2) < 0 \quad \text{and} \quad h_{0}G^{(0=0)}_{\partial \Omega_i}(x_m, q^{(0)}_1) > 0. \tag{160}
\]

(ii) A boundary flow of \(x^{(0)}\) cannot be switched into the leaving flow of \(x^{(0)}\) in a source flow on the
Fig. 22. The boundary flow barrier windows in the \((m_\alpha, m_\beta)\)-source flow on both sides of the boundary \(\partial \Omega_{ij}\): (a) partial flow barrier and (b) full flow barrier. The red curves are the \(G\)-functions relative to the flow barrier. The dark and blue surfaces are the flow barrier surfaces. The hatched area is for the zoomed boundary. The dark blue curves are coming flows \((m_\alpha, m_\beta) \in \{0, 1, 2, \ldots\}\).

\(\alpha\)-side if and only if

\[ h_{a} G_{\partial \Omega_{ij}}^{(m_\alpha, 0, 0)}(x_{m}, q_{1}^{(\alpha)}) = 0, \]

\[ h_{a} G_{\partial \Omega_{ij}}^{(l_{\alpha}, 0, 0)}(x_{m}, q_{1}^{(\alpha)}) > 0 \quad \text{and} \quad h_{a} G_{\partial \Omega_{ij}}^{(0, 0, 0)}(x_{m}, q_{2}^{(\alpha)}) < 0. \]  

\text{(iii) A boundary flow of } x^{(0)} \text{ is switched into the leaving flow of } x^{(\alpha)} \text{ in a source flow on the } \alpha\text{-side if and only if}

\[ h_{a} G_{\partial \Omega_{ij}}^{(m_\alpha, 0, 0)}(x_{m}, q_{1}^{(\alpha)}) = 0, \]

\[ h_{a} G_{\partial \Omega_{ij}}^{(l_{\alpha}, 0, 0)}(x_{m}, q_{1}^{(\alpha)}) < 0 \quad \text{and} \quad h_{a} G_{\partial \Omega_{ij}}^{(0, 0, 0)}(x_{m}, q_{2}^{(\alpha)}) < 0. \]  

\text{Proof}

(i) From Definition 23, the necessary and conditions in Eq. (160) are obtained.

(ii) An auxiliary flow of the boundary flow barrier is introduced as a fictitious flow of \(x^{(0)}_{\alpha, l_{\alpha}}(t)\).
For $x^{(0, 0, 0)}(t_{m+1}) = x^{(0)}(t_{m+1})$ and $x^{(s)}(t_{m+1}) = x^{(0)}(t_{m+1})$, the $G$-function definition gives

$$\mathbf{n}_{\partial \Omega_{ij}}(x^{(0)}(t_{m+1})) \cdot [x^{(0)}(t_{m+1}) - x^{(0)}(t_{m+1})] = \sum_{s=0}^{l_{\alpha}-1} G_{\alpha}^{(s)}(x^{(0)}(t_{m+1}), q^{(s)}_{1}) \epsilon_{s+1} + G_{\alpha}^{(l_{\alpha}+1)}(x^{(0)}(t_{m+1}), q^{(l_{\alpha}+1)}_{1}) \epsilon_{l_{\alpha}+1}.$$ 

Because of $G_{\alpha}^{(s)}(x^{(0)}(t_{m+1}), q^{(s)}_{1}) = 0$ for $s = 0, 1, 2, \ldots, l_{\alpha} - 1$;

$$G_{\alpha}^{(l_{\alpha}+1)}(x^{(0)}(t_{m+1}), q^{(l_{\alpha}+1)}_{1}) \neq 0;$$

one achieves

$$\mathbf{n}_{\partial \Omega_{ij}}(x^{(0)}(t_{m+1})) \cdot [x^{(0)}(t_{m+1}) - x^{(0)}(t_{m+1})] = -G_{\alpha}^{(l_{\alpha}+1)}(x^{(0)}(t_{m+1}), q^{(l_{\alpha}+1)}_{1}) \epsilon_{l_{\alpha}+1}.$$
From definition, the conditions for which the boundary flow of \( x^{(0)} \) cannot pass through the boundary flow barrier require
\[
\mathbf{n}_{\partial\Omega, \alpha}^T(\mathbf{x}^{(0)}(t_{m+})) \cdot [\mathbf{x}^{(0)}(t_{m+}) - \mathbf{x}^{(0)}(t_{m+})] > 0 \quad \text{for } \mathbf{n}_{\partial\Omega, \alpha} \rightarrow \partial\Omega,
\]
\[
\mathbf{n}_{\partial\Omega, \alpha}^T(\mathbf{x}^{(0)}(t_{m+})) \cdot [\mathbf{x}^{(0)}(t_{m+}) - \mathbf{x}^{(0)}(t_{m+})] < 0 \quad \text{for } \mathbf{n}_{\partial\Omega, \alpha} \rightarrow \Omega_n.
\]

From the definition of \( h_a \),
\[
h_a = 1 \quad \text{for } \mathbf{n}_{\partial\Omega, \alpha} \rightarrow \partial\Omega, \quad \text{and}
\]
\[
h_a = -1 \quad \text{for } \mathbf{n}_{\partial\Omega, \alpha} \rightarrow \Omega_n.
\]

If a boundary flow of \( x^{(0)} \) cannot be switched into the leaving flow of \( x^{(n)} \) in the source flow on the \( \alpha \)-side, the conditions in Eq. (162) can be obtained vice versa.

(iii) In a similar fashion, from definition, the conditions for which the boundary flow of \( x^{(0)} \) passes through the boundary flow barrier requires
\[
\mathbf{n}_{\partial\Omega, \alpha}^T(\mathbf{x}^{(0)}(t_{m+})) \cdot [\mathbf{x}^{(0)}(t_{m+}) - \mathbf{x}^{(0)}(t_{m+})] > 0 \quad \text{for } \mathbf{n}_{\partial\Omega, \alpha} \rightarrow \partial\Omega,
\]
\[
\mathbf{n}_{\partial\Omega, \alpha}^T(\mathbf{x}^{(0)}(t_{m+})) \cdot [\mathbf{x}^{(0)}(t_{m+}) - \mathbf{x}^{(0)}(t_{m+})] < 0 \quad \text{for } \mathbf{n}_{\partial\Omega, \alpha} \rightarrow \Omega_n.
\]

If a boundary flow of \( x^{(0)} \) is switched into the leaving flow of \( x^{(n)} \) in the source flow on the \( \alpha \)-side, the conditions in Eq. (162) can be obtained, vice versa. This theorem is proved. \( \blacksquare \)

**Theorem 10.** For a discontinuous dynamical system in Eq. (17), there is a point \( x^{(0)}(t_m) \equiv x_m \in \partial\Omega \) at time \( t_m \) between two adjacent domains \( \Omega_n \) \((n = 1, j)\). Suppose a boundary flow barrier \( \mathbf{F}^{(0, m, n)}(x^{(n)}, t, x^{(n)}, q^{(n)}) \) for \( q^{(n)} \in [q_1^{(n)}, q_2^{(n)}) \) exists on the \( \alpha \)-side of the boundary \( \partial\Omega_l \) in the \( (m_n : m_j) \)-source flow with
\[
G_{\partial\Omega_l, \alpha}^{(0, m_n, m_j)}(x_m, q_1^{(n)}) = 0
\]
for \( s_n = 0, 1, \ldots, m_n - 1; \)
\[
0 \in (h_n G_{\partial\Omega_l, \alpha}^{(m_n, m_j)}(x_m, q_1^{(n)})),
\]
\[
h_n G_{\partial\Omega_l, \alpha}^{(m_n, m_j)}(x_m, q_1^{(n)}) \subset \mathbb{R}.
\]

The two leaving flows in the \((m_n : m_j)\)-source flow satisfy
\[
G_{\partial\Omega_l, \alpha}^{(m_n, m_j)}(x_m, t_{m+}) = 0 \quad \text{for } s_n = 0, 1, \ldots, m_n - 1;
\]
\[
G_{\partial\Omega_l, \alpha}^{(m_n, m_j)}(x_m, t_{m+}) = 0 \quad \text{for } s_j = 0, 1, \ldots, m_j - 1;
\]
\[
h_n G_{\partial\Omega_l, \alpha}^{(m_n, m_j)}(x_m, t_{m+}) < 0 \quad \text{and}
\]
\[
h_n G_{\partial\Omega_l, \alpha}^{(m_n, m_j)}(x_m, t_{m+}) > 0.
\]

(i) A boundary flow of \( x^{(0)} \) cannot be switched to the leaving flow of \( x^{(n)} \) on the \( \alpha \)-side in the \((m_n : m_j)\)-source flow if and only if
\[
h_n G_{\partial\Omega_l, \alpha}^{(m_n, m_j)}(x_m, q_1^{(n)}) < 0 \quad \text{and}
\]
\[
h_n G_{\partial\Omega_l, \alpha}^{(m_n, m_j)}(x_m, q_1^{(n)}) > 0.
\]

(ii) A boundary flow of \( x^{(0)} \) cannot be switched to the leaving flow of \( x^{(n)} \) on the \( \alpha \)-side in the \((m_n : m_j)\)-source flow if and only if
\[
G_{\partial\Omega_l, \alpha}^{(m_n, m_j)}(x_m, q_1^{(n)}) = 0
\]
for \( s_n = m_n, m_n + 1, \ldots, t - 1; \)
\[
h_n G_{\partial\Omega_l, \alpha}^{(m_n, m_j)}(x_m, q_1^{(n)}) > 0 \quad \text{and}
\]
\[
h_n G_{\partial\Omega_l, \alpha}^{(m_n, m_j)}(x_m, q_1^{(n)}) < 0.
\]

(iii) A boundary flow of \( x^{(0)} \) cannot be switched into the leaving flow of \( x^{(n)} \) on the \( \alpha \)-side in the \((m_n : m_j)\)-source flow if and only if
\[
G_{\partial\Omega_l, \alpha}^{(m_n, m_j)}(x_m, q_1^{(n)}) = 0
\]
for \( s_n = m_n, m_n + 1, \ldots, t - 1; \)
\[
h_n G_{\partial\Omega_l, \alpha}^{(m_n, m_j)}(x_m, q_1^{(n)}) < 0 \quad \text{and}
\]
\[
h_n G_{\partial\Omega_l, \alpha}^{(m_n, m_j)}(x_m, q_1^{(n)}) < 0.
\]

**Proof**

(i) From Definition 24, the necessary and conditions in Eq. (164) are obtained.

(ii) An auxiliary flow of the boundary flow barrier is introduced as a fictitious flow of \( x^{(0, m_n)}(t_m) \). Since \( x^{(0, m_n)}(t_{m+}) = x^{(0)}(t_{m+}) \),
the $G$-function definition gives
\[
\mathbf{n}^T_{\partial \Omega_j}(\mathbf{x}^{(i)}(t_{m+i})) \\
\cdot [\mathbf{x}^{(0)}(t_{m+i}) - \mathbf{x}^{(i)}(t_{m+i})]
\]
\[
= \sum_{s_n=0}^{n-1} G^{(i,0-\alpha_0)}_{\partial \Omega_j}(\mathbf{x}_m, t_{m+i}, q_1^{(n)}) \varepsilon_{s_n+1}
\]
\[
+ \sum_{s_n=2k_n}^{n-1} G^{(i,0-\alpha_0)}_{\partial \Omega_j}(\mathbf{x}_m, t_{m+i}, q_1^{(n)}) \varepsilon_{s_n+1}
\]
\[
+ G^{(i,0-\alpha_0)}_{\partial \Omega_j}(\mathbf{x}_m, t_{m+i}, q_1^{(n)}) \varepsilon_{1+n+1},
\]

because of
\[
G^{(i,0-\alpha_0)}_{\partial \Omega_j}(\mathbf{x}_m, t_{m+i}, q_1^{(n)}) = 0
\]
for $s_n = 0, 1, \ldots, 2k_n - 1$

\[
G^{(i,0-\alpha_0)}_{\partial \Omega_j}(\mathbf{x}_m, q_1^{(n)}) = 0
\]
for $s_n = 2k_n, 2k_n + 1, \ldots, l_n - 1$

one obtains
\[
\mathbf{n}^T_{\partial \Omega_j}(\mathbf{x}^{(i)}(t_{m+i})) \\
\cdot [\mathbf{x}^{(0)}(t_{m+i}) - \mathbf{x}^{(i)}(t_{m+i})]
\]
\[
= -G^{(i,0-\alpha_0)}_{\partial \Omega_j}(\mathbf{x}_m, q_1^{(n)}) \varepsilon_{1+n+1}.
\]

From the definition, the boundary flow of $\mathbf{x}^{(i)}(t)$ not passing over the boundary flow barrier gives
\[
\mathbf{n}^T_{\partial \Omega_j}(\mathbf{x}^{(i)}(t_{m+i}))
\]
\[
\cdot [\mathbf{x}^{(0)}(t_{m+i}) - \mathbf{x}^{(i)}(t_{m+i})]
\]
\[
< 0 \quad \text{for } \mathbf{n}_{\partial \Omega_j} \rightarrow \Omega_\beta,
\]
\[
\mathbf{n}^T_{\partial \Omega_j}(\mathbf{x}^{(i)}(t_{m+i}))
\]
\[
\cdot [\mathbf{x}^{(0)}(t_{m+i}) - \mathbf{x}^{(i)}(t_{m+i})]
\]
\[
> 0 \quad \text{for } \mathbf{n}_{\partial \Omega_j} \rightarrow \Omega_\alpha.
\]

From the definition of $h_\alpha$,
\[
h_\alpha = 1 \quad \text{for } \mathbf{n}_{\partial \Omega_j} \rightarrow \Omega_\beta \quad \text{and}
\]
\[
h_\alpha = -1 \quad \text{for } \mathbf{n}_{\partial \Omega_j} \rightarrow \Omega_\alpha.
\]
Finally, one obtains Eq. (166). On the other hand, under Eq. (166), a boundary flow of $\mathbf{x}^{(0)}$ cannot be switched into the leaving flow of $\mathbf{x}^{(0)}$ in the $(m_\alpha : m_\beta)$ source flow on the $\alpha$-side.

(iii) In a similar fashion, from the definition, the boundary flow of $\mathbf{x}^{(0)}(t)$ passing over the boundary flow barrier request
\[
\mathbf{n}^T_{\partial \Omega_j}(\mathbf{x}^{(0)}(t_{m+i}))
\]
\[
\cdot [\mathbf{x}^{(0)}(t_{m+i}) - \mathbf{x}^{(0-\alpha_0)}(t_{m+i})]
\]
\[
< 0 \quad \text{for } \mathbf{n}_{\partial \Omega_j} \rightarrow \Omega_\beta,
\]
\[
\mathbf{n}^T_{\partial \Omega_j}(\mathbf{x}^{(0)}(t_{m+i}))
\]
\[
\cdot [\mathbf{x}^{(0)}(t_{m+i}) - \mathbf{x}^{(0-\alpha_0)}(t_{m+i})]
\]
\[
> 0 \quad \text{for } \mathbf{n}_{\partial \Omega_j} \rightarrow \Omega_\alpha.
\]

If a boundary flow of $\mathbf{x}^{(0)}$ is switched into the leaving flow of $\mathbf{x}^{(0)}$ in the $(m_\alpha : m_\beta)$ source flow on the $\alpha$-side, the conditions in Eq. (167) are obtained, vice versa. This theorem is proved. \[\blacksquare\]

6.2. Leaving flow barriers in source flows

As similar to the coming flow barriers in the $(2k_\alpha : 2k_\beta)$-sink flow, the leaving flow barrier in the $(m_\alpha : m_\beta)$ source flow can be described. For convenience, the corresponding discussion of the leaving flow barriers in the source flow will be given as follows.

**Definition 34.** For a discontinuous dynamical system in Eq. (17), there is a point $\mathbf{x}^{(0)}(t_\alpha) \equiv \mathbf{x}_m \in \partial \Omega_j$ at time $t_\alpha$ between two adjacent domains $\Omega_{\alpha}(\alpha = i, j)$. There is a vector field of $\mathbf{F}^{(0-\alpha)}(\mathbf{x}^{(0)}, t, \pi, q^{(0)})$ for $q^{(0)} \in [q_1^{(0)}, q_2^{(0)}]$ on the boundary $\partial \Omega_j$ with

\[
h_\alpha G^{(0-\alpha)}_{\partial \Omega_j}(\mathbf{x}_m, q^{(0)})
\]
\[
\in [h_\alpha G^{(0-\beta)}_{\partial \Omega_j}(\mathbf{x}_m, q_2^{(0)}), h_\alpha G^{(0-\alpha)}_{\partial \Omega_j}(\mathbf{x}_m, q_1^{(0)})]
\]
\[
\subset (-\infty, 0].
\]

(\alpha, \beta \in \{i, j\} \text{ and } \alpha \neq \beta). The leaving flows in the source flow satisfy

\[
h_\alpha G^{(0-\alpha)}_{\partial \Omega_j}(\mathbf{x}_m, t_{m+i}) < 0 \quad \text{and}
\]
\[
h_\alpha G^{(0-\beta)}_{\partial \Omega_j}(\mathbf{x}_m, t_{m+i}) > 0
\]

(169)

The vector field of $\mathbf{F}^{(0-\alpha)}(\mathbf{x}^{(0)}, t, \pi, q^{(0)})$ is called the flow barrier of the leaving flow of $\mathbf{x}^{(0)}$ in the
source flow on the α-side if the following conditions are satisfied. The two critical values of \( F^{(0,0)}(x(0), t, \pi, q^{(0)}) \) for \( \sigma = 1, 2 \) are called the lower and upper limits of the leaving flow barriers on the α-side.

(i) The leaving flow of \( x^{(0)} \) cannot leave the boundary on the α-side if

\[
x^{(0)}(t_{m+}) = x^{(0)}(t_{m+}, q^{(0)}) = x_m,
\]

\[
h_\alpha G^{(0,0)}_{\partial \Omega_1}(x_m, t_{m+})
\]

\[
eq \{h_\alpha G^{(0,0)}_{\partial \Omega_1}(x_m, q^{(0)}), h_\alpha G^{(0,0)}_{\partial \Omega_1}(x_m, q^{(0)})\}
\]

\[\subset (-\infty, 0) \] (170)

(ii) The leaving flow of \( x^{(0)} \) cannot leave the boundary on the α-side at the critical points (i.e. \( q^{(0)} = q^{(0)}, \sigma \in \{1, 2\} \)) if

\[
x^{(0)}(t_{m+}) = x^{(0)}(t_{m+}, q^{(0)}) = x_m,
\]

\[
G^{(0,0)}_{\partial \Omega_1}(x_m, t_{m+}) = G^{(0,0)}_{\partial \Omega_1}(x_m, q^{(0)}) \neq 0
\]

\[
\text{for } s_a = 0, 1, 2, \ldots, t_a - 1;
\]

\[
(-1)^{\sigma} h_\alpha n_\alpha^T(x^{(0)}(t_{m+}))
\]

\[
\cdot |x^{(0)}(t_{m+}) - x^{(0)}(t_{m+}, q^{(0)})|^2 > 0.
\] (171)

(iii) The leaving flow of \( x^{(0)} \) can leave the boundary on the α-side at the critical points (i.e. \( q^{(0)} = q^{(0)}, \sigma \in \{1, 2\} \)) if

\[
x^{(0)}(t_{m+}) = x^{(0)}(t_{m+}, q^{(0)}) = x_m,
\]

\[
G^{(0,0)}_{\partial \Omega_1}(x_m, t_{m+}) = G^{(0,0)}_{\partial \Omega_1}(x_m, q^{(0)}) \neq 0
\]

\[
\text{for } s_a = 0, 1, 2, \ldots, t_a - 1;
\]

\[
(-1)^{\sigma} h_\alpha n_\alpha^T(x^{(0)}(t_{m+}))
\]

\[
\cdot |x^{(0)}(t_{m+}) - x^{(0)}(t_{m+}, q^{(0)})|^2 < 0.
\] (172)

Definition 35. For a discontinuous dynamical system in Eq. (17), there is a point \( x^{(0)}(t_{m+}) \) \( \equiv x_m \in \partial \Omega_1 \) at time \( t_m \) between two adjacent domains \( \Omega_0 \) (α = i, j). There is a vector field \( F^{(0,0)}(x^{(0)}, t, \pi, q^{(0)}) \) for \( q^{(0)} \in \{q^{(0)}_1, q^{(0)}_2\} \) on the boundary \( \partial \Omega_1 \) with

\[
G^{(m,0,0)}_{\partial \Omega_1}(x_m, q^{(0)}) = 0 \quad \text{for } s_a = 0, 1, \ldots, m_a - 1;
\]

\[
G^{(m,0,0)}_{\partial \Omega_1}(x_m, q^{(0)}) \in \{h_\alpha G^{(m,0,0)}_{\partial \Omega_1}(x_m, q^{(0)}), \quad
\]

\[
h_\alpha G^{(m,0,0)}_{\partial \Omega_1}(x_m, q^{(0)})(x_m, q^{(0)}))
\]

\[\subset (-\infty, 0) \] (173)

(α, β ∈ \{i, j\} and \( \alpha \neq \beta \)). The leaving flows in the \((m_a : m_b)\)-source flow satisfy

\[
G^{(m,0,0)}_{\partial \Omega_1}(x_m, t_{m+}) = 0 \quad \text{for } s_a = 0, 1, \ldots, m_a - 1,
\]

\[
G^{(m,0,0)}_{\partial \Omega_1}(x_m, t_{m+}) = 0 \quad \text{for } s_a = 0, 1, \ldots, m_b - 1,
\]

\[
h_\alpha G^{(m,0,0)}_{\partial \Omega_1}(x_m, t_{m+}) < 0 \quad \text{and}
\]

\[
h_\alpha G^{(m,0,0)}_{\partial \Omega_1}(x_m, t_{m+}) > 0.
\] (174)

The vector field of \( F^{(0,0)}(x^{(0)}, t, \pi, q^{(0)}) \) is called the \textit{leaving flow barrier} in the \((m_a : m_b)\)-source flow on the α-side if the following conditions are satisfied. The critical values of \( F^{(0,0)}(x^{(0)}, t, \pi, q^{(0)}) \) are called the lower and upper limits of the leaving flow barriers in the \((m_a : m_b)\)-source flow on the α-side.

(i) The leaving flow of \( x^{(0)} \) in the \((m_a : m_b)\) source flow cannot leave the boundary on the α-side if

\[
x^{(0)}(t_{m+}) = x^{(0)}(t_{m+}, q^{(0)}) = x_m,
\]

\[
G^{(m,0,0)}_{\partial \Omega_1}(x_m, t_{m+}) = G^{(m,0,0)}_{\partial \Omega_1}(x_m, q^{(0)}),
\]

\[
h_\alpha G^{(m,0,0)}_{\partial \Omega_1}(x_m, t_{m+})
\]

\[\subset (-\infty, 0) \] (175)

(ii) The leaving flow of \( x^{(0)} \) in the \((m_a : m_b)\) source flow cannot leave the boundary on the α-side at the critical points (i.e. \( q^{(0)} = q^{(0)}, \sigma \in \{1, 2\} \)) if

\[
x^{(0)}(t_{m+}) = x^{(0)}(t_{m+}, q^{(0)}) = x_m,
\]

\[
G^{(m,0,0)}_{\partial \Omega_1}(x_m, t_{m+}) = G^{(m,0,0)}_{\partial \Omega_1}(x_m, q^{(0)}) \neq 0
\]

\[
\text{for } s_a = m_a, m_a + 1, \ldots, t_a - 1;
\]
(iii) The leaving flow of $x^{(\alpha)}$ in the $(m_\alpha : m_\beta)$ source flow can leave the boundary on the $\alpha$-side at the critical points (i.e. $q^{(\alpha)} = q^{(\alpha)}_\sigma$, $\sigma \in \{1, 2\}$) if

$$x^{(\alpha)}(t_{m+}) = x^{(0-\alpha)}(t_{m+}, q^{(\alpha)}_\sigma) = x_m, \quad G^{(s_\alpha, m_\alpha)}(x_m, t_{m+}) = G^{(s_\beta, m_\beta)}(x_m, q^{(\alpha)}_\sigma) \neq 0$$

for $s_\alpha = m_\alpha, m_\alpha + 1, \ldots, l_\alpha - 1$. To explain the leaving flow barrier in the source flow, the leaving flow barriers on both sides of the boundary $\partial \Omega_{ij}$ are also presented through the $G$-functions in Fig. 24. The red dashed curves represent the $G$-function of the flows pertaining to the flow barriers in each domain $\Omega_\alpha$ ($\alpha = i, j$). The thick lines denote the $G$-functions of the flow barriers on both sides of the boundary. To show the flow barriers, consider the $G$-function on the $\beta$-side of the boundary $\partial \Omega_{ij}$ as a reference ($\beta = i, j$; $m_\alpha \in \{0, 1, 2, \ldots\}$, $\alpha = i, j$) and similarly for the $\beta$-side.

\[ (-1)^{\sigma} n_{\alpha} \nu_{\alpha}^T (x^{(0)}(t_{m+})) \cdot [x^{(\alpha)}(t_{m+}) - x^{(0-\alpha)}(t_{m+}, q^{(\alpha)}_\sigma)] > 0. \]  
\[ \text{(176)} \]

\[ (-1)^{\sigma} n_{\alpha} \nu_{\alpha}^T (x^{(0)}(t_{m+})) \cdot [x^{(\alpha)}(t_{m+}) - x^{(0-\alpha)}(t_{m+}, q^{(\alpha)}_\sigma)] < 0. \]  
\[ \text{(177)} \]
Definition 36. For a discontinuous dynamical system in Eq. (17), there is a point \( x(0) (t_m) \equiv x_m \in \partial \Omega_i \) at time \( t_m \) between two adjacent domains \( \Omega_i (i, j) \). Suppose there is a leaving flow barrier of \( F^{(0,0)}(x, t, \pi_i, \pi_j, q) \) for \( q(0) \in [q_1^{(0)}, q_2^{(0)}] \) on the \( \alpha \)-side of the boundary in the \((m_i,m_j)\)-source flow \((m_0 = 0, 1, \ldots)\).

(i) The leaving flow barrier in the \((m_i,m_j)\)-source flow is partial on the \( \alpha \)-side if \( x_m \in S \subset \partial \Omega_i \).

(ii) The leaving flow barrier in the \((m_i,m_j)\)-source flow is full on the \( \alpha \)-side if \( x_m \in S = \partial \Omega_i \).

Definition 37. For a discontinuous dynamical system in Eq. (17), there is a point \( x(0) (t_m) \equiv x_m \in \partial \Omega_i \) at time \( t_m \) between two adjacent domains \( \Omega_i (i, j) \). There is a leaving flow barrier of \( F^{(0,0)}(x, t, \pi_i, \pi_j, q) \) for \( q(0) \in [q_1^{(0)}, q_2^{(0)}] \) on the \( \alpha \)-side of the boundary \((m_i,m_j) \in \{0,1,2,\ldots\}\) in the \((m_i,m_j)\)-source flow

(i) The leaving flow barrier in the \((m_i,m_j)\)-source flow is with an upper limit on the \( \alpha \)-side if for \( x_m \in S \subset \partial \Omega_i \)

\[
\begin{align*}
  b^{(0,0)}(x_m, q_1^{(0)}) &= 0, \quad \text{and} \quad \hbar^{(0,0)}(x_m, q_2^{(0)}) \
\end{align*}
\]  \hspace{1cm} (178)

(ii) The leaving flow barrier in the \((m_i,m_j)\)-source flow is with a lower limit on the \( \alpha \)-side if for \( x_m \in S \subset \partial \Omega_i \)

\[
\begin{align*}
  b^{(0,0)}(x_m, q_1^{(0)}) &< 0, \quad \text{and} \quad \hbar^{(0,0)}(x_m, q_2^{(0)}) \
\end{align*}
\]  \hspace{1cm} (179)

(iii) The leaving flow barrier in the \((m_i,m_j)\)-source flow is strict on the \( \alpha \)-side if for \( x_m \in S \subset \partial \Omega_i \)

\[
\begin{align*}
  b^{(0,0)}(x_m, q_1^{(0)}) &= 0, \quad \text{and} \quad \hbar^{(0,0)}(x_m, q_2^{(0)}) \
\end{align*}
\]  \hspace{1cm} (180)

(iv) The leaving flow barrier is called the leaving flow barrier wall in the \((m_i,m_j)\)-source flow on the \( \alpha \)-side if the absolute leaving flow barrier exists for \( x_m \in S = \partial \Omega_i \).

(v) The flow barrier is called the leaving flow barrier fence in the \((m_i,m_j)\)-source flow on the \( \alpha \)-side if there are many partial barriers and many nonbarriers on the \( \alpha \)-side of the boundary \( \partial \Omega_i \).

Definition 38. For a discontinuous dynamical system in Eq. (17), there is a point \( x(0) (t_m) \equiv x_m \in \partial \Omega_i \) at time \( t_m \) between two adjacent domains \( \Omega_i (i, j) \). The leaving flows in the \((m_i,m_j)\)-source flow satisfy Eq. (174). Suppose there is a leaving flow barrier of \( F^{(0,0)}(x, t, \pi_i, \pi_j, q) \) on the \( \alpha \)-side of the boundary \((m_i,m_j) \in \{0,1,2,\ldots\}\) in the \((m_i,m_j)\)-source flow for \( x_m \in S \subset \partial \Omega_i \) and \( q(0) \in [q_1^{(0)}, q_2^{(0)}] \) with

\[
\begin{align*}
  b^{(0,0)}(x_m, q_1^{(0)}) &< 0, \quad \text{and} \quad \hbar^{(0,0)}(x_m, q_2^{(0)}) \
\end{align*}
\]  \hspace{1cm} (181)

For \( q(0) \in [q_1^{(0)}, q_2^{(0)}] \), no leaving flow barrier is defined at \( x_m \in S \subset \partial \Omega_i \). Thus, the leaving flow of \( x(0) \) on the \( \alpha \)-side of the boundary can directly leave the boundary if

\[
\begin{align*}
  b^{(0,0)}(x_m, q_1^{(0)}) &= 0, \quad \text{and} \quad \hbar^{(0,0)}(x_m, q_2^{(0)}) \
\end{align*}
\]  \hspace{1cm} (182)

The \( G \)-function intervals for all \( x_m \in S \subset \partial \Omega_i \) with \( q(0) \in [q_1^{(0)}, q_2^{(0)}] \) are called the window of the leaving flow barrier in the \((m_i,m_j)\)-source flow on the \( \alpha \)-side of the boundary.

In a similar fashion, the partial and full leaving flow barriers in a \((m_i,m_j)\)-source flow on both sides of the boundary \( \partial \Omega_i \) are sketched in Fig. 25 for \( x_m \in S \subset \partial \Omega_i \) through the two different colored surfaces at the \( \alpha \)- and \( \beta \)-sides of the boundary \( \partial \Omega_i \). The two leaving flow barriers are different. Again, to clearly show the leaving flow barriers, the \( G \)-function on the \( \beta \)-side of the boundary \( \partial \Omega_i \) is considered as a reference, and the \( G \)-function on the \( \alpha \)-side of the boundary \( \partial \Omega_i \) is presented by \(-G^{(0)}(x_m)\).

The infinity leaving flow barriers in the \((m_i,m_j)\)-source flow are sketched in Fig. 26.
Definition 39. For a discontinuous dynamical system in Eq. (17), there is a point \( x^{(0)}(t_m) \equiv x_m \in \partial\Omega_\alpha \) at time \( t_m \) between two adjacent domains \( \Omega_\alpha (\alpha = i, j) \). Suppose there is a leaving flow barrier of \( F^{(0)}(x^{(0)}, t, \pi, q^{(0)}) \) for \( q^{(0)} \in [q^{(0)}_1, q^{(0)}_2] \) on the \( \alpha \)-side of the boundary \((m_\alpha, m_\beta \in \{0, 1, 2, \ldots\})\) in the \((m_\alpha, m_\beta)\)-source flow.

(i) The leaving flow barrier in the \((m_\alpha, m_\beta)\)-source flow is \textit{permanent} on the \( \alpha \)-side if the flow barrier is \textit{independent} of time \( t \in [0, \infty) \).

(ii) The leaving flow barrier in the \((m_\alpha, m_\beta)\)-source flow is \textit{instantaneous} on the \( \alpha \)-side if the flow barrier is \textit{dependent} on time \( t \in [0, \infty) \).

(iii) The leaving flow barrier in the \((m_\alpha, m_\beta)\)-source flow is \textit{intermittent} on the \( \alpha \)-side if the flow barrier exists for time \( t \in [t_k, t_{k+1}] \) with \( k \in \mathbb{Z} \).

(iv) The leaving flow barrier in the \((m_\alpha, m_\beta)\)-source flow is \textit{intermittent and static} on the \( \alpha \)-side if the flow barrier is \textit{independent} of time \( t \in [t_k, t_{k+1}] \) with \( k \in \mathbb{Z} \).

Similarly, the permanent and instantaneous windows of the leaving flow barrier in the source flow on the boundary can be discussed. Further, the
Fig. 26. The infinity leaving flow barrier in the \((m_{\alpha}, m_\beta)\)-source with lower boundary on \(\partial \Omega_{ij}\): (a) partial flow barrier and (b) full flow barrier. The red curves are the \(G\)-functions relative to the flow barrier. The dark and blue surfaces are the flow barrier surfaces. The hatched area is for the zoomed boundary. The dark blue curves are coming flows \((m_{\alpha}, m_\beta \in \{0, 1, 2, \ldots\})\).

Concept of the door of the leaving flow barrier wall on the boundary can be defined.

**Definition 40.** For a discontinuous dynamical system in Eq. (17), there is a point \(x^{(\alpha)}(t_m) \equiv x_m \in \partial \Omega_{ij}\) at time \(t_m\) between two adjacent domains \(\Omega_{\alpha}\) \((\alpha = i, j)\). The leaving flows in the \((m_{\alpha}, m_\beta)\)-source flow satisfy Eq. (174). Suppose there is a leaving flow barrier of \(F^{(0n)}(x^{(\alpha)}, t, \pi_{\alpha}, q^{(\alpha)})\) for \(q^{(\alpha)} \in [q^{(\alpha)}_{2n-1}, q^{(\alpha)}_{2n}]\) in the \((m_{\alpha}, m_\beta)\)-source flow on the \(\alpha\)-side \((m_{\alpha}, m_\beta \in \{0, 1, 2, \ldots\}, n = 1, 2, \ldots\), and there is a window of the leaving flow barrier for \(S \subset \partial \Omega_{ij}\) and \(q^{(\alpha)} \in [q^{(\alpha)}_{2n-1}, q^{(\alpha)}_{2n+1}]\).

(i) The window of the leaving flow barrier in the \((m_{\alpha}, m_\beta)\)-source flow is permanent on the \(\alpha\)-side if the window is independent of time \(t \in [0, +\infty)\).

(ii) The window of the leaving flow barrier in the \((m_{\alpha}, m_\beta)\)-source flow is instantaneous on the
(iii) The window of the leaving flow barrier in the \((m_α; m_β)\)-source flow is intermittently open if the window exists for time \(t \in [t_k, t_{k+1}]\) with \(k \in Z\).

(iv) The window of the leaving flow barrier in the \((m_α; m_β)\)-source flow is intermittently and static on the \(α\)-side if the window is independent of time \(t \in [t_k, t_{k+1}]\) with \(k \in Z\).

**Definition 41.** For a discontinuous dynamical system in Eq. (17), there is a point \(x^{(0)}(t_m) \equiv x_m \in ∂Ω_α\) at time \(t_m\) between two adjacent domains \(Ω_α (α = i, j)\). There is a leaving flow barrier of \(F^{(α, β)}(x^{(0)}(t), π_m, q_{ij}^{(0)})\) for \(q_{ij}^{(0)} \in [q_{ij}^{(0)}, q_{ij}^{(0)}]\) in the \((m_α; m_β)\)-source flow on the \(α\)-side of the boundary \((m_α, m_β) \in \{0, 1, 2, \ldots\}\). Suppose a leaving flow barrier wall in the source flow exists on the \(α\)-side, but there is an intermittent and static window of the leaving flow barrier on \(S \subseteq ∂Ω_β\) for \(q_{ij}^{(0)} \in [q_{ij}^{(0)}, q_{ij}^{(0)}]\) and \(t \in [t_k, t_{k+1}]\) with \(k \in Z\).

(i) The window of the leaving flow barrier is termed the door of the leaving flow barrier wall in the \((m_α; m_β)\)-source flow on the \(α\)-side if the window and the barrier exist alternatively.

(ii) The door of the leaving flow barrier wall in the \((m_α; m_β)\)-source flow is open on the \(α\)-side if the window exists for time \(t \in [t_k, t_{k+1}]\) with \(k \in Z\).

(iii) The door of the leaving flow barrier wall in the \((m_α; m_β)\)-source flow is closed if the leaving flow barrier exists for time \(t \in [t_k, t_{k+1}]\) with \(k \in Z\).

(iv) The door of the leaving flow barrier wall in the \((m_α; m_β)\)-source flow is permanently open on the \(α\)-side if the window exists for time \(t \in [t_k, t_{k+1}]\) with \(k \in Z\).

(v) The door of the leaving flow barrier wall in the \((m_α; m_β)\)-source flow is permanently closed on the \(α\)-side if the leaving flow barrier exists for \(t \in [t_k, t_{k+1}]\).

From the previous definition, the window of the leaving flow barrier in the source flow is sketched in Fig. 27. On the window area, the source flow should satisfy the conditions in [Luo, 2006, 2008a, 2008b]. The door of the leaving flow barrier wall in the \((m_α; m_β)\)-source flow on the boundary \(∂Ω_β\) is sketched in Fig. 28. In Fig. 28(a), the door of the flow barrier wall is open. Thus the leaving flow of \(x^{(0)}\) can leave the boundary. However, in Fig. 28(b), the door of the leaving flow barrier is closed, and the leaving flow of \(x^{(0)}\) cannot leave the boundary.

If the leaving flow barrier in the source flow exists on the \(α\)-side of such a boundary with a subset \(S \subseteq ∂Ω_β\), the leaving flow of \(x^{(0)}\) cannot leave the boundary if for \(x_m \in S \subseteq ∂Ω_β\)

\[
|h_0G^{(m_α, m_β)}(x_m)| = \begin{cases} h_0G^{(m_α, m_β)}(x_m, q_{ij}^{(0)}), & \text{if } x_m \in S \subseteq ∂Ω_β; \\
|\begin{pmatrix} h_0G^{(m_α, m_β)}(x_m, q_{ij}^{(0)}) \\
(\alpha_0, q_{ij}^{(0)}) \end{pmatrix}|, & \text{otherwise.} \end{cases}
\]

For this case, the dynamical system on the \(α\)-side of the boundary will be constrained by the boundary \(∂Ω_β\), i.e.

\[
\dot{x}^{(0)} = F^{(0, 0)}(x^{(0)}, t, π_m, q_{ij}^{(0)}) \text{ in } Ω_α (α = i, j),
\]

\[
\text{with } φ_{ij}(x^{(0)}, t, λ) = 0 \text{ on } ∂Ω_β.
\]

The leaving flow of \(x^{(0)}\) will be along the boundary in the domain \(Ω_α\) until the condition in Eq. (183) cannot be satisfied.

**Theorem 11.** For a discontinuous dynamical system in Eq. (17), there is a point \(x^{(0)}(t_m) \equiv x_m \in ∂Ω_α\) at time \(t_m\) between two adjacent domains \(Ω_α (α = i, j)\). For \(x_m \in S \subseteq ∂Ω_β\), there is a leaving flow barrier \(F^{(α, β)}(x^{(0)}(t), π_m, q_{ij}^{(0)})\) for \(q_{ij}^{(0)} \in [q_{ij}^{(0)}, q_{ij}^{(0)}]\) in the source flow on the \(α\)-side of the boundary \(∂Ω_β\) with

\[
|h_0G^{(m_α, m_β)}(x_m, q_{ij}^{(0)})| = \begin{cases} |h_0G^{(m_α, m_β)}(x_m, q_{ij}^{(0)}), h_0G^{(m_α, m_β)}(x_m, q_{ij}^{(0)})| & \text{if } x_m \in S \subseteq ∂Ω_β; \\
(−∞, 0], & \text{otherwise.} \end{cases}
\]

The leaving flow in the source flow satisfies

\[
|h_0G^{(m_α, m_β)}(x_m, t_m, x_m)| < 0 \text{ and } h_0G^{(m_α, m_β)}(x_m, t_m, x_m) > 0.
\]

(i) A leaving flow of \(x^{(0)}\) in the domain \(Ω_α\) cannot leave the boundary \(∂Ω_β\) if and only if

\[
|h_0G^{(m_α, m_β)}(x_m, t_m)| < 0 \text{ and } h_0G^{(m_α, m_β)}(x_m, t_m, x_m) > 0.
\]

(ii) A leaving flow of \(x^{(0)}\) in the domain \(Ω_α\) cannot leave the boundary at the critical points.
Fig. 27. The leaving flow barrier windows in the \((m_\alpha, m_\beta)\)-source flow on \(\partial \Omega_{ij}\): (a) partial flow barrier and (b) full flow barrier. The red curves are the \(G\)-functions relative to the flow barrier. The dark and blue surfaces are the flow barrier surfaces. The hatched area is for the zoomed boundary. The dark blue curves are coming flows \((m_\alpha, m_\beta \in \{0, 1, 2, \ldots\})\).

\[
q^\alpha_\sigma (\sigma = 1, 2) \text{ if and only if } \quad G^{(s_\alpha, \alpha)}(x_m, t_m + 1) = G^{(s_\alpha, 0 > \alpha)}(x_m, t_m + q^{\alpha}_\sigma) \neq 0, \quad s_\alpha = 0, 1, \ldots, l_\alpha - 1; \quad (188)
\]

\[
(-1)^s h_\alpha [G^{(s_\alpha, \alpha)}(x_m, t_m + 1) - G^{(s_\alpha, 0 > \alpha)}(x_m, t_m + q^{\alpha}_\sigma)] > 0.
\]

(iii) A leaving flow of \(x^{(\alpha)}\) in the domain can leave the boundary at the critical points \(q^{\alpha}_\sigma\) \((\sigma = 1, 2)\) if and only if

\[
G^{(s_\alpha, \alpha)}(x_m, t_m + 1) = G^{(s_\alpha, 0 > \alpha)}(x_m, t_m + q^{\alpha}_\sigma) \neq 0, \quad s_\alpha = 0, 1, \ldots, l_\alpha - 1; \quad (189)
\]

\[
(-1)^s h_\alpha [G^{(s_\alpha, \alpha)}(x_m, t_m + 1) - G^{(s_\alpha, 0 > \alpha)}(x_m, t_m + q^{\alpha}_\sigma)] < 0.
\]

\[
q^{\alpha}_\sigma (\sigma = 1, 2) \text{ if and only if } \quad G^{(s_\alpha, \alpha)}(x_m, t_m + 1) = G^{(s_\alpha, 0 > \alpha)}(x_m, t_m + q^{\alpha}_\sigma) \neq 0, \quad s_\alpha = 0, 1, \ldots, l_\alpha - 1; \quad (189)
\]

\[
(-1)^s h_\alpha [G^{(s_\alpha, \alpha)}(x_m, t_m + 1) - G^{(s_\alpha, 0 > \alpha)}(x_m, t_m + q^{\alpha}_\sigma)] < 0.
\]

\[
q^{\alpha}_\sigma (\sigma = 1, 2) \text{ if and only if } \quad G^{(s_\alpha, \alpha)}(x_m, t_m + 1) = G^{(s_\alpha, 0 > \alpha)}(x_m, t_m + q^{\alpha}_\sigma) \neq 0, \quad s_\alpha = 0, 1, \ldots, l_\alpha - 1; \quad (189)
\]

\[
(-1)^s h_\alpha [G^{(s_\alpha, \alpha)}(x_m, t_m + 1) - G^{(s_\alpha, 0 > \alpha)}(x_m, t_m + q^{\alpha}_\sigma)] < 0.
\]

Proof. The proof of this theorem is similar to Theorem 1. ■
Theorem 12. For a discontinuous dynamical system \(x_t\) (17), there is a point \(x(0)_{m} \in \partial \Omega_{ij}\) at time \(t_{m}\) between two adjacent domains \(\Omega_{\alpha}\) \((\alpha = i, j)\). Suppose a leaving barrier \(F(0 \mapsto \alpha)\) \((x(\alpha), t, \pi, q(\alpha))\) in the \((m_{\alpha} : m_{\beta})\)-source flow for \(q(\alpha) \in [q_{1}(\alpha), q_{2}(\alpha)]\) exists on the \(\alpha\)-side of the boundary \(\partial \Omega_{ij}\) with

\[
G^{\alpha_{\alpha_{0} \mapsto \alpha}}(x, q(\alpha)) = 0
\]

for \(s_{\alpha} = 0, 1, \ldots, m_{\alpha} - 1\); (190)

\[
h_{\alpha}G^{\alpha_{\alpha_{0} \mapsto \alpha}}(x, q(\alpha)) \in \left[ h_{\alpha}G^{\alpha_{\alpha_{0} \mapsto \alpha}}(x, q_{1}(\alpha)), h_{\alpha}G^{\alpha_{\alpha_{0} \mapsto \alpha}}(x, q_{2}(\alpha)) \right]
\subset (-\infty, 0]. \quad (191)

The leaving flows in the \((m_{\alpha} : m_{\beta})\)-source flow satisfy

\[
G^{\alpha_{\alpha_{0} \mapsto \alpha}}(x, \pi_{m} + p_{\alpha}, \lambda) = 0
\]

for \(s_{\alpha} = 0, 1, \ldots, m_{\alpha} - 1\);
and also there is a flow barrier $\mathbf{F}^{(\text{no})}(\mathbf{x}^{(i)}, t, \sigma, q^{(\alpha)})$ at $q^{(\alpha)} \in [q_{1}^{(\alpha)}, q_{2}^{(\alpha)}]$ on the $\alpha$-side of the boundary $\partial \Omega_{ij}$ with the $G$-function

$$h_{a}G_{\alpha|\beta}^{(\text{no})}(\mathbf{x}_{m}, q^{(\alpha)}_{\alpha}) \in \left[h_{a}G_{\alpha|\beta}^{(\text{no})}(\mathbf{x}_{m}, q_{1}^{(\alpha)}), h_{a}G_{\alpha|\beta}^{(\text{no})}(\mathbf{x}_{m}, q_{2}^{(\alpha)})\right] \subset [0, +\infty); \quad (196)$$

and also there is a flow barrier $\mathbf{F}^{(\text{no})}(\mathbf{x}^{(i)}, t, \sigma, q^{(\alpha)})$ at $q^{(\alpha)} \in [q_{1}^{(\alpha)}, q_{2}^{(\alpha)}]$ on the $\beta$-side of the boundary $\partial \Omega_{ij}$ with the $G$-function

$$h_{a}G_{\alpha|\beta}^{(\text{no})}(\mathbf{x}_{m}, q^{(\alpha)}_{\beta}) \in \left[h_{a}G_{\alpha|\beta}^{(\text{no})}(\mathbf{x}_{m}, q_{1}^{(\alpha)}), h_{a}G_{\alpha|\beta}^{(\text{no})}(\mathbf{x}_{m}, q_{2}^{(\alpha)})\right] \subset (-\infty, 0]. \quad (197)$$

The coming flows in the sink flow satisfy

$$h_{a}G_{\alpha|\beta}^{(\text{no})}(\mathbf{x}_{m}, t_{m}+) > 0 \quad \text{and} \quad h_{a}G_{\alpha|\beta}^{(\text{no})}(\mathbf{x}_{m}, t_{m}-) < 0. \quad (198)$$

A coming flow of $\mathbf{x}^{(\alpha)}$ switches to the boundary flow of $\mathbf{x}^{(0)}$ to form a sink flow on the boundary $\partial \Omega_{ij}$ if and only if for $\sigma_{\alpha}, \sigma_{\beta} \in [1, 2]$.
either
\[ h_{\alpha}G^{(\alpha)}_{\partial\Omega_{ij}}(x_m, t_{m-}) \notin \{ h_{\alpha}G^{(\alpha)}_{\partial\Omega_{ij}}(x_m, q_{\alpha}^{(\alpha)}), h_{\alpha}G^{(\gamma)}_{\partial\Omega_{ij}}(x_m, q_{\gamma}^{(\gamma)}) \} \]

or
\[ G^{(\alpha)}_{\partial\Omega_{ij}}(x_m, t_{m-}) = G^{(\gamma)}_{\partial\Omega_{ij}}(x_m, q_{\alpha}^{(\gamma)}) \neq 0 \quad \text{for } s_{\alpha} = 0, 1, \ldots, l_{\alpha} - 1; \]
\[ (-1)^{s_{\alpha}}[h_{\alpha}G^{(\alpha)}_{\partial\Omega_{ij}}(x_m, t_{m-}) - h_{\alpha}G^{(\gamma)}_{\partial\Omega_{ij}}(x_m, q_{\alpha}^{(\gamma)})] > 0; \] (199)

either
\[ h_{\beta}G^{(\beta)}_{\partial\Omega_{ij}}(x_m, t_{m-}) \notin \{ h_{\beta}G^{(\beta)}_{\partial\Omega_{ij}}(x_m, q_{\beta}^{(\beta)}), h_{\beta}G^{(\gamma)}_{\partial\Omega_{ij}}(x_m, q_{\gamma}^{(\gamma)}) \} \]

or
\[ G^{(\beta)}_{\partial\Omega_{ij}}(x_m, t_{m-}) = G^{(\gamma)}_{\partial\Omega_{ij}}(x_m, q_{\beta}^{(\gamma)}) \neq 0 \quad \text{for } s_{\beta} = 0, 1, \ldots, l_{\beta} - 1; \]
\[ (-1)^{s_{\beta}}[h_{\beta}G^{(\beta)}_{\partial\Omega_{ij}}(x_m, t_{m-}) - h_{\beta}G^{(\gamma)}_{\partial\Omega_{ij}}(x_m, q_{\beta}^{(\gamma)})] < 0. \] (200)

Proof. The proof of theorem is completed from the definition. ■

Theorem 14. For a discontinuous dynamical system in Eq. (17), there is a point \( x^{(0)}(t_m) \equiv x_m \in \partial\Omega_{ij} \) at time \( t_m \) between two adjacent domains \( \Omega_{ij} \) (\( \alpha = i, j \)). For \( x_m \in S \subseteq \partial\Omega_{ij} \), there is a flow barrier \( F^{(\alpha,0)}(x^{(\alpha)}, t, \pi, q^{(\alpha)}) \) at \( q^{(\alpha)} = [q_1^{(\alpha)}, q_2^{(\alpha)}] \) on the \( \alpha \)-side of the boundary \( \partial\Omega_{ij} \) with the \( G \)-function
\[ G^{s_{\alpha},0}_{\partial\Omega_{ij}}(x_m, q_{\alpha}) = 0 \quad \text{for } s_{\alpha} = 0, 1, 2, \ldots, 2k_{\alpha} - 1; \]
\[ h_{\alpha}G^{(\alpha)}_{\partial\Omega_{ij}}(x_m, q_{\alpha}) \in \{ h_{\alpha}G^{(\alpha)}_{\partial\Omega_{ij}}(x_m, q_{\alpha}^{(0)}), h_{\alpha}G^{(\gamma)}_{\partial\Omega_{ij}}(x_m, q_{\gamma}^{(\gamma)}) \} \subset [0, +\infty); \]
\[ G^{s_{\beta},0}_{\partial\Omega_{ij}}(x_m, q_{\beta}) = 0 \quad \text{for } s_{\beta} = 0, 1, 2, \ldots, 2k_{\beta} - 1; \]
\[ h_{\beta}G^{(\beta)}_{\partial\Omega_{ij}}(x_m, q_{\beta}) \in \{ h_{\beta}G^{(\beta)}_{\partial\Omega_{ij}}(x_m, q_{\beta}^{(0)}), h_{\beta}G^{(\gamma)}_{\partial\Omega_{ij}}(x_m, q_{\gamma}^{(\gamma)}) \} \subset (-\infty, 0). \] (201)

and also there is a flow barrier \( F^{(\beta,0)}(x^{(\beta)}, t, \pi, q^{(\beta)}) \) at \( q^{(\beta)} = [q_1^{(\beta)}, q_2^{(\beta)}] \) on the \( \beta \)-side of the boundary \( \partial\Omega_{ij} \) with the \( G \)-function
\[ G^{s_{\beta},0}_{\partial\Omega_{ij}}(x_m, q_{\beta}) = 0 \quad \text{for } s_{\beta} = 0, 1, 2, \ldots, 2k_{\beta} - 1; \]
\[ h_{\beta}G^{(\beta)}_{\partial\Omega_{ij}}(x_m, q_{\beta}) \in \{ h_{\beta}G^{(\beta)}_{\partial\Omega_{ij}}(x_m, q_{\beta}^{(0)}), h_{\beta}G^{(\gamma)}_{\partial\Omega_{ij}}(x_m, q_{\gamma}^{(\gamma)}) \} \subset (-\infty, 0). \] (202)

The coming flows in the \((2k_{\alpha} - 2k_{\beta})\)-sink flow satisfy
\[ G^{s_{\alpha},0}_{\partial\Omega_{ij}}(x_m, t_{m-}) = 0 \quad \text{for } s_{\alpha} = 0, 1, 2, \ldots, 2k_{\alpha} - 1; \]
\[ h_{\alpha}G^{(\alpha)}_{\partial\Omega_{ij}}(x_m, t_{m-}) = 0 \quad \text{for } s_{\alpha} = 0, 1, 2, \ldots, 2k_{\alpha} - 1; \]
\[ h_{\alpha}G^{(\alpha)}_{\partial\Omega_{ij}}(x_m, t_{m-}) > 0 \quad \text{and } h_{\alpha}G^{(\alpha)}_{\partial\Omega_{ij}}(x_m, t_{m-}) < 0. \] (203)

A coming flow of \( x^{(\alpha)} \) switches to the boundary flow of \( x^{(0)} \) to form a \((2k_{\alpha} - 2k_{\beta})\)-sink flow on the boundary \( \partial\Omega_{ij} \) if and only if for \( s_{\alpha}, s_{\beta} \in \{1, 2\} \)

either
\[ h_{\alpha}G^{(\alpha)}_{\partial\Omega_{ij}}(x_m, t_{m-}) \notin \{ h_{\alpha}G^{(\alpha)}_{\partial\Omega_{ij}}(x_m, q_{\alpha}^{(\alpha)}), h_{\alpha}G^{(\gamma)}_{\partial\Omega_{ij}}(x_m, q_{\gamma}^{(\gamma)}) \}; \]

or
\[ G^{s_{\alpha},0}_{\partial\Omega_{ij}}(x_m, t_{m-}) = G^{s_{\alpha},0}_{\partial\Omega_{ij}}(x_m, q_{\alpha}^{(\alpha)}) \neq 0 \quad \text{for } s_{\alpha} = 2k_{\alpha}, 2k_{\alpha} + 1, \ldots, l_{\alpha} - 1; \]
\[ (-1)^{s_{\alpha}}[h_{\alpha}G^{(\alpha)}_{\partial\Omega_{ij}}(x_m, t_{m-}) - h_{\alpha}G^{(\alpha)}_{\partial\Omega_{ij}}(x_m, q_{\alpha}^{(\alpha)})] > 0. \] (204)

\[ h_{\beta}G^{(\beta)}_{\partial\Omega_{ij}}(x_m, t_{m-}) \notin \{ h_{\beta}G^{(\beta)}_{\partial\Omega_{ij}}(x_m, q_{\beta}^{(\beta)}), h_{\beta}G^{(\gamma)}_{\partial\Omega_{ij}}(x_m, q_{\gamma}^{(\gamma)}) \}; \]

or
\[ G^{s_{\beta},0}_{\partial\Omega_{ij}}(x_m, t_{m-}) = G^{s_{\beta},0}_{\partial\Omega_{ij}}(x_m, q_{\beta}^{(\beta)}) \neq 0 \quad \text{for } s_{\beta} = 2k_{\beta}, 2k_{\beta} + 1, \ldots, l_{\beta} - 1; \]
\[ (-1)^{s_{\beta}}[h_{\beta}G^{(\beta)}_{\partial\Omega_{ij}}(x_m, t_{m-}) - h_{\beta}G^{(\beta)}_{\partial\Omega_{ij}}(x_m, q_{\beta}^{(\beta)})] < 0. \] (205)
Proof. The proof of Theorem is completed from the definition of higher-order singular sink flow barriers.

Once the sink flow is formed, the vanishing of the sink flow on the boundary is of great interest herein. When the boundary flow in the sink flow disappears from the boundary and enters the α-domain, the corresponding conditions are from [Luo, 2008a, 2008b]. However, the following will give the vanishing condition for the sink flow on the boundary with flow barriers. Theorem 15. For a discontinuous dynamical system in Eq. (17), there is a point \( x^{(i)}(t_m) \equiv x_m \in \partial \Omega_i \) at time \( t_m \) between two adjacent domains \( \Omega_i \) (\( \alpha = i,j \)). Suppose the boundary flow in the sink flow on the boundary is formed under certain conditions. There is a boundary flow barrier of \( F^{(i,j)}(x^{(i)}, t, \pi_{\alpha}, q^{(i)}) \) at \( q^{(i)} \in [q_i^{(i)}, q_2^{(i)}] \) on the α-side of the boundary \( \partial \Omega_i \) for \( x_m \in S \subseteq \partial \Omega_i \), with the G-function

\[
0 \in [h_{\alpha}G^{(i,j)}(x_m, q_1^{(i)}), h_{\alpha}G^{(i,j)}(x_m, q_1^{(i)})] < R
\] (206)

and also there is a boundary flow barrier of \( F^{(i,j)}(x^{(i)}, t, \pi_{\beta}, q^{(i)}) \) at \( q^{(i)} \in [q_i^{(i)}, q_2^{(i)}] \) with G-function

\[
0 \in [h_{\beta}G^{(i,j)}(x_m, q_{1}^{(i)}), h_{\beta}G^{(i,j)}(x_m, q_2^{(i)})] < R
\] (207)

on the β-side of the boundary \( \partial \Omega_i \) (\( \alpha, \beta \in \{i,j\} \) and \( \alpha \neq \beta \)). The boundary flow of \( x^{(i)} \) disappears on the α-side if and only if

\[
\text{Proof. This theorem can be proved from the boundary flow barrier in the source flow.} \]

Theorem 16. For a discontinuous dynamical system in Eq. (17), there is a point \( x^{(i)}(t_m) \equiv x_m \in \partial \Omega_i \) at time \( t_m \) between two adjacent domains \( \Omega_i \) (\( \alpha = i,j \)). Suppose the boundary flow in the (2k, 2l, 2m)-sink flow on the boundary is formed under certain conditions. There is a boundary flow barrier \( F^{(i,j)}(x^{(i)}, t, \pi_{\alpha}, q^{(i)}) \) at \( q^{(i)} \in [q_i^{(i)}, q_2^{(i)}] \) with G-function

\[
G^{(i,j)}(x_m, q_1^{(i)}) = 0 \quad \text{for } s_{\alpha} = 0, 1, 2, \ldots, m_{\alpha} - 1
\] (210)

\[
0 \in [h_{\alpha}G^{(i,j)}(x_m, q_1^{(i)}), h_{\alpha}G^{(i,j)}(x_m, q_1^{(i)})] < R
\]

for the leaving flow of \( x^{(i)} \) on the α-side and also there is a flow barrier \( F^{(i,j)}(x^{(i)}, t, \pi_{\beta}, q^{(i)}) \) at

\[
q^{(i)} \in [q_i^{(i)}, q_2^{(i)}] \text{ with G-function}
\]

\[
G^{(i,j)}(x_m, q_i^{(i)}) = 0 \quad \text{for } s_{\beta} = 0, 1, 2, \ldots, m_{\beta} - 1
\] (211)

\[
0 \in [h_{\beta}G^{(i,j)}(x_m, q_1^{(i)}), h_{\beta}G^{(i,j)}(x_m, q_2^{(i)})] < R
\]

for the leaving flow of \( x^{(i)} \) on the β-side (\( \alpha, \beta \in \{i,j\} \) and \( \alpha \neq \beta \)),

\[
G^{(i,j)}(x_m, q_1^{(i)}) = 0 \quad \text{for } s_{\alpha} = 0, 1, 2, \ldots, m_{\alpha} - 1;
\] (212)

\[
0 \in [h_{\alpha}G^{(i,j)}(x_m, q_1^{(i)}), h_{\alpha}G^{(i,j)}(x_m, q_2^{(i)})] < R
\]

for the leaving flow of \( x^{(i)} \) on the α-side and also there is a flow barrier \( F^{(i,j)}(x^{(i)}, t, \pi_{\alpha}, q^{(i)}) \) at

\[
q^{(i)} \in [q_i^{(i)}, q_2^{(i)}] \text{ with G-function}
\]

\[
G^{(i,j)}(x_m, q_i^{(i)}) = 0 \quad \text{for } s_{\beta} = 0, 1, 2, \ldots, m_{\beta} - 1;
\] (213)

\[
0 \in [h_{\beta}G^{(i,j)}(x_m, q_1^{(i)}), h_{\beta}G^{(i,j)}(x_m, q_2^{(i)})] < R
\]

for the leaving flow of \( x^{(i)} \) on the β-side (\( \alpha, \beta \in \{i,j\} \) and \( \alpha \neq \beta \)).
The boundary flow of \( \alpha^{(0)} \) disappears on the \( \alpha \)-side if and only if

\[
\begin{align*}
\text{both} & \quad h_{a} G_{\alpha \beta 1}^{(m,a)}(x_{m}, m_{a} + 1, \ldots, m_{a} - 1) < 0 \\
\text{or} & \quad h_{a} G_{\alpha \beta 1}^{(m,a)}(x_{m}, q_{1}^{a}) = 0 \quad \text{for } s_{a} = m_{a}, m_{a} + 1, \ldots, m_{a} - 1.
\end{align*}
\]

(213)

either \( h_{a} G_{\alpha \beta 1}^{(m,a)}(x_{m}, q_{1}^{a}) > 0 \) but \( h_{a} G_{\alpha \beta 1}^{(m,a)}(x_{m}, t_{m+}) < 0 \)

or

\[
\begin{align*}
\text{for } s_{3} = m_{3}, m_{3} + 1, \ldots, m_{3} - 1 & \quad \text{and} \quad h_{a} G_{\alpha \beta 1}^{(m,a)}(x_{m}, q_{1}^{a}) > 0
\end{align*}
\]

(214)

on the \( \beta \)-side.

Proof. This theorem can be proved from the theorems for the boundary flow barriers in the source flows with higher-order singularity.

8. An Application

As in [Luo, 2006, 2007], consider a periodically-forced, friction-induced oscillator in Fig. 29(a). The dynamic system consists of a mass \( m \), a spring of stiffness \( k \), and a damper of viscous damping coefficient \( r \). The moving mass rests on the horizontal belt surface traveling with a constant speed \( V \). The coordinate system \((x, t)\) is absolute with displacement \( x \) and time \( t \). The periodic excitation force \( Q_{0} \cos \Omega t \) is exerted on the mass, where \( Q_{0} \) and \( \Omega \) are the excitation strength and frequency, respectively. The nonlinear friction is approximated by a piecewise linear model, as shown in Fig. 29(b).

\[
\begin{align*}
\mathcal{F}(\dot{x}) & \in \left[ -\mu_{s} N_{0}, \mu_{s} N_{0} \right], \\
& = \begin{cases} \\
& \mu_{s}(\dot{x} - V) - F_{N} N_{0}, \quad \dot{x} \in (V, V_{2}) \\
& -\mu_{s}(\dot{x} - V) - F_{N} N_{0}, \quad \dot{x} \in (V_{2}, V) \\
& \mu_{s}(\dot{x} - V_{1}) - \mu_{k}(V_{2} - V) - F_{N} N_{0}, \quad \dot{x} \in (V_{1}, \infty) \\
& -\mu_{s}(\dot{x} - V_{1}) + F_{N} N_{0}, \quad \dot{x} \in (-\infty, V)
\end{cases}
\end{align*}
\]

(215)

where \( \dot{x} \equiv dx/dt \). The parameters \( (\mu_{s}, \mu_{k}, \text{ and } F_{N}) \) are static and kinetic friction coefficients and a normal force to the contact surface, respectively. The coefficients \( \mu_{j} \) \((j = 1, 2, 3, 4)\) are the slope for friction force with velocity. For this problem, the normal force is \( F_{N} = mg \) where \( g \) is the gravitational acceleration. The static friction force is in the interval of \([-\mu_{s} N_{0}, \mu_{s} N_{0}]\). The amplitude of the static friction force is \( \mu_{s} N_{0} \). The dynamic friction forces just for the beginning of the relative motion are \( \pm \mu_{k} F_{N} \). Two boundaries for the piecewise continu-ity of the friction force are at \( \dot{x} = V \) and \( \dot{x} = V_{1} \).

The third boundary is at \( \dot{x} = V \). For this boundary, the dynamical friction force for the passable motion is discontinuous, which can be referred into [Luo, 2006] (also see [Luo & Gegg, 2006a, 2006b]).

Once the mass and the translation belt stick together, the relative motion does not exist between the mass and the belt. Only when the nonfriction force is greater than the static friction force, the relative motion between the mass and belt can start. Three separation boundaries give four velocity regions on which the vector fields are different. In the aforementioned model, the normal force \( F_{N} \) can be exerted externally and arbitrarily instead of \( F_{N} = mg \). In addition, the nonfriction forces per unit mass in the \( x \)-direction is determined by

\[
F_{s} = A_{0} \cos \omega t - 2m - c \dot{x}, \quad \text{for } \dot{x} = V
\]

(216)

where \( A_{0} = Q_{0} / m d = r / 2 m \) and \( c = k / m \). For stick motions, the nonfriction force per unit mass
is less than the static friction force amplitude per unit mass $F_{fs}$ (i.e. $|F_s| \leq F_{fs}$ and $F_{fs} = \mu_s F_N/m$).

The mass does not have any relative motion to the belt. Therefore, no acceleration exists because the belt speed is constant, i.e.

$$\ddot{x} = 0, \quad \text{for } \dot{x} = V. \quad \text{(217)}$$

If the nonfriction force per unit mass is greater than the static friction force per unit mass (i.e. $|F_s| > F_{fs}$), the nonstick motion occurs. For the nonstick motion, the total force per unit mass is

$$F = A_0 \cos \Omega t - F_f \text{sgn}(\dot{x} - V) - 2d\dot{x} - cx,$$

for $\dot{x} \neq V; \quad \text{(218)}$

where the friction force per unit mass $F_f = F_f/m$ for $x \neq V$. Therefore, the equation of the nonstick motion for such a dynamical system with a piecewise linear friction is

$$\ddot{x} + 2d\dot{x} + cx = A_0 \cos \Omega t - F_f \text{sgn}(\dot{x} - V),$$

for $\dot{x} \neq V. \quad \text{(219)}$

8.1. Switchability conditions

For simplicity, the vectors for the flow and the corresponding vector field in such a system are introduced by

$$x \triangleq (x, \dot{x})^T \equiv (x, y)^T \quad \text{and} \quad F \triangleq (y, F)^T. \quad \text{(220)}$$

The discontinuities in this dynamical system are caused by the jumping from static to dynamic friction forces and piecewise linear dynamical friction model. As discussed before, there are four velocity regions caused by three velocity boundaries. Therefore, the phase space can be partitioned into four subdomains by the three velocity boundaries. Such a phase space partition is sketched in Fig. 30. Among the three velocity boundaries, the friction force jumping is as a main discontinuity at $\dot{x} = V$. So the naming of the subdomains in phase space starts from the domain near the main discontinuous boundary $\dot{x} = V$. In fact, the subdomains can be named arbitrarily. From the direction of trajectories of mass motion in phase space, the corresponding boundaries are also named, as shown in Fig. 30(a). The boundary with the friction force jumping is represented by a dotted line. The remaining boundaries are depicted by two dashed lines, respectively. The named domains and the
oriented boundaries are expressed by
\[ \Omega_1 = \{(x,y) | y \in (V_1)\}, \]
\[ \Omega_2 = \{(x,y) | y \in (V_1,\infty)\}, \]
\[ \Omega_3 = \{(x,y) | y \in (V_2,\infty)\}, \]
\[ \Omega_4 = \{(x,y) | y \in (-\infty, V_2)\}; \]
\[ \partial \Omega_{13} = \{(x,y) | \varphi_{13}(x,y) = y - V_2 = 0\}, \]
where \( \rho = 1 \) if \( \alpha, \beta \in \{1,2\} \);
\( \rho = 0 \) if \( \alpha, \beta \in \{1,3\} \) and \( \rho = 2 \) if \( \alpha, \beta \in \{2,3\} \). \( V_2 \) is the sliding vector field on the boundary \( \partial \Omega_{13} \).

The boundary flow barriers on the \( \alpha \)-side of the discontinuous force boundary are accessible with a specific vector field. On the boundary \( \partial \Omega_{13} \) or \( \partial \Omega_{34} \), the vector fields are \( C^\infty \)-discontinuous, but on the boundaries \( \partial \Omega_{12} \) and \( \partial \Omega_{34} \), the vector fields are \( C^0 \)-continuous. The vector fields for all the subdomains are sketched in Fig. 30(b) and are labeled by \( F^{(0)} \) for

\[
F^{(0)}_f(x,t) = \nu_1(y-V_1) - \nu_2(V_1-V) + F_Nv_k, \quad y \in (V_1,\infty)
\]
\[
F^{(0)}_f(x,t) = -\nu_2(y-V) + F_Nv_k, \quad y \in (V_2,V_1)
\]
\[
F^{(0)}_f(x,t) = -\nu_1(y-V) - \nu_2(V_2-V) - F_Nv_k, \quad y \in (-\infty,V_2)
\]
\[
F^{(0)}_f(x,t) = \nu_1(y-V_2) - \nu_2(V_2-V) - F_Nv_k, \quad y \in (V_2,\infty)
\]

where \( \nu_1 = \mu_1/m \) \( (i = 1,\ldots,4) \) and \( v_k = \mu_k/m \) are the slope coefficients of friction forces and dynamic friction coefficient per unit mass. The two force boundaries relative to \( V_{i2} \) (i.e. \( y = V_1 \) or \( V_2 \)) are \( C^0 \)-continuous. However, the boundary relative to the velocity \( V \) is a discontinuous force boundary.

The time \( t_m \) represents the moment for the motion just on the separation boundary and the time \( t_m = t_m \pm 0 \) reflects the flows in the regions instead of the separation boundary. However, the boundary flow barriers on the boundary \( \partial \Omega_{13} \) are for \( x_m \in \partial \Omega_{13}(\alpha, \beta \in \{1,3\}) \)

\[
\begin{align*}
F^{(0)-}_f(x_m, t_m) &= (y_m, F^{(0)}(x_m, t_m))^T, \\
F^{(0)-}_f(x_m, t_m) &= (y_m, F^{(0)}(x_m, t_m))^T, \\
F^{(0)-}_f(x_m, t_m) &= A_0 \cos \Omega_m - F^{(0)}(x_m, t_m, \varphi_m), \\
F^{(0)-}_f(x_m, t_m) &= -2d_0y_m - c_m x_m.
\end{align*}
\]

From Eq. (216), the static friction forces per unit mass on the boundary \( \partial \Omega_{13} \) are

\[
F^{(0)+}_f(x_m, t_m) \in (-\infty, F_Nv_k] \quad \text{and} \quad F^{(0)-}_f(x_m, t_m) \in [-F_Nv_k, +\infty).
\]

The boundary flow barrier on the \( \alpha \)-side of the discontinuous force boundary \( \partial \Omega_{13} \) are for \( x_m \in \partial \Omega_{13}(\alpha, \beta \in \{1,3\}) \)

\[
\begin{align*}
F^{(0)-}_f(x_m, q_m) &= (y_m, F^{(0)-}_f(x_m, q_m))^T, \\
F^{(0)-}_f(x_m, q_m) &= A_0 \cos \Omega_m - F^{(0)}(x_m, q_m), \\
F^{(0)-}_f(x_m, q_m) &= -2d_0y_m - c_m x_m.
\end{align*}
\]

Before discussion of the analytical conditions, the \( G \)-function can be reduced for the special boundary. The normal vector of the boundary \( \partial \Omega_{13} \) is

\[
\mathbf{n}_{\partial \Omega_{13}} = \nabla \varphi_{13} = (\partial_x \varphi_{13}, \partial_y \varphi_{13})^T |_{(x_m, t_m)}.
\]

where \( \nabla = (\partial_x, \partial_y)^T \) is the Hamilton operator. From Eq. (222), the boundaries are straight lines.
in phase space, which imply that the normal vectors are constant vectors. Furthermore, one obtains \(D\Omega_{\alpha\beta} = 0\). Thus

\[
\begin{align*}
G^{(\alpha)}_{\Omega_{\alpha\beta}}(x^{(\alpha)}, t) &= n^{T}_{\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\alpha)}(x^{(\alpha)}, t), \\
G^{(\alpha)}_{\Omega_{\alpha\beta}}(x^{(\alpha)}, t) &= n^{T}_{\Omega_{\alpha\beta}} \cdot \mathbf{D}\mathbf{F}^{(\alpha)}(x^{(\alpha)}, t); \\
G^{(0,0)}_{\Omega_{\alpha\beta}}(x^{(\alpha)}, t) &= n^{T}_{\Omega_{\alpha\beta}} \cdot \mathbf{E}^{(0,0)}(x^{(\alpha)}, t), \\
G^{(1,0-\alpha)}_{\Omega_{\alpha\beta}}(x^{(\alpha)}, t) &= n^{T}_{\Omega_{\alpha\beta}} \cdot \mathbf{D}\mathbf{F}^{(1,0-\alpha)}(x^{(\alpha)}, t),
\end{align*}
\] (232)

where

\[
\begin{align*}
\mathbf{D}\mathbf{F}^{(\alpha)}(x, t) &= (\mathbf{F}^{(\alpha)}(x, t), \mathbf{D}\mathbf{F}^{(\alpha)}(x, t))^T, \\
\mathbf{D}\mathbf{F}^{(0,0)}(x, t) &= \nabla \mathbf{F}^{(0,0)}(x, t) \cdot \mathbf{F}^{(0,0)}(x, t) \\
&+ \partial_t \mathbf{F}^{(0,0)}(x, t); \\
\mathbf{D}\mathbf{F}^{(0,0)}(x, t) &= (\mathbf{F}^{(0,0)}(x, t), \mathbf{D}\mathbf{F}^{(0,0)}(x, t))^T, \\
\mathbf{D}\mathbf{F}^{(0,0)}(x, t) &= \nabla \mathbf{F}^{(0,0)}(x, t) \cdot \mathbf{F}^{(0,0)}(x, t) \\
&+ \partial_t \mathbf{F}^{(0,0)}(x, t).
\end{align*}
\] (233)

Note that \(\partial_t(\cdot) = \partial(\cdot)/\partial t\). From [Luo, 2005, 2006], the existence condition of the stick motion (or sliding flow in mathematics) between oscillator and the translation belt on the boundary \(\partial\Omega_{13}\) is

\[
\begin{align*}
\mathbf{h}_{\alpha} G^{(\alpha)}_{\Omega_{\alpha\beta}}(x_m, t_{m-}) &= \mathbf{n}^{T}_{\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\alpha)}(x_m, t_{m-}) > 0; \\
\mathbf{h}_{\alpha} G^{(\beta)}_{\Omega_{\alpha\beta}}(x_m, t_{m-}) &= \mathbf{n}^{T}_{\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\beta)}(x_m, t_{m-}) < 0.
\end{align*}
\] (234)

From [Luo, 2008a, 2008b], the necessary and sufficient conditions for the nonstick motion (or passable motion) are on the boundary \(\partial\Omega_{13}\)

\[
\begin{align*}
\mathbf{h}_{\alpha} G^{(\alpha)}_{\Omega_{\alpha\beta}}(x_m, t_{m-}) &= \mathbf{n}^{T}_{\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\alpha)}(x_m, t_{m-}) > 0, \\
\mathbf{h}_{\alpha} G^{(\beta)}_{\Omega_{\alpha\beta}}(x_m, t_{m-}) &= \mathbf{n}^{T}_{\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\beta)}(x_m, t_{m-}) > 0.
\end{align*}
\] (235)

The foregoing equations give the necessary and sufficient conditions for the motion without sliding. It indicates that the friction oscillator on the boundary will not stick with the translation belt together. From [Luo, 2008a, 2008b], the switching bifurcation from the nonstick motion to the stick motion is

\[
\begin{align*}
\mathbf{h}_{\alpha} G^{(\alpha)}_{\Omega_{\alpha\beta}}(x_m, t_{m-}) &= \mathbf{n}^{T}_{\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\alpha)}(x_m, t_{m-}) > 0; \\
\mathbf{h}_{\alpha} G^{(\beta)}_{\Omega_{\alpha\beta}}(x_m, t_{m-}) &= \mathbf{n}^{T}_{\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\beta)}(x_m, t_{m-}) = 0, \\
\mathbf{h}_{\alpha} G^{(1,0)}_{\Omega_{\alpha\beta}}(x_m, t_{m-}) &= \mathbf{n}^{T}_{\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(1,0)}(x_m, t_{m-}) < 0.
\end{align*}
\] (236)

Once the stick motion (or the sink flow) is formed under Eq. (234), the boundary flow will control the motion on the boundary, which are independent of the vector fields except for the conditions in Eq. (234). For this problem, the boundary flow on the boundary possesses a boundary flow barrier caused by the static friction force. To obtain a new nonstick motion on the belt, the nonfriction force must be greater than the static friction force. From Theorem 13, the necessary and sufficient conditions for vanishing the sink flow (or sliding flow) on the \(\alpha\)-side are

\[
\begin{align*}
\text{both } \mathbf{h}_{\alpha} G^{(\alpha)}_{\Omega_{\alpha\beta}}(x_m, t_{m-}) &= \mathbf{n}^{T}_{\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\alpha)}(x_m, t_{m-}) < 0, \\
\text{and } \mathbf{h}_{\alpha} G^{(1,0-\alpha)}_{\Omega_{\alpha\beta}}(x_m, q_{\alpha}^{(\alpha)}) &= \mathbf{n}^{T}_{\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(1,0-\alpha)}(x_m, q_{\alpha}^{(\alpha)}) = 0, \\
\text{with } \mathbf{h}_{\alpha} G^{(1,0-\alpha)}_{\Omega_{\alpha\beta}}(x_m, q_{\alpha}^{(\alpha)}) &= \mathbf{n}^{T}_{\Omega_{\alpha\beta}} \cdot \mathbf{D}\mathbf{F}^{(1,0-\alpha)}(x_m, q_{\alpha}^{(\alpha)}) < 0;
\end{align*}
\] (257)

\[
\begin{align*}
\text{either } \mathbf{h}_{\alpha} G^{(0,0)}_{\Omega_{\alpha\beta}}(x_m, q_{\beta}^{(\beta)}) &= \mathbf{n}^{T}_{\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(0,0)}(x_m, q_{\beta}^{(\beta)}) > 0, \\
\text{but } \mathbf{h}_{\alpha} G^{(\beta)}_{\Omega_{\alpha\beta}}(x_m, t_{m-}) &= \mathbf{n}^{T}_{\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(\beta)}(x_m, t_{m-}) < 0, \\
\text{or } \mathbf{h}_{\alpha} G^{(0,0)}_{\Omega_{\alpha\beta}}(x_m, q_{\beta}^{(\beta)}) &= \mathbf{n}^{T}_{\Omega_{\alpha\beta}} \cdot \mathbf{F}^{(0,0)}(x_m, q_{\beta}^{(\beta)}) < 0 \\
\text{with } \mathbf{h}_{\alpha} G^{(1,0-\alpha)}_{\Omega_{\alpha\beta}}(x_m, q_{\beta}^{(\beta)}) &= \mathbf{n}^{T}_{\Omega_{\alpha\beta}} \cdot \mathbf{D}\mathbf{F}^{(1,0-\alpha)}(x_m, q_{\beta}^{(\beta)}) < 0.
\end{align*}
\] (238)
From [Luo, 2008a, 2008b], the necessary and sufficient conditions for grazing motion to the boundary in Eq. (222) are

\[
\begin{align*}
G^{(a)}_{\partial\Omega_{13}}(x_m, t_{m+}) = n^T_{\partial\Omega_{13}} \cdot F^{(a)}(x_m, t_{m+}) = 0 \quad &\text{for } a \neq \beta; \\
G^{(1,a)}_{\partial\Omega_{13}}(x_m, t_{m+}) = n^T_{\partial\Omega_{13}} \cdot DF^{(a)}(x_m, t_{m+}) > 0 \\
&\text{for } a = 2, 3, 4 \quad &\text{on } \partial\Omega_{13} \in \{\partial\Omega_{12}, \partial\Omega_{13}, \text{ and } \partial\Omega_{14}\}; \\
G^{(1,a)}_{\partial\Omega_{13}}(x_m, t_{m+}) = n^T_{\partial\Omega_{13}} \cdot DF^{(a)}(x_m, t_{m+}) < 0 \\
&\text{for } a = 2, 3, 4 \quad &\text{on } \partial\Omega_{13} \in \{\partial\Omega_{12}, \partial\Omega_{13}, \text{ and } \partial\Omega_{14}\}. 
\end{align*}
\]  

Using Eq. (231), the normal vector of boundary \(\partial\Omega_{13}\) with \(\alpha, \beta \in \{1, 2, 3, 4\}\) is

\[
n_{\partial\Omega_{13}} = n_{\partial\Omega_{13}} = (0, 1)^T.
\]

The normal vectors of the boundaries \((\partial\Omega_{12} \text{ and } \partial\Omega_{13}), \partial\Omega_{13} \text{ and } \partial\Omega_{14}\) and \((\partial\Omega_{14} \text{ and } \partial\Omega_{13})\) point to the domains \(\Omega_2, \Omega_1\) and \(\Omega_2\), respectively. Therefore, for \(\alpha \in \{i, j\}\), one obtains

\[
\begin{align*}
&n^T_{\partial\Omega_{13}} \cdot F^{(a)}(x, t) = F^{(a)}(x, t); \\
&n^T_{\partial\Omega_{13}} \cdot DF^{(a)}(x, t) = DF^{(a)}(x, t); \\
&n^T_{\partial\Omega_{13}} \cdot F^{(0,3a)}(x, t, q^{(a)}) = F^{(0,3a)}(x, t, q^{(a)}); \\
&n^T_{\partial\Omega_{13}} \cdot DF^{(0,3a)}(x, t, q^{(a)}) = DF^{(0,3a)}(x, t, q^{(a)}).
\end{align*}
\]

With Eq. (240), the conditions in Eqs. (234) and (235) to form sink and passable motions to the boundary give the force conditions as:

\[
\begin{align*}
F^{(1)}(x_m, t_{m-}) < 0 \quad &\text{and } F^{(3)}(x_m, t_{m-}) > 0 \quad &\text{on } \partial\Omega_{13}; \\
F^{(1)}(x_m, t_{m-}) < 0 \quad &\text{and } F^{(3)}(x_m, t_{m-}) < 0 \quad &\text{for } \Omega_3 \rightarrow \Omega_2. \\
F^{(1)}(x_m, t_{m+}) > 0 \quad &\text{and } F^{(3)}(x_m, t_{m+}) > 0 \quad &\text{for } \Omega_1 \rightarrow \Omega_2.
\end{align*}
\]

The force condition for onset of the sink motion on \(\partial\Omega_{13}\) is

\[
\begin{align*}
F^{(1)}(x_m, t_{m-}) < 0, \\
F^{(3)}(x_m, t_{m+}) = 0 \quad &\text{with } DF^{(1)}(x_m, t_{m+}) < 0 \quad &\text{for } \Omega_1 \rightarrow \partial\Omega_{13}, \\
F^{(1)}(x_m, t_{m-}) > 0, \\
F^{(3)}(x_m, t_{m+}) = 0 \quad &\text{with } DF^{(1)}(x_m, t_{m+}) > 0 \quad &\text{for } \Omega_3 \rightarrow \partial\Omega_{13}.
\end{align*}
\]

Owing to the flow barrier, the force conditions for vanishing of the sink motion are either

\[
\begin{align*}
F^{(0,3\alpha)}(x_m, q^{(1)}) > 0 \quad &\text{but } F^{(1)}(x_m, t_{m+}) < 0 \\
or \quad &F^{(0,3\alpha)}(x_m, q^{(1)}) < 0
\end{align*}
\]

or

\[
\begin{align*}
F^{(0,3\alpha)}(x_m, q^{(1)}) = 0 \quad &\text{with } DF^{(0,3\alpha)}(x_m, q^{(1)}) < 0; \\
F^{(1)}(x_m, t_{m+}) < 0, \\
F^{(0,3\alpha)}(x_m, q^{(3)}) = 0 \quad &\text{with } DF^{(0,3\alpha)}(x_m, q^{(3)}) < 0
\end{align*}
\]
The force conditions for grazing motions are

\[
\begin{align*}
&\text{either } F^{(0-\alpha)}(x_m, q_1^{(0)}) < 0 \text{ but } F^{(3)}(x_m, t_{m-}) > 0 \\
&\text{or } F^{(0-\alpha)}(x_m, q_1^{(0)}) > 0 \\
&\text{or } F^{(0-\alpha)}(x_m, q_1^{(0)}) = 0 \text{ with } D F^{(0-\alpha)}(x_m, q_1^{(3)}) > 0; \\
&F^{(3)}(x_m, t_{m+}) > 0, \\
&F^{(0-\alpha)}(x_m, q_1^{(1)}) = 0 \text{ with } D F^{(0-\alpha)}(x_m, q_1^{(1)}) > 0
\end{align*}
\]

from \(\partial \Omega_{13} \to \Omega_3\), and

The force conditions for passable motions on the boundary \(\partial \Omega_{m\beta}\) are

\[
\begin{align*}
F^{(\alpha)}(x_m, t_{m-}) &< 0 \quad \text{and} \quad F^{(\beta)}(x_m, t_{m+}) < 0 \quad \text{for } (\alpha, \beta) \in \{(2,1), (1,3), (3,4)\}; \\
F^{(\alpha)}(x_m, t_{m-}) &> 0 \quad \text{and} \quad F^{(\beta)}(x_m, t_{m+}) > 0 \quad \text{for } (\alpha, \beta) \in \{(1,2), (3,1), (4,3)\}.
\end{align*}
\]

The force conditions for grazing motions are

\[
\begin{align*}
F^{(\alpha)}(x_m, \Omega_{m\beta}) &= 0 \\
DF^{(\alpha)}(x_m, t_{m\beta}) &> 0 \quad \text{for } \alpha = 2, 1, 3 \quad \text{at } \partial \Omega_{m\beta} \in \{\partial \Omega_{21}, \partial \Omega_{13}\}, \\
DF^{(\alpha)}(x_m, t_{m+}) &< 0 \quad \text{for } \alpha = 1, 3, 4 \quad \text{at } \partial \Omega_{m\beta} \in \{\partial \Omega_{12}, \partial \Omega_{31}\}.
\end{align*}
\]

8.2. Illustrations

To illustrate the motion with flow barriers in non-smooth dynamic systems, the basic mappings are introduced as in [Luo & Zwiegart, 2008]. The mappings are determined by the close-form solution of the differential equation in the corresponding domain. With an initial condition \((t_4, x_k, V)\), the direct integration of Eq. (217) yields

\[
x = V \times (t - t_4) + x_k.
\]

Substitution of Eq. (250) into (225) produces the forces for the very small-neighborhood of the stick motion in the domains \(\Omega_j (j \in \{1, 3\})\). Because of the static friction jumping, the forces at \((x_m, t_m)\) for the coming and leaving flows and the boundary flow barriers on the boundary \(\partial \Omega_{13}\) are:

\[
\begin{align*}
F^{(j)}(x_m, t_{m\beta}) &= -2d_j V - c_j x_m \\
&\quad + A_0 \cos \Omega_{m\beta} - a_j, \\
F^{(0-\alpha)}(x_m, q_1^{(j)}) &= -2d_j V - c_j x_m \\
&\quad + A_0 \cos \Omega_{m\beta} - a_j^{(0-\alpha)},
\end{align*}
\]

where \(a_1 = -a_3 = v_3 F_N\) and \(a_1^{(0-\alpha)} = -a_3^{(0-\alpha)} = v_3 F_N\).

Fig. 31. Regular and stick mappings: (a) local and stick mappings and (b) global mappings.
To label the motion in this discontinuous dynamical system, the generic mappings are introduced. The switching sets on the boundary should be numbered first. The switching set for the discontinuous force boundary is represented by \( \Omega_1 \), and the other separation boundaries are \( \Omega_2 \) and \( \Omega_3 \). The switching sets for the three boundaries are

\[
\Sigma_a = \Sigma_a^0 \cup \Sigma_a^+ \cup \Sigma_a^- \quad \text{for } a = 1, 2, 3. \tag{253}
\]

The corresponding, switching subsets are defined as

\[
\Sigma_a^0 = \{(x_k, \Omega_k)|x_k = V_a^0\} \quad \text{and} \quad \Sigma_a^\pm = \{(x_k, \Omega_k)|x_k = V_a^\pm}\; \tag{254}
\]

where \( V_a^\pm = \text{lim}_{\delta \to 0}(V_a \pm \delta) \) for an arbitrarily small \( \delta > 0 \) and \( \rho = \{0, 1, 2\} \) for \( a = 1, 2, 3 \). In phase space, the trajectories in \( \Omega_1 \) starting and ending at the separation boundaries are sketched in Fig. 31. The starting and ending points for mappings \( P_{j\rho,\beta} \) in \( \Omega_1 \) are \((x_k, x_k, t_k)\) on \( \Sigma_0 \) and \((x_{k+1}, x_{k+1}, t_{k+1})\) on \( \Sigma_0 \), respectively. The indices \( j = 1, 2, 3 \) and \( \alpha, \beta = 1, 2, 3 \) are for domains and boundaries, respectively. The stick mapping is \( P_{B_1} \). Thus, the mappings are defined as

\[
P_{11} : \Sigma_1^+ \ni \rho_1, \Sigma_1^- \quad \text{and} \quad P_{B_1} : \Sigma_1^+ \ni \rho_1, \Sigma_1^-.
\]

for the local mappings,

\[
P_{B_1} : \Sigma_1^+ \ni \rho_1, \Sigma_1^- \quad \tag{255}
\]

for the global mappings and

\[
P_{B_1} : \Sigma_1^0 \ni \Omega_1, \Sigma_1^0 \quad \tag{256}
\]

for the stick mapping.

The governing equations of \( P_{B_1} \) for a sink flow to leave for \( \Omega_j \) with \( j \in \{1, 3\} \) are

\[
2d_jV + c_j[V \times (t_{k+1} - t_k) + x_k] - A_0 \cos \Omega_{k+1} + a_j^{(0, 0)} = 0. \tag{258}
\]

is considered first. Using the above conditions, the periodic motions in such an oscillator can be obtained as in [Luo & Ziegert, 2007]. The phase plane, force distributions, and the responses of displacement, velocity and acceleration are presented respectively in Figs. 32(a)–32(l) for the periodic motion of mapping \( P_{B_{13}} \) with \( \Omega = 5 \), \( Q_0 = 70 \) and the initial condition \( (\Omega_k, x_k, x_k) \approx (0.0458, 3.0183, 1.50) \). The responses in \( \Omega_3 \) and \( \Omega_4 \) are depicted through the thin and dark curves, accordingly. The circles are switching points, and the gray filled cycle is the starting point of the periodic motion. The arrows give the direction of the periodic motion. In addition, the corresponding mappings are labeled in plots. In Fig. 32(a), the periodic trajectory in phase plane is clearly shown and the periodic motion does not have any intersection with the boundary \( \partial \Omega_3 \). This periodic motion only intersects with the boundary \( \partial \Omega_4 \). Consider the force in domain \( \Omega_a (a = 3, 4) \) as

\[
F(t) \equiv F(t)(x, t) = -2d_xx - c_1x + A_1 \cos \Omega t + \nu_1(x - V) + \mu_1g.
\]

Therefore, with \( \dot{x}_m = V_2 \), the force conditions on the boundary \( \partial \Omega_4 \) from domain \( \Omega_3 \) to \( \Omega_4 \) are from Eq. (248) at time \( t_m \) and \( t_{m+1} \)
Because $c_3 = c_4$ and $d_3 = d_4$, one obtains $F^{(3)}_\pm = F^{(4)}_\mp$. From Eqs. (261) and (262), the total force on the boundary $\partial\Omega_{43}$ is continuous, but from Eq. (260), the derivative of the forces (i.e., $F^{(4)}_\mp$ and $F^{(4)}_\mp$) is discontinuous. Such force characteristics of the periodic flow can be observed in Figs. 32(b) and 32(c). The forces at the switching points on the boundary $\partial\Omega_{43}$ and $\partial\Omega_{43}$ are labeled by $F^{(3)}_\pm$ and $F^{(4)}_\mp$. The force distributions in domain $\Omega_3$ and $\Omega_4$ are presented through the thin and dark curves, respectively. In addition, such force distributions in domain $\Omega_3$ and $\Omega_4$ are labeled by $F^{(3)}_\pm$ and $F^{(4)}_\mp$. Note that the forces in Figs. 32(b) and 32(c) are the total force acting on the mass instead of the total force per unit mass. The displacement and velocity responses in Figs. 32(d) and 32(e) are very smooth owing to the continuity of the forces at the boundary. However, the nonsmoothness of the acceleration is observed in Fig. 32(f). If the initial point is selected at the another switching point, the mapping structure of the periodic motion becomes $P_{3343} = P_{3343} \circ P_{3343}$. However, the two mapping structures present the same periodic motion except for the different initial conditions.

Consider a stick periodic motion relative to mapping $P_{3343}$ for the excitation frequency ($\Omega = 1$) and excitation amplitude ($Q_0 = 70$) with the initial conditions ($\Omega t, x_0, \dot{x}_0$) $\approx (0.6072, 6.5814, 1.50)$. The other parameters are the same as in the first example. The phase plane, force distributions, and displacement, velocity and acceleration responses for the periodic motion of $P_{3343}$ are shown in Figs. 33(a)–33(f), respectively. In Fig. 33(a), the stick motion in phase plane is
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a straight line along the discontinuous boundary. The trajectory of this periodic motion exists in domains $\Omega_3$ and $\Omega_4$. The force description in domains $\Omega_3$ and $\Omega_4$ is given in Eq. (260). The forces on the switching points on the boundary $\partial \Omega_{34}$ or $\partial \Omega_{43}$ can be determined by Eqs. (261) and (262). Since the friction force on $\partial \Omega_{13}$ is $C^0$-discontinuous, such a force discontinuity causes the existence of the sliding (stick) motion along the boundary $\partial \Omega_{13}$. From Eq. (242), the condition for the stick motion appearing on $\partial \Omega_{13}$ is

\[
\begin{align*}
F^{(1)} &\equiv F^{(1)}(x_m, t_m) = -2d_1 V - c_1 x_m + A_0 \cos \Omega t_m - \mu g < 0, \\
F^{(3)} &\equiv F^{(3)}(x_m, t_m) = -2d_3 V - c_3 x_m + A_0 \cos \Omega t_m + \mu g > 0
\end{align*}
\]

(263)
Because the static and kinetic friction forces are different, the flow barriers exist in this dynamical system. Therefore, once the stick appears between the mass and the translation belt, the stick motion disappearance requires

\[
F^{(0-0)}(x_m, q_{(1)}^1) = -2d_i V - c_i x_m + A_0 \cos \Omega t_m - \mu_s g < 0
\]

\[
F^{(0-0)}(x_m, q_{(3)}^3) = -2d_i V - c_i x_m + A_0 \cos \Omega t_{m+2} + \mu_s g = 0
\]

\[
\Delta F^{(0-0)} = DF^{(0-0)}(x_m, q_{(3)}^3) = -c_i V - A_0 \Omega \sin \Omega t_{m+2} < 0
\]

\[
F^{(0-0)}(x_m, q_{(3)}^3) = -2d_i V - c_i x_m + A_0 \cos \Omega t_m + \mu_s g > 0
\]

\[
\Delta F^{(0-0)} = DF^{(0-0)}(x_m, q_{(1)}^1) = -c_i V - A_0 \Omega \sin \Omega t_{m+2} > 0
\]

For simplicity, \( F^{(0-0)}(x_m, q_{(3)}^3) \) is depicted to observe the force criteria for the disappearance of the stick motion. In Fig. 33(b), the force \( F^{(0)}(\alpha = 1, 3) \) for coming flow is presented by the dashed curves. Since \( F_{(3)} > 0 \) and \( F^{(1)} < 0 \), the stick motion appears on the boundary of the flow barrier is plotted by the thick-solid curves. The stick motion disappears at \( F^{(0-0)}(\alpha = 1, 3) \) < 0, which satisfies the conditions in Eq. (264), and such a condition indicates that the nonfriction force must be greater than the conditions in Eq. (263) are satisfied. The force \( F^{(0-0)}(\alpha = 1, 3) \) of the boundary flow barrier is plotted by the thick-solid curves. The stick motion disappears at \( F^{(0-0)}(\alpha = 1, 3) < 0 \), which satisfies the conditions in Eq. (264), and such a condition indicates that the nonfriction force must be greater than the conditions in Eq. (263) are satisfied. The force \( F^{(0-0)}(\alpha = 1, 3) \) of the boundary flow barrier is plotted by the thick-solid curves. The stick motion disappears at \( F^{(0-0)}(\alpha = 1, 3) < 0 \), which satisfies the conditions in Eq. (264), and such a condition indicates that the nonfriction force must be greater than the conditions in Eq. (263) are satisfied. The force \( F^{(0-0)}(\alpha = 1, 3) \) of the boundary flow barrier is plotted by the thick-solid curves. The stick motion disappears at \( F^{(0-0)}(\alpha = 1, 3) < 0 \), which satisfies the conditions in Eq. (264), and such a condition indicates that the nonfriction force must be greater than the conditions in Eq. (263) are satisfied.

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**Fig. 33.** Periodic responses of mapping \( P_{h_0} \circ P_{h_1} \circ P_{h_2} \circ P_{h_3} \): (a) phase plane, (b) force distribution along displacement, (c) force distribution along velocity, (d) displacement, (e) velocity and (f) acceleration for \( \Omega = 1 \) and \( Q_0 = 70 \) with \( (\Omega_1, x_0, \mu_s) = (0.6672, 6.5814, 1.50) \).
than the static friction force (i.e., flow barriers). Furthermore, the oscillator will relatively oscillate on the moving belt. Once the relative motion starts between the oscillator and the belt, the kinetic friction force will control the motion in domain $\Omega_3$. So the corresponding force jumps from zero to the negative one (i.e., $F^{(3)} < 0$), which is observed in Fig. 33(b). The nonsmoothness of the forces on the boundary $\partial\Omega_{34}$ is also observed in Fig. 33(b) because of the piecewise continuity of the forces at the boundary. The forces at the switching points on the boundary $\partial\Omega_{44}$ and $\partial\Omega_{34}$ are labeled by $F^{(4)}$ and $F^{(3)}$. The force distributions in domains $\Omega_3$ and $\Omega_4$ are presented through the thin and thick curves, respectively. In addition, such force distributions in domains $\Omega_3$ and $\Omega_4$ are labeled...
by $F^{(3)}$ and $F^{(4)}$. The force discontinuity on the boundary $\partial \Omega_3$ between the two domains $\Omega_3$ and $\Omega_4$ is clearly observed. Such force characteristics of stick motion and switching are presented in Fig. 33(c). In Fig. 33(d), the displacement is continuous because the velocity is $C^\infty$-continuous. The nonsmoothness of the velocity response is observed because the force is $C^0$-discontinuous. The stick motion in the velocity response is clearly observed. Because the belt possesses a constant speed, the corresponding acceleration for the stick motion is zero. Therefore, in Fig. 33(f), the acceleration is zero for the stick motion on the boundary $\partial \Omega_3$ and nonsmooth at $\partial \Omega_4$.

The grazing phenomenon is independent of the flow barriers in discontinuous dynamical systems, such phenomena for this problem is identical to the friction oscillator without flow barriers, which can be found in references (e.g. [Luo & Gegg, 2006c, 2007]). The grazing conditions in Eqs. (249) are not employed for numerical illustrations. However, such conditions were employed to determine the parameter map for different periodic motions in [Luo & Zwiegart, 2008]. The grazing bifurcations were also discussed in [Luo, 2006, 2007].

9. Conclusions

In this paper, a theory for flow barriers in discontinuous dynamical systems was presented. The coming and leaving flow barriers in passable flows were discussed. Because the flow barriers in the passable flow exist, the leaving and coming flows cannot reach the boundary flow, and they will slide along the corresponding sides of the boundary in domains. The coming flow barriers to the boundary flow in the sink flow were presented. If a coming flow in the sink flow is blocked by its flow barrier, the coming flow will slide along or stand at the boundary in the corresponding domain. However, once the sink flow is formed, the boundary flow barrier in the source flow may exist. The flow switchability of a boundary flow on the boundary with flow barriers was discussed. Once the boundary flow leaves the boundary flow, the flow barrier for leaving flows in the source flow may exist. Thus, the flow barriers for the leaving flows in the source flow were also presented. A practical discontinuous system with the flow barrier of the boundary flow was presented for a better understanding of the flow barrier theory in the discontinuous dynamical systems. The flow barrier theory in discontinuous dynamical systems is a new theory, which will provide a useful tool for one to design desired dynamical systems to satisfy engineering-oriented complex systems. The flow barrier theory may provide a theoretical base for stabilized dynamical systems in control theory.

References


