Nonlocal impulsive fractional differential inclusions with fractional sectorial operators on Banach spaces

JinRong Wang\textsuperscript{a,b}, Ahmed Gamal Ibrahim\textsuperscript{c}, Michal Fečkan\textsuperscript{d,e,*}

\textsuperscript{a} Department of Mathematics, Guizhou University, Guiyang, Guizhou 550025, PR China
\textsuperscript{b} School of Mathematics and Computer Science, Guizhou Normal College, Guiyang, Guizhou 550018, PR China
\textsuperscript{c} Department of Mathematics, Faculty of Science, King Faisal University, Al-Ahsa 31982, Saudi Arabia
\textsuperscript{d} Department of Mathematical Analysis and Numerical Mathematics, Faculty of Mathematics, Physics and Informatics, Comenius University, Mlynská dolina, 842 48 Bratislava, Slovakia
\textsuperscript{e} Mathematical Institute, Slovak Academy of Sciences, Štefánikova 49, 814 73 Bratislava, Slovakia

\section{Abstract}

This paper investigates existence of \( PC\)-mild solutions of impulsive fractional differential inclusions with nonlocal conditions when the linear part is a fractional sectorial operators like in Bajlekova (2001) \cite{1} on Banach spaces. We derive two existence results of \( PC\)-mild solutions when the values of the semilinear term \( F\) is convex as well as another existence result when its values are nonconvex. Further, the compactness of the set of solutions is characterized.

\section{Introduction}

Fractional differential equations and fractional differential inclusions arise in many engineering and scientific disciplines as the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, electrodynamics of a complex medium, polymer rheology, etc., involves derivatives of fractional order. Fractional differential equations also serve as an excellent tool for the description of hereditary properties of various materials and processes. For some applications of fractional differential equations, one can see \cite{26,28,33–36,40} and the references therein. It seems that El-Sayed and Ibrahim \cite{22} initiated the study of fractional multivalued differential inclusions. Recently, some basic theory for initial value problems for fractional differential equations and inclusions was discussed by Kilbas et al. \cite{33}, Lakshimkantham et al. \cite{34}, Miller et al. \cite{36}, Podlubny \cite{40} and the papers \cite{1–3,5,12,18,23,27,35,42–49} and the references therein.

The theory of impulsive differential equations and impulsive differential inclusions has been an object interest because of its wide applications in physics, biology, engineering, medical fields, industry and technology. The reason for this applicability arises from the fact that impulsive differential problems are an appropriate model for describing process which at certain moments change their state rapidly and which cannot be described using the classical differential problems. For some of these applications we refer to \cite{6,10}. During the last ten years, impulsive differential inclusions with different conditions
have intensely student by many mathematicians. At present, the foundations of the general theory of impulsive differential equations and inclusions are already laid, and many of them are investigated in details in the book of Benchohra et al. [13]. Moreover, a strong motivation for investigating the nonlocal Cauchy problems, which is a generalization for the classical Cauchy problems with initial condition, comes from physical problems. For example, it used to determine the unknown physical parameters in some inverse heat condition problems. For the applications of nonlocal conditions problems we refer to [11,25]. In the few past years, several papers have been devoted to study the existence of solutions for differential equations or differential inclusions with nonlocal conditions [7]. For impulsive differential equation or inclusions with nonlocal conditions of order one we refer to [16,24]. For impulsive differential equation or inclusions of fractional order we refer to [4,21,41,45] and the references therein.

In this paper we are concerned with the existence of mild solutions to the following nonlocal impulsive fractional differential inclusions of the type

\[
\begin{aligned}
\mathcal{D}^\alpha x(t) &\in Ax(t) + f(t, x(t)), \quad t \in (0, 1), \quad \text{a.e. on } J - \{t_1, t_2, \ldots, t_m\}, \\
x(0) &\in x_0 - g(x), \\
x(t_i^-) &\in x(t_i) + I_i(x(t_i)), \quad i = 1, 2, \ldots, m,
\end{aligned}
\tag{1}
\]

where \( J := [0, b] \) with \( b > 0 \) is fixed, \( \mathcal{D}^\alpha \) is the Caputo fractional derivative of the order \( \alpha \in (0, 1) \) with the lower limit zero, \( A \) is a fractional sectorial operator like in [1] defined on a separable Banach space \( E, F : J \times E \to \mathbb{R}^E - \{0\} \) is a multifunction, \( 0 = t_0 < t_1 < \cdots < t_m < t_{m+1} = b \), \( I_i : E \to E \) are impulsive functions which characterize the jump of the solutions at impulse points \( t_i \), \( g : PC(J, E) \to E \) is a nonlinear function related to the nonlocal condition at the origin and \( x(t_i^-), \ldots, x(t_i^+) \) are the right and left limits of \( x \) at the point \( t_i \), respectively and \( PC(J, E) \) will be define later.

Concerning with the main problem (1), we have to study the following impulsive fractional evolution equations with nonlocal conditions:

\[
\begin{aligned}
\mathcal{D}^\alpha x(t) &\in Ax(t) + f(t, x(t)), \quad t \in (0, 1), \quad \text{a.e. on } J - \{t_1, t_2, \ldots, t_m\}, \\
x(0) &\in x_0 - g(x), \\
x(t_i^-) &\in x(t_i) + I_i(x(t_i)), \quad i = 1, 2, \ldots, m.
\end{aligned}
\tag{2}
\]

In [44], Wang et al. introduced a new concept of mild solutions for (2) and derived existence and uniqueness results concerning the PC-mild solutions for (2) when \( f \) is a Lipschitz single-valued function or continuous and maps bounded sets into bounded sets and \( A \) is the infinitesimal generator of a compact semigroup \( \{T(t) \, t \geq 0\} \).

After reviewing the previous research on the fractional evolution equations, we find that the operator in the linear part is the infinitesimal generator of a strongly continuous semigroup, an analytic semigroup, or compact semigroup, or a Hille-Yosida operator, much less is known about the fractional evolution (differential) inclusions with sectorial or almost sectorial operators.

In order to do a comparison between our obtained results in this paper and the known recent results in the same topic, we would like to mention that, recently, the study of evolution equations involving sectorial or almost sectorial operators has been investigated to a large extent. For example, Shu et al. [42] introduced a new concept of mild solutions for impulsive fractional evolution equations and derived existence results concerning the mild solutions for (2) when \( F \) is a completely continuous single-valued function, \( g = 0 \) and \( A \) is a sectorial operator such that the operators families \( \{S_i(t), \, t \geq 0\} \) and \( \{T_i(t), \, t \geq 0\} \) are compact. We will explain in Remark 2.21 why does the definition given in [42] not suitable in some sense. So, we will give another definition for the PC-mild solutions for (1) based on the definition given by Wang et al. [44]. Periago and Straub [39] gave a functional calculus for almost sectorial operator, and using the semigroup of growth \( 1 + \gamma \) which is defined by this functional calculus, obtained the existence and uniqueness for Cauchy problems of abstract evolution equations involving almost sectorial operator, that is by constructing an evolution process of growth \( 1 + \gamma \).

More recently, Wang et al. [46] considered abstract fractional Cauchy problem when \( F \) is a single valued, \( g = 0 \) and \( A \) is an almost sectorial operator whose resolvent satisfies the estimate of growth \( \gamma \left( -1 < \gamma < 0 \right) \) in a sector of the complex plane. Agarwal et al. [3] proved an existence result for (1) without impulses when \( A \) is a sectorial operator and the dimension of \( E \) is finite. They studied the dimension of the set of mild solutions. Ouahab [38, 11] proved a version of Filippov’s Theorem for (1) without impulse, \( g = 0 \) and \( A \) is an almost sectorial operator. The study of differential equations or evolution equations in which the linear part is the infinitesimal generator of \( C_0 \)-semigroup has been investigated by many authors. Cardinali and Rubbioni [16] proved the existence of mild solutions to (1) when \( x = 1 \) and the multivalued function \( F \) satisfies the lower Scorza-Dragoni property and \( \{A(t), \, t \geq 0\} \) is a family of linear operator, generating a strongly continuous evolution operators. Henderson and Ouahab [29] considered the problem (1) when \( A = 0 \), and Zhou et al. [47, 48] introduced a suitable definition of mild solution for (1) based on Laplace transformation and probability density functions for (1) without impulses when \( A \) is the infinitesimal generator of \( C_0 \)-semigroup, \( F \) is single-valued function. Very recently, Wang and Ibrahim [45] proved existence and controllability results for (1) when \( A \) is the infinitesimal generator of \( C_0 \)-semigroup and \( \{T(t), \, t > 0\} \) is strongly equicontinuous \( C_0 \)-semigroup. In addition, Ibrahim et al. [27] proved the existence of mild solutions to the problem (1) when the multivalued function \( F \) satisfies the lower Scorza-Dragoni property and \( A \) is the infinitesimal generator of a compact semigroup \( \{T(t), \, t > 0\} \).
In this paper, motivated by the works mentioned above, we derive two existence results of PC-mild solutions for (1) when the values of the semilinear term $F$ are convex as well as nonconvex and linear term $A$ is a fractional sectorial operator like in [1].

2. Preliminaries and notation

Let $PC(J,E)$ be the space of $E$-valued bounded functions on $J$ with the uniform norm $\|x\| = \sup \{\|x(t)\|, t \in J\}$ such that $x(t^i_i)$ exist for any $i = 0, \ldots, m$ and $x(t)$ is continuous on $J_i$, $i = 0, \ldots, m$, where $J_i = (t_i, t_{i+1})$ and $t_0 = 0, t_{m+1} = b$. $L^1(J,E)$ be the space of $E$-valued Bochner integrable functions on $J$ with the norm $\|f\|_{L^1(J,E)} = \int_0^b \|f(t)\|\,dt$. $P(E) = \{B \subseteq E : B$ is nonempty, closed and bounded $\}$, $P_0(E) = \{B \subseteq E : B$ is nonempty and compact $\}$, $P_\infty(E) = \{B \subseteq E : B$ is nonempty and closed $\}$.

Let $X$ be (not necessarily separable) Banach spaces, and let $F : J \times X \rightarrow P_\infty(E)$ be the multifunction.

**Definition 2.1** ([8, 17, 30, 31]). Let $X$ and $Y$ be two topological spaces. A multifunction $G : X \rightarrow P(Y)$ is said to be upper semicontinuous if

$$G^{-1}(V) = \{x \in X : G(x) \subseteq V\}$$

is an open subset of $X$ for every open $V \subseteq Y$. $G$ is called closed if its graph $\Gamma_G = \{(x, y) \in X \times Y : y \in G(x)\}$ is closed subset of the topological space $X \times Y$. $G$ is said to be completely continuous if $G(B)$ is relatively compact for every bounded subset $B$ of $X$. If the multifunction $G$ is completely continuous with non empty compact values, then $G$ is u.s.c. if and only if $G$ is closed.

**Lemma 2.2** [32, Theorem 1.3.5]. Let $X_0, X$ be (not necessarily separable) Banach spaces, and let $F : J \times X_0 \rightarrow P_\infty(X)$ be such that

(i) for every $x \in X_0$ the multifunction $F(\cdot, x)$ has a strongly measurable selection;

(ii) for a.e. $t \in J$ the multifunction $F(t, \cdot)$ is upper semicontinuous.

Then for every strongly measurable function $z : J \rightarrow X_0$ there exists a strongly measurable function $f : J \rightarrow X$ such that $f(t) \in F(t,z(t))$ a.e. $t \in J$.

**Remark 2.3** ([32, Theorem 1.3.1]). For single-valued or compact-valued multifunctions acting on a separable Banach space the notions measurability and strongly measurable coincide. So, if $X_0, X$ are separable Banach spaces we can replace strongly measurable with measurable in the above lemma, and by Theorem 1.3.4 of [32], for every measurable multifunction $G : J \rightarrow P_\infty(X)$ the multifunction $\Omega : J \rightarrow P_\infty(X)$, $\Omega(t) = F(t,G(t))$ is measurable.

**Lemma 2.4** ([17], Generalized Cantor’s intersection). If $(B_n)_{n \geq 1}$ is a decreasing sequence of nonempty closed subsets of $E$ and $\lim_{n \to \infty} \chi(B_n) = 0$, then $\bigcap_{n=1}^{\infty} B_n$ is nonempty and compact.

**Definition 2.5** ([8]). A sequence $(f_n : n \in \mathbb{N}) \subset L^1(J,E)$ is said to be semicompact if

(i) It is integrable bounded, i.e., there is $q \in L^1(J, \mathbb{R}^+)$ such that $\|f_n(t)\| \leq q(t)$ a.e. $t \in J$.

(ii) The set $(f_n(t) : n \in \mathbb{N})$ is relatively compact in $E$ a.e. $t \in J$.

We need the following simple result related to [8, Theorem 1.1.4].

**Lemma 2.6.** Let $(Z_n)_{n \geq 1}$ be a sequence of subsets of $E$. Suppose there is a compact and convex subset $Z \subset X$ such that for any neighborhood $U$ of $Z$ there is an $n$ so that for any $m \geq n : Z_m \subset U$. Then $\bigcap_{n=1}^{\infty} \text{conv}(\bigcup_{n \geq N} Z_n) \subset Z$.

We recall one fundamental result which follows from Dunford–Pettis Theorem.

**Lemma 2.7** ([32]). Every semicompact sequence in $L^1(J,E)$ is weakly compact in $L^1(J,E)$.

For more about multifunctions we refer to [20, 30–34].
Lemma 2.8 ([16]). Let \( \chi \) be the Hausdorff measure of noncompactness on \( E \). If \( (B_n)_{n \geq 1} \) is a decreasing sequence of nonempty closed subsets of \( E \) and \( \lim_{n \to \infty} \chi(B_n) = 0 \), then \( \bigcap_{n=1}^{\infty} B_n \) is nonempty and compact.

Lemma 2.9 ([14]). Let \( B \) be a bounded set in \( E \). Then for every \( \varepsilon > 0 \) there is a sequence \( (x_n)_{n \geq 1} \) in \( B \) such that

\[
\chi(B) \leq 2\chi(x_n : n \geq 1) + \varepsilon.
\]

Lemma 2.10 ([37]). Let \( \chi_{C(J,E)} \) be the Hausdorff measure of noncompactness on \( C(J,E) \). If \( W \subseteq C(J,E) \) is bounded, then for every \( t \in J \),

\[
\chi(W(t)) \leq \chi_{C(J,E)}(W),
\]

where \( W(t) = \{ x(t) : x \in W \} \). Furthermore, if \( W \) is equicontinuous on \( J \), then the map \( t \mapsto \chi(x(t) : x \in W) \) is continuous on \( J \) and

\[
\chi_{C(J,E)}(W) = \sup_{t \in J} \chi(x(t) : x \in W).
\]

Lemma 2.11 ([9, Lemma 4]). Let \( \{ f_n : n \in \mathbb{N} \} \subset L^p(J,E), p > 1 \) be an integrable bounded sequence such that

\[
\chi(f_n : n \geq 1) \leq \gamma(t), \quad \text{a.e. } t \in J,
\]

where \( \gamma \in L^1(J,\mathbb{R}^+) \). Then for each \( \epsilon > 0 \) there exists a compact \( K_\epsilon \subseteq E \), a measurable set \( J_\epsilon \subset J \), with measure less than \( \epsilon \), and a sequence of functions \( \{ g_n \} \subset L^p(J,E) \) such that

\[
\{ g_n(t) : n \geq 1 \} \subseteq K_\epsilon, \quad t \in J_\epsilon
\]

and

\[
\| f_n(t) - g_n(t) \| < 2\gamma(t) + \epsilon, \quad \text{for every } n \geq 1 \text{ and every } t \in J - J_\epsilon.
\]

Definition 2.12 ([33]). The fractional integral of order \( \gamma \) with the lower limit zero for a function \( f \in L^1(J,E) \) is defined as

\[
I_0^\gamma f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t \frac{f(s)}{(t-s)^{1-\gamma}} ds, \quad t > 0, \quad \gamma > 0
\]

provided the right side is point-wise defined on \([0,\infty)\), where \( \Gamma(\cdot) \) is the gamma function.

Definition 2.13 ([33]). The Riemann–Liouville derivative of order \( \gamma \) with the lower limit zero for a function \( f \in L^1(J,E) \) can be written as

\[
D_0^\gamma f(t) = \frac{1}{\Gamma(n-\gamma)} \frac{d^n}{dt^n} \int_0^t \frac{f(s)}{(t-s)^{n-\gamma}} ds, \quad t > 0, \quad n-1 < \gamma < n,
\]

Definition 2.14 ([33]). The Caputo derivative of order \( \gamma \) for a function \( f \in L^1(J,E) \) can be written as

\[
D_0^\gamma f(t) = D_0^\gamma [f(t) - f(0)], \quad t > 0, \quad 0 < \gamma < 1.
\]

For further readings and details on fractional calculus, we refer to the books and papers by Kilbas et al. [33] and Podlubny [40].

Next, we are ready to recall some facts of fractional Cauchy problem. Bajlekova [1] studied the following linear fractional Cauchy problem

\[
\begin{cases}
{}^cD_t^\gamma x(t) = Ax(t), & t \geq 0, \\
x(0) = x_0 \in E,
\end{cases}
\]

where \( A \) is linear closed and \( D(A) \) is dense.

Definition 2.15 (see [1, Definition 2.3]). A family \( \{ S_\gamma(t) : t \geq 0 \} \subset L(E) \) is called a solution operator for (3) if the following conditions are satisfied:

(a) \( S_\gamma(t) \) is strongly continuous for \( t \geq 0 \) and \( S_\gamma(0) = I \);
(b) \( S_\gamma(t)A \subset D(A) \) and \( A S_\gamma(t)x = S_\gamma(t)Ax \) for all \( x \in D(A) \) and \( t \geq 0 \);
(c) \( S_\gamma(t)x \) is a solution of (3) for all \( x \in D(A) \) and \( t \geq 0 \).
\textbf{Definition 2.16} (see [1, Definition 2.4]). An operator $A$ is said to be belong to $e^x(M, \omega)$ if the solution operator $S_x(\cdot)$ of (3) satisfies
\[ \|S_x(t)\|_{L(E)} \leq Me^{\omega t}, \quad t \geq 0 \]
for some constants $M \geq 1$ and $\omega \geq 0$.

\textbf{Definition 2.17} (see [1, Definition 2.13]). A solution operator $S_x(t)$ of (3) is called analytic if it admits an analytic extension to a sector $\Sigma_{\theta_0} = \{ \lambda \in C - \{0\} : |\arg \lambda| < \theta_0 \}$ for some $\theta_0 \in (0, \frac{\pi}{2}]$. An analytic solution operator is said to be of analyticity type $(\theta_0, \omega_0)$ if for each $\theta < \theta_0$ and $\omega > \omega_0$ there is an $M = M(\theta, \omega)$ such that
\[ \|S_x(t)\|_{L(E)} \leq Me^{\omega t}, \quad t \in \Sigma_{\theta}. \]

Set
\[ e^x(\omega) := \bigcup \{e^x(M, \omega) : M \geq 1 \} \quad \text{and} \quad e^x := \bigcup \{e^x(\omega) : \omega \geq 0 \}, \]
\[ A^x(\theta_0, \omega_0) = \{ A \in e^x : A \text{ generates an analytic solution operator } S_x \text{ of type } (\theta_0, \omega_0) \}. \]

\textbf{Remark 2.18} ([1, Theorem 2.14]). Let $x \in (0, 2)$. A linear closed densely defined operator $A$ belongs to $A^x(\theta_0, \omega_0)$ if and only if $\lambda^x \in \rho(A)$ for each $\lambda \in \Sigma_{\theta_0+\frac{\pi}{2}}(\omega_0) = \{ \lambda \in C - \{0\} : |\arg(\lambda - \omega_0)| < \theta_0 + \frac{\pi}{2} \}$ and for any $\omega > \omega_0$, $\theta < \theta_0$ there is a constant $C = C(\theta, \omega)$ such that
\[ \|\lambda^{x-1}R(\lambda^x, A)\|_{L(E)} \leq \frac{C}{|\lambda - \omega|} \]
for $\lambda \in \Sigma_{\theta_0+\frac{\pi}{2}}(\omega)$.

According to the proof of Theorem 2.14 in [1], if $A \in A^x(\theta_0, \omega_0)$ for some $\theta_0 \in (0, \pi)$ and $\omega_0 \in \mathbb{R}$, the solution operator for the Eq. (3) is given by
\[ S_x(t) = \frac{1}{2\pi i} \int_{\Gamma} e^{\pi i \lambda^{x-1}R(\lambda^x, A)} d\lambda. \] (4)

for a suitable path $\Gamma$. Next following [1,3,42], a mild solution of the Cauchy problem
\[
\begin{align*}
\partial_t x(t) &= Ax(t) + f(t), & t \in J, \\
x(0) &= x_0 \in E
\end{align*}
\]
can be defined by
\[ u(t) = S_x(t)x_0 + \int_0^t T_x(t-s)f(s)ds, \]
where
\[ T_x(t) = \frac{1}{2\pi i} \int_{\Gamma} e^{\pi i R(\lambda^x, A)} d\lambda \]
for a suitable path $\Gamma$ and $f : J \rightarrow E$ is continuous. We need the following estimates

\textbf{Lemma 2.19} ([1, (2.26)], [42]). If $A \in A^x(\theta_0, \omega_0)$ then
\[ \|S_x(t)\|_{L(E)} \leq Me^{\omega t} \quad \text{and} \quad \|T_x(t)\|_{L(E)} \leq Ce^{\omega t}(1 + t^{x-1}) \]
for every $t > 0$, $\omega > \omega_0$. So putting
\[ M^x := \sup_{0 \leq t \leq b} \|S_x(t)\|_{L(E)}, \quad M^x_T := \sup_{0 \leq t \leq b} Ce^{\omega t}(1 + t^{1-x}), \]
we get
\[ \|S_x(t)\|_{L(E)} \leq M^x, \quad \|T_x(t)\|_{L(E)} \leq t^{x-1}M^x_T. \] (5)

Based on the above consideration (see also [44]), we introduce the definition of mild solution for (1).

\textbf{Definition 2.20}. Let $A \in A^x(\theta_0, \omega_0)$ with $\theta_0 \in (0, \frac{\pi}{2}]$ and $\omega_0 \in \mathbb{R}$. A function $x \in PC(J, E)$ is called a mild solution of (1) if
\[ x(t) = \begin{cases} 
S_2(t)(x_0 - g(x)) + \int_0^t T_x(t - s)f(s)ds, & t \in J_0, \\
S_2(t)(x_0 - g(x)) + S_2(t - t_1)I_1((t_1^{-})) + \int_0^t T_x(t - s)f(s)ds, & t \in J_1, \\
\vdots \\
S_2(t)(x_0 - g(x)) + \sum_{i=1}^{i=m} S_2(t - t_i)I_i(x(t_i^{-})) + \int_0^t T_x(t - s)f(s)ds, & t \in J_m, 
\end{cases} \quad (6) \]

where \( f \in S^1_{\mathbb{R}[\mathbb{R}^n]} \).

**Remark 2.21.**

(i) In Definition 2.2 of [42], Shu et al. introduced the following definition of mild solutions for (1) when \( F \) is a single valued function and \( g = 0 \)
\[ x(t) = \begin{cases} 
S_2(t)x_0 + \int_0^t T_x(t - s)f(s)ds, & t \in J_0, \\
S_2(t)(x(t_1^{-})) + I_1((t_1^{-})) + \int_0^t T_x(t - s)f(s)ds, & t \in J_1, \\
\vdots \\
S_2(t)(x(t_m^{-})) + I_m((t_m^{-})) + \int_0^t T_x(t - s)f(s)ds, & t \in J_m, 
\end{cases} \quad (7) \]

where \( f \in S^1_{\mathbb{R}[\mathbb{R}^n]} \).

(ii) Our Definition 2.20 is more suitable than the definition given by Shu et al. [42]. In fact, if \( x = 1 \) then (6) reduces to
\[ \begin{align*}
T(t)(x_0 - g(x)) + \int_0^t T_x(t - s)f(s)ds, & \quad t \in J_0, \\
T(t)(x_0 - g(x)) + T(t - t_1)I_1((t_1^{-})) + \int_0^t T_x(t - s)f(s)ds, & \quad t \in J_1, \\
\vdots \\
T(t)(x_0 - g(x)) + \sum_{i=1}^{i=m} T(t - t_i)I_i(x(t_i^{-})) + \int_0^t T_x(t - s)f(s)ds, & \quad t \in J_m,
\end{align*} \]

which is the standard formula of PC-mild solutions of
\[ \begin{align*}
x'(t) & \in Ax(t) + F(t, x(t)), \quad \text{a.e. on } J \setminus \{t_1, t_2, \ldots, t_m\}, \\
x(0) & = x_0 - g(x), \\
x(t_i^{-}) & = x(t_i) + I_i(x(t_i)), \quad i = 1, 2, \ldots, m.
\]

(iii) There are also other concepts of solutions for impulsive fractional differential equations, see [50–53]. We are now aware according to the paper [54] that the definition of solutions of impulsive Caputo fractional equations (ICFE) is questionable. We tried to explain our attitude in [55]. It is now clear that the definition of solutions of ICFE is not so simple as for natural-order differential equations with impulses. Our approach is based on the fact the lower limit in the Caputo derivative is given, so it is fixed. This means that a family of solutions is set at the beginning by the Cauchy initial conditions. Then at each impulses the solution is kicked by the impulse on one of these solutions. This was first demonstrated in derivation of formula (10) in [56, Lemma 2.7] for non-homogeneous linear Caputo fractional differential equation. Latter this formula is adapted for determining a mild solution for linear Caputo fractional abstract differential equation with impulses in [44, formula (3.4)] (see also [57, Definition 4.1]). We follow this path also in this paper by introducing Definition 2.20. Summarizing, we do not claim that our definition of solutions for ICFE is the best one, but we tried to follow a similar way as for ordinary differential equations with impulses, since there, for ODE, a solution is kicked by an impulse on a solution with different Cauchy condition at the beginning.

The following fixed point theorems are crucial in the proof of our main results.

**Lemma 2.22 ([32, Corollary 3.3.1]).** Let \( W \) be a closed convex subset of a Banach space \( X \) and \( R: W \to P_{ccl}(X) \) be a closed multifunction which is \( \chi \)-condensing, where \( \chi \) is a non singular measure of noncompactness defined on subsets of \( W \), then \( R \) has a fixed point.

**Lemma 2.23 ([32, Proposition 3.5.1]).** Let \( W \) be a closed subset of a Banach space \( X \) and \( R: W \to P_c(X) \) be a closed multifunction which is \( \chi \)-condensing on every bounded subset of \( W \), where \( \chi \) is a monotone measure of noncompactness defined on \( X \). If the set of fixed points for \( R \) is a bounded subset of \( X \) then it is compact.
Lemma 2.24 ([19]). Let \((X, d)\) be a complete metric space. If \(R : X \rightarrow P_{cl}(X)\) is contraction, then \(R\) has a fixed point.

3. Existence results for convex case

In this section, we give some existence results for (1) when \(A\) is a sectorial operator.

Theorem 3.1. Let \(A \in A^2([0, \omega_0) \cap [\omega_0, \infty) \cap \mathbb{R})\) with \(0_0 \in (0, \frac{\omega_0}{2})\) and \(\omega_0 \in \mathbb{R}, F : J \times E \rightarrow P_{cl}(E)\) a multifunction, \(g : PC(J, E) \rightarrow E\) and \(I_i : E \rightarrow E\) \((i = 1, 2, \ldots, m)\).

We assume the following conditions:

(HF1) For every \(x \in E\), \(t \rightarrow F(t, x)\) is measurable, for a.e. \(t \in J\), \(x \rightarrow F(t, x)\) is upper semicontinuous.

(HF2) There exists a function \(\varphi \in L^1(J, \mathbb{R}^+)\), \(q \in (0, \omega)\) and a nondecreasing continuous function \(\Omega : \mathbb{R}^+ \rightarrow \mathbb{R}^+\) such that for any \(x \in E\)
\[
\|F(t, x)\| \leq \varphi(t)\Omega(\|x\|), \quad \text{a.e. } t \in J.
\]

(HF3) There exists a function \(\beta \in L^1(J, \mathbb{R}^+)\), \(q \in (0, \omega)\) satisfying
\[
2\eta M \beta ||\beta||_{L^1(J, \mathbb{R}^+)} < 1,
\]
where \(\eta = \frac{\omega^+}{\omega^{-}}\) and \(\sigma = \frac{\omega^+}{\omega^{-}}\) and for every bounded subset \(D \subseteq E\),
\[
\chi(F(t, D)) \leq \beta(t)\chi(D)
\]
for a.e. \(t \in J\), where \(\chi\) is the Hausdorff measure of noncompactness in \(E\).

(Hg) The function \(g\) is continuous, compact and there are two positive constants \(a, b\) such that
\[
\|g(x)\| \leq a\|x\|_{PC(J,E)} + b, \quad \text{for all } x \in PC(J,E).
\]

(HI) For every \(i = 1, 2, \ldots, m\), \(I_i\) is continuous and compact and there exists a positive constant \(h_i\)

Then the problem (1) has a mild solution provided that there is \(r > 0\) such that
\[
M \beta \left(\|x_0\| + ar + d + hr\right) + M \eta \Omega(r) \|\varphi\|_{L^1(J, \mathbb{R}^+)} \leq r,
\]
where \(h = \sum_{i=1}^{m} h_i\).

Proof. We turn the problem (1) into fixed point problem and define a multifunction \(R : PC(J, E) \rightarrow 2^{PC(J, E)}\) as follows: for \(x \in PC(J,E), R(x)\) is the set of all functions \(y \in R(x)\) such that
\[
y(t) = \begin{cases}
S_x(t)(x_0 - g(x)) + \int_0^t T_x(t-s)f(s)ds, & t \in J_0, \\
\vdots \\
S_x(t)(x_0 - g(x)) + \sum_{k=1}^{i-1} S_x(t - t_k)k(x(t_k)) + \int_0^t T_x(t-s)f(s)ds, & t \in J_i, 1 \leq i \leq m,
\end{cases}
\]
where \(f \in S_{f(x(t),J)}\). In view of (HF1) the values of \(R\) are nonempty. It is easy to see that any fixed point for \(R\) is a mild solution for (1). So our goal is to prove, by using Lemma 2.22, that \(R\) has a fixed point. The proof will be given in several steps.

Step 1. The values of \(R\) are closed.
Let \(x \in PC(J,E)\) and \((y_n : n \geq 1)\) be a sequence in \(R(x)\) and converging to \(y\) in \(PC(J,E)\). Then, according to the definition of \(R\), there is a sequence \((f_n : n \geq 1)\) in \(S_{f(x(t),J)}\) such that for any \(t \in J_i, i = 0, 1, \ldots, m\)
\[
y_n(t) = \begin{cases}
S_x(t)(x_0 - g(x)) + \int_0^t T_x(t-s)f_n(s)ds, & t \in J_0, \\
\vdots \\
S_x(t)(x_0 - g(x)) + \sum_{k=1}^{i-1} S_x(t - t_k)k(x(t_k)) + \int_0^t T_x(t-s)f_n(s)ds, & t \in J_i, 1 \leq i \leq m.
\end{cases}
\]
In view of (HF2) for every \(n \geq 1\), and for a.e. \(t \in J\)
\[
\|f_n(t)\| \leq \varphi(t)\Omega(\|x(t)\|) \leq \varphi(t)\Omega(\|x\|_{PC(J,E)}).
\]
This show that the set \( \{ f_k : n \geq 1 \} \) is integrably bounded. Moreover, because \( \{ f_s(t) : n \geq 1 \} \subset F(t,x(t)) \), for a.e. \( t \in J \), the set \( \{ f_s(t) : n \geq 1 \} \) is relatively compact in \( E \) for a.e. \( t \in J \). Therefore, the set \( \{ f_s : n \geq 1 \} \) is semicompact and then, by Lemma 2.7 it is weakly compact in \( L^1(J,E) \). So, without loss of generality we can assume that \( f_s \) converges weakly to a function \( f \in L^1(J,E) \). From Mazur’s lemma, for every natural number \( n \) there is a natural number \( k_0(j) > J \) and a sequence of nonnegative real numbers \( \lambda_{jk} \), \( k = J \ldots , k_0(j) \) such that \( \sum_{k=J}^{k_0(j)} \lambda_{jk} = 1 \) and the sequence of convex combinations \( z_j = \sum_{k=J}^{k_0(j)} \lambda_{jk} f_k, j \geq 1 \) converges strongly to \( f \) in \( L^1(J,E) \) as \( j \to \infty \). So we may suppose that \( z_j(t) \to f(t) \) for a.e. \( t \in J \). Since \( F \) takes convex and closed values, we obtain for a.e. \( t \in J \)

\[
f(t) \in \bigcap_{j=1}^{\infty} \{ z_k(t) : k \geq j \} \subseteq \bigcap_{j=1}^{\infty} \text{conv} \{ f_k : k \geq j \} \subseteq F(t,x(t)).
\]

Note that, by (5) for every \( t,s \in J, s \in (0,t) \) and every \( n \geq 1 \)

\[
\| T_s(t-s)z_n(s) \| \leq \| t-s \|^{-1} M_f \phi(s) \Omega(\|x\|_{PC(J,E)}) \in L^1((0,t], \mathbb{R}^+).
\]

Next taking \( \tilde{y}_n(t) = \sum_{k=J}^{k_0(j)} \lambda_{jk} y_k(j), (11) \) implies

\[
\tilde{y}_n(t) = \begin{cases} 
S_x(t)(x_0 - g(x)) + \int_0^1 T_s(t-s)z_n(s)ds, & t \in J_0, \\
: \\
S_x(t)(x_0 - g(x)) + \sum_{k=J}^{k_0(j)} \lambda_{jk} y_k(j)+ \int_0^1 T_s(t-s)z_n(s)ds, & t \in J_i, 1 \leq i \leq m.
\end{cases}
\]

But \( \tilde{y}_n(t) \to y(t) \) and \( z_n(t) \to f(t) \) for a.e. \( t \in J \), therefore, by passing to the limit as \( n \to \infty \) in (12), we obtain from the Lebesgue dominated convergence theorem that, for every \( i = 0,1, \ldots , m \),

\[
y(t) = \begin{cases} 
S_x(t)(x_0 - g(x)) + \int_0^1 T_s(t-s)f(s)ds, & t \in J_0, \\
: \\
S_x(t)(x_0 - g(x)) + \sum_{k=J}^{k_0(j)} \lambda_{jk} y_k(j)+ \int_0^1 T_s(t-s)f(s)ds, & t \in J_i, 1 \leq i \leq m.
\end{cases}
\]

This proves that \( R(x) \) is closed.

Step 2. Set \( B_0 = \{ x \in PC(J,E) : \| x \| \leq r \} \). Obviously, \( B_0 \) is a bounded, closed and convex subset of \( PC(J,E) \). We claim that \( R(B_0) \subseteq B_0 \). To prove that, let \( x \in B_0 \) and \( y \in R(x) \). By using (5), (9), (11), (HF2), (Hg) and Hölder’s inequality, we get for \( t \in J_0 \)

\[
\| y(t) \| \leq M_S(\| x_0 \| + ar + d) + M_T \Omega(r) \int_0^1 (t-s)^{-q} \phi(s)ds \\
\leq M_S(\| x_0 \| + ar + d) + M_T \Omega(r) \| \phi \|_{\ell^q(J,R^+)} \left( \int_0^1 (t-s)^{-q}ds \right)^{1-q} \\
\leq M_S(\| x_0 \| + ar + d) + M_T \eta \Omega(r) \| \phi \|_{\ell^q(J,R^+)} \leq r.
\]

Similarly, by using (HI) in addition, we get for \( t \in J_i, i = 1, 2, \ldots , m \)

\[
\| y(t) \| \leq M_S(\| x_0 \| + ar + d + hr) + M_T \eta \Omega(r) \| \phi \|_{\ell^q(J,R^+)} \leq r.
\]

Therefore, \( R(B_0) \subseteq B_0 \).

Step 3. Let \( Z = R(B_0) \). We claim that the set \( Z \) is equicontinuous for every \( i = 0, 1, 2, \ldots , m \), where

\[
Z_i = \{ y^* \in C(J,E) : y^*(t) = y(t), t \in J_i, y^*(t_i) = y(t_i^+) \, , \, y \in Z \}.
\]

Let \( y \in Z \). Then there is \( x \in B_0 \) with \( y \in R(x) \). By recalling the definition of \( R \), there is \( f \in S_{F_{L_{\Omega}(\cdot,x)}}(j) \) such that

\[
y(t) = \begin{cases} 
S_x(t)(x_0 - g(x)) + \int_0^1 T_s(t-s)f(s)ds, & t \in J_0, \\
: \\
S_x(t)(x_0 - g(x)) + \sum_{k=J}^{k_0(j)} \lambda_{jk} y_k(j)+ \int_0^1 T_s(t-s)f(s)ds, & t \in J_i, 1 \leq i \leq m.
\end{cases}
\]

We consider the following cases:

Case 1. When \( i = 0 \), let \( t, t + \delta \) be two points in \( J_0 \), then
\[ \|y'(t + \delta) - y'(t)\| = \|y(t + \delta) - y(t)\| \leq \|S_x(t + \delta)(x_0 - g(x)) - S_x(t)(x_0 - g(x))\| + \int_0^{t+\delta} \|T_x(s + \delta - s)f(s)\| ds - \int_0^t \|T_x(t - s)f(s)\| ds \]
\[ \leq \|S_x(t + \delta)(x_0 - g(x)) - S_x(t)(x_0 - g(x))\| + \bigg\| \int_0^{t+\delta} T_x(s + \delta - s)f(s)ds - \int_0^t T_x(t - s)f(s)ds \bigg\| \]
\[ + \bigg\| \int_0^t [T_x(t + \delta - s) - T_x(t - s)]f(s)ds \bigg\| := G_1 + G_2 + G_3, \]  

where
\[ G_1 = \|S_x(t + \delta)(x_0 - g(x)) - S_x(t)(x_0 - g(x))\|, \]
\[ G_2 = \bigg\| \int_0^{t+\delta} T_x(s + \delta - s)f(s)ds \bigg\|, \]
\[ G_3 = \bigg\| \int_0^t [T_x(t + \delta - s) - T_x(t - s)]f(s)ds \bigg\|. \]

We only need to check \( G_i \to 0 \) as \( \delta \to 0 \) for every \( i = 1, 2, 3 \).

For \( G_1 \) we have
\[ \lim_{\delta \to 0} G_1 = \lim_{\delta \to 0} \|S_x(t + \delta)(x_0 - g(x)) - S_x(t)(x_0 - g(x))\| \leq \lim_{\delta \to 0} \|S_x(t + \delta) - S_x(t)\| (\|x_0\| + ar + d) = 0 \]
uniformly for \( x \in B_0 \).

For \( G_2 \), by the Hölder’s inequality we have
\[ \lim_{\delta \to 0} G_2 = \lim_{\delta \to 0} \bigg\| \int_0^{t+\delta} T_x(s + \delta - s)f(s)ds \bigg\| \leq \|M_1\| \lim_{\delta \to 0} \int_0^{t+\delta} (t + \delta - s)^{-\alpha} |f(s)| ds \]
\[ \leq \|M_1\| \lim_{\delta \to 0} \int_0^{t+\delta} (t + \delta - s)^{-\alpha} \phi(s) ds \leq \|M_1\| \lim_{\delta \to 0} \int_0^{t+\delta} (t + \delta - s)^{-\alpha} ds \]
\[ \leq \|M_1\| \lim_{\delta \to 0} [\frac{\delta^\alpha}{\alpha}]^{-1} \|\phi\|_{L^q(J_{\beta+1})} = 0 \]
uniformly for \( x \in B_0 \).

For \( G_3 \), by using and the Lebesgue dominated convergence theorem and the definition of \( T_x \) we get
\[ \lim_{t \to 0} G_3 \leq \lim_{\delta \to 0} \bigg\| \int_0^t [T_x(t + \delta - s) - T_x(t - s)]f(s)ds \bigg\| \leq \int_0^t \lim_{\delta \to 0} \|T_x(t + \delta - s)f(s) - T_x(t - s)f(s)\| ds = 0, \]

independently of \( x \).

Case 2. When \( i \in \{1, 2, \ldots, m\} \), let \( t, t + \delta \) be two points in \( J_i \). Invoking to the definition of \( R \), we have
\[ \|y'(t + \delta) - y'(t)\| = \|y(t + \delta) - y(t)\| \]
\[ \leq \|S_x(t + \delta)(x_0 - g(x)) - S_x(t)(x_0 - g(x))\| + \sum_{k=1}^{k-i} \|S_x(t + \delta - t_k)I_k(t_\sigma) - S_x(t - t_k)I_k(t_\sigma)\| \]
\[ + \bigg\| \int_0^{t+\delta} T_x(s + \delta - s)f(s)ds - \int_0^t T_x(t - s)f(s)ds \bigg\|. \]

Arguing as in the first case we get
\[ \lim_{\delta \to 0} \|y(t + \delta) - y(t)\| = 0. \]

Case 3. When \( t = t_i, i = 1, \ldots, m \), let \( \lambda > 0 \) be such that \( t_i + \lambda \in J_i \) and \( \sigma > 0 \) such that \( t_i < \sigma < t_i + \delta \leq t_{i+1} \), then we have
\[ \|y'(t_i + \delta) - y'(t_i)\| = \lim_{\sigma \to t_i} \|y(t_i + \delta) - y(\sigma)\|. \]

According the definition of \( R \) we get
\[ \|y(t_i + \delta) - y(\sigma)\| \leq \|S_x(t_i + \delta)(x_0 - g(x)) - S_x(\sigma)(x_0 - g(x))\| + \sum_{k=1}^{k-i} \|S_x(t_i + \delta - t_k)I_k(t_\sigma) - S_x(\sigma - t_k)I_k(t_\sigma)\| \]
\[ + \bigg\| \int_0^{t_i+\delta} (t + \delta - s)^{-\alpha} T_x(t + \delta - s)f(s)ds - \int_0^{\sigma} (\sigma - s)^{-\alpha} T_x(\sigma - s)f(s)ds \bigg\|. \]
Arguing as in the first case we can see that
\[
\lim_{\delta \to 0} \|y(t_1 + \delta) - y(\sigma)\| = 0. \tag{18}
\]

From (13)-(18) we conclude that \(Z_{\Gamma}^i\) is equicontinuous for every \(i = 0, 1, 2, \ldots, m\).

Now for every \(n \geq 1\), set \(B_n = \overline{\operatorname{conv}}(B_{n-1})\). From Step 1, \(B_n\) is a nonempty, closed and convex subset of \(PC(J, E)\). Moreover, \(B_1 = \overline{\operatorname{conv}}(B_0) \subseteq B_0\). Also \(B_2 = \overline{\operatorname{conv}}(B_1) \subseteq \overline{\operatorname{conv}}(B_0) \subseteq B_1\). By induction, the sequence \((B_n)\), \(n \geq 1\) is a decreasing sequence of nonempty, closed and bounded subsets of \(PC(J, E)\).

Our goal is to show that the subset \(B = \cap_{n=1}^\infty B_n\) is nonempty and compact in \(PC(J, E)\). By Lemma 2.8, it is enough to show that
\[
\lim_{n \to \infty} \chi_{PC}(B_n) = 0, \tag{19}
\]
where \(\chi_{PC}\) is the Hausdorff measure of noncompactness on \(PC(J, E)\) defined in Section 2. In the following step we prove (19).

**Step 4.** Let \(n \geq 1\) be a fixed natural number and \(\varepsilon > 0\). In view of Lemma 2.9, there exists a sequence \((y_k)\), \(k \geq 1\) in \(R(B_{n-1})\) such that
\[
\chi_{PC}(B_n) = \chi_{PC}(R(B_{n-1})) \geq 2\chi_{PC}(y_k : k \geq 1) + \varepsilon.
\]

From the definition of \(\chi_{PC}\), the above inequality becomes
\[
\chi_{PC}(B_n) \leq 2 \max_{i=0,1,\ldots,m} \chi_i(S_{\Gamma}^i) + \varepsilon, \tag{20}
\]
where \(S = \{y_k : k \geq 1\}\) and \(\chi_i\) is the Hausdorff measure of noncompactness on \(C(J, E)\). Arguing as in the previous step we can show that \(B_n^i, i = 0, 1, \ldots, m\) is equicontinuous. Then, by applying Lemma 2.10 we obtain
\[
\chi_i(S_{\Gamma}^i) = \sup_{t \in J} \chi(S(t)), \tag{21}
\]
where \(\chi\) is the Hausdorff measure of noncompactness on \(E\). Therefore, by using the nonsingularity of \(\chi\), (20) becomes
\[
\chi_{PC}(B_n) \leq 2 \max_{i=0,1,\ldots,m} \left[ \sup_{t \in J} \chi(S(t)) \right] + \varepsilon = 2 \sup_{t \in \overline{J}} \chi(S(t)) + \varepsilon = 2 \sup_{t \in J} \chi(y_k(t) : k \geq 1) + \varepsilon.
\]

Now, since \(y_k \in R(B_{n-1})\), \(k \geq 1\) there is \(x_k \in B_{n-1}\) such that \(y_k \in R(x_k)\), \(k \geq 1\). By recalling the definition of \(R\) for every \(k \geq 1\) there is \(f_k \in S_{\Gamma}(x_k)\) such that for every \(t \in J\),
\[
\begin{align*}
\chi(y_k(t) : k \geq 1) &\leq \left\{ \begin{array}{ll}
\chi(S_x(t)(x_0 - g(x_k)) : k \geq 1) + \chi(\int_0^t T_x(t - s)f_k(s)ds) & : k \geq 1, \quad \text{if } t \in J_0, \\
\vdots & \\
\chi(S_x(t)(x_0 - g(x_k)) : k \geq 1) + \sum_{p=1}^{n-1} \chi(S_x(t - t_p)(f_k(t_p) : k \geq 1) \\
+ \chi(\int_0^t T_x(t - s)f_k(s)ds) : k \geq 1, \quad \text{if } t \in J, \quad i = 1, \ldots, m.
\end{array} \right.
\end{align*}
\]

Since \(g\) is compact, the set \(\{g(x_k) : k \geq 1\}\) is relatively compact. Hence, for every \(t \in J\) we have
\[
\chi(S_x(t)(x_0 - g(x_k)) : k \geq 1) = 0. \tag{23}
\]

Furthermore, condition \((Hi)\) implies, for every \(p = 1, 2, \ldots, m\) and every \(t \in J\),
\[
\chi(S_x(t - t_p)(f_k(t_p)) : k \geq 1) = 0. \tag{24}
\]

In order to estimate
\[
\chi\left\{ \int_0^t T_x(t - s)f_k(s)ds : k \geq 1 \right\},
\]
we observe that, from \((HF3)\) it holds that for \(a.e. \ t \in J\)
\[
\chi(\{f_k(t) : k \geq 1\} \leq \chi(\{F(t, x_k(t)) : k \geq 1\} \leq \beta(t) \chi(x_k(t) : k \geq 1) \leq \beta(t) \chi(B_{n-1}(t)) \leq \beta(t) \chi_{PC}(B_{n-1}) = \gamma(t).
\]

Furthermore, for any \(k \geq 1\), by \((HF2)\), for almost \(t \in J\), \(\|f_k(t)\| \leq \phi(t)\Omega(r)\). Consequently, \(f_k \in L^2(J, E)\), \(k \geq 1\). Note that \(\gamma \in L^2(J, R^+).\) Then, by virtue of Lemma 2.11, there exists a compact \(K_\varepsilon \subseteq E\), a measurable set \(J_\varepsilon \subseteq J\), with measure less than \(\varepsilon\), and a sequence of functions \((g_k^\varepsilon) \subset L^2(J, E)\) such that for all \(s \in J\), \(g_k^\varepsilon(s) : k \geq 1 \subset K_\varepsilon\), and
Then using Minkowski's inequality, we get
\[
\left\| \int_{J^c} T_s(t-s)(f_k(s) - g_k(s))ds \right\| \leq \overline{M}_r \eta \left[ \int_{J^c} (2\gamma(s) + \varepsilon t^q)ds \right]^{\frac{1}{q}} \leq \overline{M}_r \eta \|2\gamma(s) + \varepsilon t^q\|_{L^1(J,R^+)}
\]
\[
\leq \overline{M}_r \eta \left( \|2\gamma\|_{L^1(J,R^+)} + \|\varepsilon t^q\|_{L^1(J,R^+)} \right) \leq 2\overline{M}_r \eta \left( \|\gamma\|_{L^1(J,R^+)} + \varepsilon t^q \right)
\]
\[
= 2\overline{M}_r \eta \left( \chi_{PC}(B_{n-1}) \|\beta\|_{L^1(J,R^+)} + \varepsilon t^q \right)
\]
and Hölder's inequality
\[
\left\| \int_{J^c} T_s(t-s)f_k(s)ds \right\| \leq \overline{M}_r \eta \Omega(v) \left[ \int_{J^c} \varphi(s)^\frac{1}{q}ds \right]^{\frac{1}{q}}.
\]
So by (26) and (27), we derive
\[
\chi \left( \left\{ \int_{J^c} T_s(t-s)f_k(s)ds : k \geq 1 \right\} \right) \leq \chi \left( \left\{ \int_{J^c} T_s(t-s)f_k(s)ds : k \geq 1 \right\} \right) + \chi \left( \left\{ \int_{J^c} T_s(t-s)f_k(s)ds : k \geq 1 \right\} \right) + \chi \left( \left\{ \int_{J^c} T_s(t-s)f_k(s)ds : k \geq 1 \right\} \right)
\]
\[
\leq 2\overline{M}_r \eta \left( \chi_{PC}(B_{n-1}) \|\beta\|_{L^1(J,R^+)} + \varepsilon t^q \right) + \overline{M}_r \eta \Omega(v) \left[ \int_{J^c} \varphi(s)^\frac{1}{q}ds \right]^{\frac{1}{q}}.
\]
By taking into account that \( \varepsilon \) is arbitrary, we get for all \( t \in J \)
\[
\chi \left( \left\{ \int_{J^c} T_s(t-s)f_k(s)ds : k \geq 1 \right\} \right) \leq 2\overline{M}_r \eta \chi_{PC}(B_{n-1}) \|\beta\|_{L^1(J,R^+)}.
\]
Then, by (22) and (24), for every \( t \in J \)
\[
\chi (y_k(t) : k \geq 1) \leq 2\overline{M}_r \eta \chi_{PC}(B_{n-1}) \|\beta\|_{L^1(J,R^+)}.
\]
This inequality with (21) and with the fact that \( \varepsilon \) is arbitrary, imply
\[
\chi_{PC}(B_n) \leq 2\overline{M}_r \eta \chi_{PC}(B_{n-1}) \|\beta\|_{L^1(J,R^+)}.
\]
By means of a finite number of steps, we can write
\[
0 \leq \chi_{PC}(B_n) \leq \left( 2\overline{M}_r \eta \|\beta\|_{L^1(J,R^+)} \right)^{n-1} \chi_{PC}(B_1), \quad \text{for all } n \geq 1.
\]
Since this inequality is true for every \( n \in \mathbb{N} \), by (8) and by passing to the limit as \( n \to +\infty \), we obtain (19) and so our aim in this step is verified.

**Step 5.** At this point, we are in position to apply **Lemma 2.8.** We claim that the set \( B = \bigcap_{n=0}^{\infty} B_n \) is a nonempty and compact subset of \( PC(J,E) \). Moreover, every \( B_n \) being bounded, closed and convex, \( B \) is also bounded closed and convex. Let us verify that \( R(B) \subseteq B \). Indeed,
\[
R(B) \subseteq R(B_n) \subseteq \overline{CUTP} R(B_n) = B_{n+1}
\]
for every \( n \geq 1 \). Therefore, \( R(B) \subseteq \bigcap_{n=2}^{\infty} B_n \). On the other hand \( B_n \subseteq B_1 \) for every \( n \geq 1 \). So,
\[
R(B) \subseteq \bigcap_{n=2}^{\infty} B_n = \bigcap_{n=1}^{\infty} B_n = B.
\]
**Step 6.** The graph of the multi-valued function \( R_B : B \to 2^B \) is closed. Consider a sequence \( \{x_n\}_{n=1}^{\infty} \) in \( B \) with \( x_n \to x \) in \( B \) and let \( y_n \in R(x_n) \) with \( y_n \to y \) in \( PC(J,E) \). We have to show that \( y \in R(x) \). By recalling the definition of \( R \), for any \( n \geq 1 \), there is \( f_n \in S_{P(I_1 \chi_{\{x_n\}})} \) such that
\[
\|f_n(s) - g_n(s)\| < 2\gamma(s) + \varepsilon, \quad \text{for every } k \geq 1 \quad \text{and every } s \in J^c = J - J.
\]
(25)
$$y_n(t) = \begin{cases} S_2(t)(x_0 - g(x_n)) + \int_0^t T_2(t - s)f_n(s)ds, & t \in J_0, \\ \vdots \\ S_2(t)(x_0 - g(x_n)) + \sum_{k=1}^{k-j} S_2(t - t_k)I_k(x(t_k)) + \int_0^t T_2(t - s)f_n(s)ds, & t \in J_i, \quad 1 \leq i \leq m. \end{cases}$$

(28)

Observe that for every $n \geq 1$ and for a.e. $t \in J$

$$\|f_n(t)\| \leq \varphi(t)\Omega(\|x_n(t)\|) \leq \varphi(t)\Omega(r).$$

This show that the set $\{f_n : n \geq 1\}$ is integrably bounded. In addition, the set $\{f_n(t) : n \geq 1\}$ is relatively compact for a.e. $t \in J$ because assumption (HF3) both with the convergence of $\{x_n\}_{n\geq 1}$ imply that

$$\chi(f_n(t) : n \geq 1) \leq \chi(F(t, x_n(t) : n \geq 1)) \leq \beta(t)\chi(x_n(t) : n \geq 1) = 0.$$ 

So, the sequence $\{f_n\}_{n \geq 1}$ is semi-compact, hence by Lemma 2.7 it is weakly compact in $L^1(J, E).$ So, without loss of generality we can assume that $f_n$ converges weakly to a function $f \in L^1(J, E).$ From Mazur’s lemma, for every natural number $j$ there is a natural number $k_0(j) > j$ and a sequence of nonnegative real numbers $\lambda_{i,k}, k = j, 1, \ldots, k_0(j)$ such that $\sum_{k=j}^{k_0(j)} \lambda_{i,k} = 1$ and the sequence of convex combinations $z_j = \sum_{k=j}^{k_0(j)} \lambda_{i,k}t_k, j \geq 1$ converges strongly to $f$ in $L^1(J, E)$ as $j \to \infty.$ So we may suppose that $z_j(t) \to f(t)$ for a.e. $t \in J.$

Let $t$ be such that $F(t, \cdot)$ is upper semicontinuous. Then, for any neighborhood $U$ of $F(t, \cdot),$ there is a natural number $n_0$ so that for any $n \geq n_0$ we have

$$F(t, x_n(t)) \subseteq U.$$ 

Because the values of $F$ are convex and compact, Lemma 2.6 tells us that

$$\bigcap_{j \geq 1} \text{conv} \left( \bigcup_{n \geq j} F(t, x_n(t)) \right) \subseteq F(t, x(t)).$$

As in Step 1, from Mazur’s theorem, there is a sequence $\{z_n : n \geq 1\}$ of convex combinations of $f_n$ such that for a.e. $t \in J$

$$f(t) \in \bigcap_{j \geq 1} \begin{cases} [z_n(t) : n \geq j] \subseteq \bigcap_{j \geq 1} \text{conv} \{f_n(t) : n \geq j\} \subseteq \\
 \bigcup_{n \geq j} \text{conv} \left( \bigcup_{n \geq j} F(t, x_n(t)) \right) \subseteq F(t, x(t)).
\end{cases}$$

Then, by the continuity of $g,$ $S_2,$ $T_2,$ $I_k (k = 1, 2, \ldots, m)$ and by the same arguments used in Step 1, we conclude from relation (28) that

$$y(t) = \begin{cases} S_2(t)(x_0 - g(x)) + \int_0^t (t - s)^{\alpha-1} T_2(t - s)g(s)ds, & t \in J_0, \\
 \vdots \\
 S_2(t)(x_0 - g(x)) + \sum_{k=1}^{k-j} S_2(t - t_k)I_k(x(t_k)) + \int_0^t (t - s)^{\alpha-1} T_2(t - s)f(s)ds, & t \in J_i, \quad 1 \leq i \leq m. \end{cases}$$

Therefore, $y \in R(x).$ This show that the graph of $R$ is closed.

As a result of the Steps 1–5 the multivalued $R_B : B \to 2^B$ is closed and $\chi_{PC}$-condensing, with nonempty convex compact values. Applying the fixed point theorem, Lemma 2.22, there is $x \in B$ such that $x \in R(x).$ Clearly $x$ is a $PC$-mild solution for the problem (1). \qed

In the following theorem we prove that the set of mild solutions of (1) is compact.

**Theorem 3.2.** If the function $\Omega$ in (HF2) is given of the form $\Omega(r) = r + 1,$ and under the assumptions (HA), (HF1), (HF3), (Hg), (Hl) of Theorem 3.1, then the set of mild solutions of (1) is compact in $PC(J, E)$ provided that

$$\left[ M_5(a + h) + M_7 \eta \| \varphi \|_{L^1(J, \mathbb{R}^+)} \right] < 1.$$ 

(29)

**Proof.** Not by Theorem 3.1 the set of solutions of (1) is nonempty. Indeed, we take

$$r = \frac{M_5(\|x_0\| + d) + M_7 \eta \| \varphi \|_{L^1(J, \mathbb{R}^+)}}{1 - \left( M_5(a + h) + M_7 \eta \| \varphi \|_{L^1(J, \mathbb{R}^+)} \right)}.$$
Theorem 3.3. Let $E$ be a separable Banach space, $F : J \times E \rightarrow P_{cb}(E)$ be a multifunction, $g : PC(J, E) \rightarrow E$ and $I_i : E \rightarrow E$ ($i = 1, 2, \ldots, m$). We assume the following conditions:

(HA) $A \in A^2(\theta_0, \omega_0)$ with $\theta_0 \in (0, \frac{\pi}{2})$ and $\omega_0 \in \mathbb{R}$.

(H1) For every $x \in E$, $t \rightarrow F(t, x)$ is measurable.

(H2) There is a function $\zeta \in L^1(J, \mathbb{R}^+)$, $q \in (0, x)$ such that

(i) For every $x, y \in E$

$$h(F(t, x), F(t, y)) \leq \zeta(t)||x - y||, \quad \text{for a.e. } t \in J.$$

where $h : P_{cb}(E) \times P_{cb}(E) \rightarrow \mathbb{R}^+$ is the Hausdorff distance.

(ii) For every $x \in E$

$$\sup\{|x| : x \in F(t, 0)\} \leq \zeta(t), \quad \text{for a.e. } t \in J.$$

(H3) There is a positive constant $\xi$ such that

$$\|g(x_1) - g(x_2)\| \leq \xi\|x_1 - x_2\|_{PC(J, E)}, \quad \text{for all } x_1, x_2 \in PC(J, E).$$

(H4) For each $i = 1, 2, \ldots, m$, there is $\xi_i > 0$ such that

$$\|I_i(x) - I_i(y)\| \leq \xi_i\|x - y\|, \quad \text{for all } x, y \in E.$$

(H5) The following inequality hold

$$\mathcal{M}_{\xi}(a + \zeta) + \mathcal{M}_1(\zeta) \eta < 1, \quad \xi = \sum_{i=1}^{m} \xi_i.$$

Then the problem (1) has a PC-mild solution.

Proof. For $x \in PC(J, E)$, set $S_{F,(x(i))}^1 = \{f \in L^1(J, E) : f(t) \in F(t, x(t))$ for a.e. $t \in J\}$ by Lemma 2.2.2, (H1) and (H2)(i). [32, Theorems 1.1.9 and 1.3.1] $F \frac{}{(i)} \frac{}{(i)} \frac{}{(i)} x(\cdot)$ has a measurable selection $\chi$, by hypothesis (H1)(ii), II $L^1(J, E)$. Thus $S_{F,(x(i))}^1$ is nonempty. Let us transform the problem into a fixed point problem. Consider the multifunction map, $R : PC(J, E) \rightarrow 2^{PC(J, E)}$ defined as follows: for $x \in PC(J, E), R(x)$ is the set of all functions $y \in R(x)$ given by (10). It is easy to see that any fixed point for $R$ is a mild solution for (1). So, we shall show that $R$ satisfies the assumptions of Lemma 2.2.2. The proof will be given in two steps.

Step 1. The values of $R$ are closed. By [32, Theorem 1.1.9], assumptions (H1) and (H2)(ii) imply assumption (HF1) of Theorem 3.1. Next, since by (H2)

$$\|F(t, x)\| = h(F(t, x), \{0\}) \leq h(F(t, x), F(t, 0)) + h(F(t, 0), \{0\}) \leq \zeta(t)||x|| + ||F(t, 0)|| \leq \zeta(t)(1 + ||x||).$$

assumption (HF2) of Theorem 3.1 holds as well. So the statement follows from the 1th step of the proof of Theorem 3.1.

Step 2. $R$ is a contraction. Let $x_1, x_2 \in PC(J, E)$ and $y_1 \in R(x_1)$. Then, there is $f \in S_{F,(x_1(i))}^1$ such that for any $t \in J, i = 0, 1, 2, \ldots, m,$
Consider the multifunction $Z : J \to 2^E$ defined by

$$Z(t) = \{ u \in E : \| f(t) - u \| \leq \zeta(t) \| x_2(t) - x_1(t) \| \}. $$

For each $t \in J$, $Z(t)$ is nonempty. Indeed, let $t \in J$, from (H2)(i), we have

$$h(F(t, x_2(t)), F(t, x_1(t))) \leq \zeta(t) \| x_1(t) - x_2(t) \|. $$

Hence, there exists $u \in F(t, x_1(t))$ such that

$$\| u_t - f(t) \| \leq \zeta(t) \| x_1(t) - x_2(t) \|. $$

Since the functions $f$, $\zeta$, $x_1$, $x_2$ are measurable, Proposition 3.4 in [17] (or [32, Corollary 1.3.1(a)]), tells us that the multifunction $V : t \to Z(t) \cap F(t, x_2(t))$ is measurable. Because its values are nonempty and compact, by [30, Theorem 1.3.1], there is $h \in S_{I_1, x_2(\cdot)}$ with

$$\| h(t) - f(t) \| \leq \zeta(t) \| x_2(t) - x_1(t) \|, \quad \text{a.e. } t \in J. $$

Let us define

$$y_2(t) = \begin{cases} 
S_2(t)(x_0 - g(x_2)) + \int_0^t T_2(t - s)h(s)ds, & t \in J_0, \\
S_2(t)(x_0 - g(x_2)) + S_2(t - t_1)I_1(x_1(t_1)) + \int_0^t T_2(t - s)h(s)ds, & t \in J_1, \\
\vdots \\
S_2(t)(x_0 - g(x_2)) + \sum_{k=1}^{m} S_2(t - t_k)I_k(x_1(t_k)) + \int_0^t T_2(t - s)h(s)ds, & t \in J_i, \quad 1 \leq i \leq m. 
\end{cases} $$

Obviously, $y_2 \in R(x_2)$ and if $t \in J_0$ we get from (30)-(32), (H3) and (H4)

$$\| y_2(t) - y_1(t) \| \leq M_2 \| g(x_1) - g(x_2) \| + M_T \int_0^t (t - s)^{q-1} \| h(s) - f(s) \| ds$$

$$\leq M_2 a \| x_1 - x_2 \|_{PC(J, E)} + M_T \| x_1 - x_2 \|_{PC(J, E)} \int_0^t (t - s)^{q-1} \| h(s) - f(s) \| ds$$

$$\leq M_2 a \| x_1 - x_2 \|_{PC(J, E)} + M_T \| x_1 - x_2 \|_{PC(J, E)} \| \zeta \|_{L^2(J, \mathbb{R}^+)} \eta \leq \left[ M_2 a + M_T \| \zeta \|_{L^2(J, \mathbb{R}^+)} \eta \right] \| x_1 - x_2 \|_{PC(J, E)}. $$

Similarly, if $t \in J_i$, $i = 1, \ldots, m$, we get from (30)-(32), (H3) and (H4)

$$\| y_2(t) - y_1(t) \| \leq \left[ M_2 a + M_T \| \zeta \|_{L^2(J, \mathbb{R}^+)} \eta \right] \| x_1 - x_2 \|_{PC(J, E)}. $$

By interchanging the role of $y_2$ and $y_1$, we obtain from (33) and (34)

$$h(R(x_2), R(x_1)) \leq \left[ M_2 a + M_T \| \zeta \|_{L^2(J, \mathbb{R}^+)} \eta \right] \| x_1 - x_2 \|_{PC(J, E)}. $$

Therefore, $R$ is a contraction due to (H3) and thus by Lemma 2.23, $R$ has a fixed point which is a mild solution for (1). This completes the proof. \qed

4. Existence results for nonconvex case

In this section, we give a nonconvex version for Theorem 3.3. Our hypothesis on the orient field is the following:

(HF$^-$) $F : J \times E \to P_c(E)$ is a multifunction such that

(i) $(t, x) \to F(t, x)$ is graph measurable and $x \to F(t, x)$ is lower semicontinuous.

(ii) There exists a function $\varphi \in L^q(J, \mathbb{R}^+), 0 < q < \infty$ such that for any $x \in E$,

$$\| F(t, x) \| \leq \varphi(t), \quad \text{a.e. } t \in J. $$

**Theorem 4.1.** Let $A \in A^2(\theta_0, \omega_0)$ with $\theta_0 \in (0, \frac{\pi}{2}]$ and $\omega_0 \in \mathbb{R}, F : J \times E \to P_c(E)$ a multifunction, $g : PC(J, E) \to E$ and $I : E \to E$ ($i = 1, 2, \ldots, m$). If the conditions (HF3), (Hg), (HI) and (HF$^-$) hold, then the problem (1) has a PC-mild solution provided that there is $r > 0$ provided that

...
\[ M_S (\|x_0\| + ar + d + h \, r) + M_T \eta \|\phi\|_E \leq r. \]  
(35)

**Proof.** Consider the multivalued Nemitsky operator \( N : PC(J, E) \to 2^{L^1(J, E)}, \) defined by

\[ N(x) = S^1_{t(x)} = \{ f \in L^1(J, E) : f(t) \in F(t, x(t)), \ a.e. \ t \in J \}. \]

First we note that \( F \) is superpositionally measurable by [58]. Next, we shall prove that \( N \) has nonempty closed decomposable value and l.s.c. Let \( x \in PC(J, E). \) Since \( F \) has closed values, \( S^1_{t(x)} \) is closed [30]. Because \( F \) is integrably bounded, \( S^1_{t(x)} \) is non-empty (see Theorem 3.2 of [30]). It is readily verified, \( S^1_{t(x)} \) is decomposable (see Theorem 3.1 of [30]). To check the lower semicontinuity of \( N, \) we need to show that, for every \( u \in L^1(J, E), \ x \to d(u, N(x)) \) is upper semicontinuous (see Proposition 1.2.26 of [31]). To this end let \( u \in L^1(J, E) \) be fixed. From Theorem 2.2 in [30] with \( \phi(t, v) = \|u(t) - v\|, \) we have that

\[ d(u, N(x)) = \inf_{v \in N(x)} \|u - v\|_E^2 = \inf_{v(t) \in F(t, x(t))} \int_0^b \|u(t) - v(t)\| \, dt = \int_0^b \inf_{z(t) \in F(t, x(t))} \|u(t) - z(t)\| \, dt = \int_0^b d(u(t), F(t, x(t))) \, dt. \]

(36)

We shall show that for any \( \lambda \geq 0, \) the set

\[ U_\lambda = \{ x \in PC(J, E) : d(u, N(x)) \geq \lambda \} \]

closed. For this purpose, let \( \{x_n\} \subseteq U_\lambda, \) and assume that \( x_n \to x \) in \( PC(J, E). \) Then, for all \( t \in J, x_n(t) \to x(t) \) in \( E. \) By virtue of (HF) of \( (i) \) the function \( z \to d(u(t), F(t, z)) \) is upper semicontinuous. So, via the Fatou lemma, and (36) we have

\[ \lambda \leq \lim_{n \to \infty} \sup \int_0^b d(u(t), F(t, x_n(t))) \, dt \leq \int_0^b \lim_{n \to \infty} \sup \int_0^b d(u(t), F(t, x_n(t))) \, dt \leq \int_0^b d(u(t), F(t, x(t))) \, dt = d(u, N(x)). \]

Therefore \( x \in U_\lambda, \) and this proves the lower semicontinuity of \( N. \) This allows us to apply Theorem 3 of [15] and obtain a continuous map \( Z : PC(J, E) \to L^1(J, E) \) such that \( Z(x) \in N(x), \) for every \( x \in PC(J, E). \) Then, \( Z(x)(s) \in F(s, x(s)) \) a.e. \( s \in J. \)

Consider a map \( \Phi : PC(J, E) \to PC(J, E) \) defined by

\[ (\Phi x)(t) = \begin{cases} 0_S (t) (t) (x_0 - g(x_1)) + \int_0^t (t - s)^{\alpha - 1} T_S (t) (t - s) Z(x)(s) \, ds, & t \in J_0, \\ S_2 (t) (t) (x_0 - g(x_1)) + \sum_{k=1}^{i-1} S_2 (t) (t - t_k) k (x_1 (t_k)) + \int_0^t (t - s)^{\alpha - 1} T_S (t) (t - s) Z(x)(s) \, ds, & t \in J_i, 1 \leq i \leq m. \end{cases} \]

**Acknowledgments**

The authors thanks the referees for their careful reading of the manuscript and insightful comments, which help to improve the quality of the paper. We would also like to acknowledge the valuable comments and suggestions from the editors, which vastly contributed to improve the presentation of the paper.

**References**


