Characterizing Contextual Equivalence in Calculi with Passivation

Serguei Lenglet\textsuperscript{a}, Alan Schmitt\textsuperscript{b}, Jean-Bernard Stefani\textsuperscript{b}

\textsuperscript{a}Université Joseph Fourier, Grenoble, France
\textsuperscript{b}INRIA, Grenoble, France

Abstract

We study the problem of characterizing contextual equivalence in higher-order languages with passivation. To overcome the difficulties arising in the proof of congruence of candidate bisimilarities, we introduce a new form of labelled transition semantics together with its associated notion of bisimulation, which we call \textit{complementary semantics}. Complementary semantics allows to apply the well-known Howe’s method for proving the congruence of bisimilarities in a higher-order setting, even in presence of an early form of bisimulation. We use complementary semantics to provide a coinductive characterization of contextual equivalence in the HO\(\pi\)P calculus, an extension of the higher-order \(\pi\)-calculus with passivation, obtaining the first result of this kind. We then study the problem of defining a more effective variant of bisimilarity that still characterizes contextual equivalence, along the lines of Sangiorgi’s notion of \textit{normal bisimilarity}. We provide partial results on this difficult problem: we show that a large class of test processes cannot be used to derive a normal bisimilarity in HO\(\pi\)P, but we show that a form of normal bisimilarity can be defined for HO\(\pi\)P without restriction.

1. Introduction

1.1. Characterizing contextual equivalence in higher-order concurrent languages

A natural notion of program equivalence in concurrent languages is a form of contextual equivalence called \textit{barbed congruence}, introduced by Milner and Sangiorgi [29]. Roughly, given an operational semantics defined by means of a small-step reduction relation, two processes are barbed congruent if they have the same reductions and the same observables (or \textit{barbs}), under any context.

The definition of barbed congruence, however, is impractical to use in proofs because of its quantification on contexts. An important question, therefore, is to find more effective characterizations of barbed congruence. A powerful method for proving program equivalence is the use of coinduction with the definition of

\textit{Email addresses: serguei.lenglet@inria.fr (Serguei Lenglet), alan.schmitt@inria.fr (Alan Schmitt), jean-bernard.stefani@inria.fr (Jean-Bernard Stefani)}

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an appropriate notion of **bisimulation**. The question of characterizing barbed congruence to enable the use of coinduction becomes that of finding appropriate bisimulation relations such that their resulting behavioral equivalences, called **bisimilarities**, are **sound** (i.e., included in) and **complete** (i.e., containing) with respect to barbed congruence.

For first-order languages, such as CCS or the π-calculus, the behavioral theory and the associated proof techniques, e.g., for proving congruence, are well developed [36]. Characterizing contextual equivalence in these languages, i.e., finding a bisimilarity relation that is both sound and complete with respect to barbed congruence, is a reasonably well understood proposition.

The situation is less satisfactory for higher-order concurrent languages. Bisimilarity relations that coincide with barbed congruence have only been given for some higher-order concurrent languages. They usually take the form of **context bisimilarities**, building on a notion of **context bisimulation** introduced by D. Sangiorgi for a higher-order π-calculus, HOπ [34]. Context bisimilarity has been proven to coincide with contextual equivalence for higher-order variants of the π-calculus: Sangiorgi’s HOπ [33, 34, 19], a concurrent ML with local names [18], a higher-order distributed π-calculus called SafeDpi [15], Mobile Ambients [27], and some of Mobile Ambients’s variants such as Boxed Ambients [5]. A sound but incomplete form of context bisimilarity has been proposed for the Seal calculus [10]. For the Homer calculus [13], strong context bisimilarity is proven sound and complete, but weak context bisimilarity is not complete. A sound and complete context bisimilarity has been defined for the Kell calculus [38], but for the strong case only.

Context bisimilarity is not entirely satisfactory, however. Its definition still involves quantification on processes (or on abstractions and concretions, following Milner’s terminology [28], that can be understood, respectively, as receiving processes and emitting processes)¹. For this reason, Sangiorgi has introduced in his study of HOπ [34] an alternative form of bisimulation, called **normal bisimulation**, that replaces the universal quantification on processes in the input and output clauses in the definition of context bisimulation with a single test process.² To the best of our knowledge, the only higher-order concurrent language for which normal bisimilarity has been defined and proved to coincide with context bisimilarity, is HOπ and its typed variant [33, 34, 19].

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¹Despite this quantification on processes, the use of context bisimulation as a proof technique is still an improvement over the direct use of barbed congruence, as argued in [27]. Removing this quantification would pave the way to automated proof support.

²For instance, the definition of an early strong contextual bisimulation $\mathcal{R}$ in HOπ has the following input clause:

\[
\bullet \text{ for all } P \xrightarrow{a} F, \forall C, \exists F' \text{ such that } Q \xrightarrow{a} F' \text{ and } F \bullet C \mathcal{R} F' \bullet C
\]

This input clause requires to find a matching transition for all emitting processes (actually concretions) $C$. The corresponding clause in the definition of strong normal bisimilarity takes the form:

\[
\bullet \text{ for all } P \xrightarrow{a} F, \exists F' \text{ such that } Q \xrightarrow{a} F' \text{ and } F \bullet C_0 \mathcal{R} F' \bullet C_0
\]

where $C_0$ is a fixed (up to the choice of a fresh name) emitting test process (concretion).
1.2. Process calculi with passivation

The difficulties in characterizing contextual equivalence are particularly acute in calculi featuring process passivation, such as the Homer calculus, the Kell calculus, and, to some extent, the Seal calculus.

Let us motivate first our interest in higher-order languages with strong process mobility and process passivation. Strong process mobility refers to the possibility of moving a running process from one locus of computation (or locality) to another. This feature typically occurs in languages or calculi intended for distributed programming such as the Join calculus [25], Mobile Ambients [8], or Nomadic Pict [41]. Process passivation refers to the ability to suspend the execution of a named running process and to pass around the suspended process, typically as a higher-order parameter in messages. This capability is featured in the Homer calculus [13], the M-calculus [37], and the Kell calculus [38]. Passivation actually subsumes strong mobility, as discussed in [38], since strong mobility amounts to a sequence of passivation, transfer of the suspended process between localities, and reactivation. Strong mobility is a linear operation that moves a computation from one locality to another, whereas passivation may be non-linear: a passivated process can be reactivated several times. The Seal calculus [10] provides an intermediate form, with a combined migrate and replicate (and hence non-linear) operation.

Strong mobility is one of several paradigms for mobile code. It has been introduced as a primary feature in several languages, including Obliq [7], Nomadic Pict [41], and JoCaml [11]. It potentially allows interesting performance and design trade-offs [9, 12], and its use can be compelling in certain application areas such as network and distributed system management [3]. Process passivation provides basic support for dynamic reconfiguration: with passivation, named parts of a system can be replaced during execution. Dynamic reconfiguration is useful to support patches and system updates while limiting system downtime and increasing availability; to support fault recovery and fault tolerance by providing a basic mechanism for checkpointing computations and replicating them; and to support adaptive behaviors, whereby a system changes its configuration to adapt to varying operating conditions, with the aim of improving performance and/or dependability. A form of process passivation has been introduced in the Acute programming language [39] for the same reasons. There, it is called thunkification and applies to designated groups of threads.

In this paper, we work with the HOπP calculus, a minimal extension of HOπ with passivation. An example of process passivation in HOπP is given by the following reduction:

\[ a[P] \mid a(X)Q \rightarrow Q(P/X) \]

where \( a[P] \) is a locality named \( a \) that contains a process \( P \), and \( a(X)Q \) is a receiver process. The passivation above removes the locality \( a \), and passes process \( P \) as an argument to the receiver process \( a(X)Q \). A locality \( a[\ ] \) is an execution context and is transparent: if \( P \) can evolve into \( P' \) (i.e. \( P \rightarrow P' \)), then we have \( a[P] \rightarrow a[P'] \). Also, if \( P \) can emit a message, then \( a[P] \) can also emit the same message. This form of passivation in HOπP is a simplification.
of the passivation constructs present in the Kell calculus and in the Homer calculus. In particular, we eschew the use of join patterns of the Kell calculus, and of communication paths of the Homer calculus.

1.3. Contributions

This paper contributes to the study of the interrelated issues of proving the congruence of bisimilarity relations and of characterizing barbed congruence in higher-order concurrent languages\(^3\), notably those featuring strong process mobility and process passivation capabilities such as the Seal calculus, the Homer calculus, or the Kell calculus. Specifically, this paper makes two sets of contributions: positive ones and negative ones.

On the positive side, we develop a new form of labelled transitions semantics and its associated bisimulation, which we call complementary semantics, that is devised to overcome the difficulties that appear when trying to apply Howe’s method in proving the congruence and soundness of bisimulation relations defined in an early style. Howe’s method is a systematic technique for proving the congruence of bisimilarity relations \([17, 1, 14]\). Unfortunately, Howe’s method is originally well suited for bisimulations that are defined in both a late and a delay style, either of which generally breaks the correspondence with contextual equivalence.\(^4\) In their work on the Homer calculus, Godskesen and Hildebrandt have managed to extend Howe’s method to a version of context bisimulation in an input-early style \([13]\), but the resulting weak bisimilarity is not complete with respect to weak barbed congruence. To our knowledge, our work is the first one to exploit Howe’s method to prove congruence with bisimulation relations defined in an early style. We then show that complementary semantics and complementary bisimilarity can be used successfully to characterize barbed congruence in \(\text{HO}^\pi_\text{P}\), a minimal extension of Sangiorgi’s \(\text{HO}^\pi\) with passivation. This is also, to our knowledge, the first result of its kind.

On the negative side, we show that we we cannot readily exploit Sangiorgi’s notion of normal bisimulation to derive more effective forms of bisimilarities than

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\(^3\)Proving the congruence of a candidate bisimilarity is typically the key step in proving its soundness with respect to barbed congruence.

\(^4\)The early or late style of a bisimulation relation refers to the order of certain quantifiers in its definition. For instance, the definition of an early strong contextual bisimulation \(\mathcal{R}\) in \(\text{HO}^\pi\) has the following two clauses:

- **input clause:** for all \(P \xrightarrow{a} F, \forall C, \exists F'\) such that \(Q \xrightarrow{a} F'\) and \(F \cdot C \mathcal{R} F' \cdot C\);
- **output clause:** for all \(P \xrightarrow{a} C, \forall F, \exists C'\) such that \(Q \xrightarrow{a} C'\) and \(F \cdot C \mathcal{R} F \cdot C'\).

The late variant of strong contextual bisimulation can be obtained by exchanging the order of the quantifiers \(\forall C, \exists F'\) in the input clause, and of the quantifiers \(\forall F, \exists C'\) in the output clause. In other words, the “early” and late styles in a contextual bisimulation game define when a test process is selected with respect to the move of the adversary: in the early style, a test process \(C\) or \(F\) is selected before (hence the term *early*) the adversary has to pick a matching move \(F'\) or \(C'\).

The qualifier *delay* is used in relation with weak forms of contextual bisimulations. In the definition of a delay bisimulation, internal actions are allowed before but not after a visible action.
contextual or complementary bisimilarity for concurrent higher-order languages with process passivation. Specifically, we show that a large class of test processes cannot be used to define for HO\pi P a notion of normal bisimilarity similar to the one defined for HO\pi . The difficulty seems to be linked to the interplay between passivation and restriction. Indeed, we show that a form of normal bisimilarity can be defined for HOP, a calculus which is essentially HO\pi P without restriction, and that it coincides with barbed congruence.

1.4. Organization of the paper

The paper is organized as follows. In Section 2, we recall the main results on HO\pi , the higher-order \pi-calculus. We then introduce the HO\pi P calculus, a minimal extension of HO\pi with passivation. In Section 3, we review two existing techniques for proving congruence of context bisimilarities: the technique used in the proof of congruence of strong context bisimilarity in the Kell calculus, and Howe’s method. We explain why the Kell calculus method fails when trying to prove the congruence of weak context bisimilarities, and why Howe’s method fails when using early context bisimilarities. In Section 4, we present our notion of complementary semantics, using the HO\pi calculus as an example. In Section 5, we present a complementary semantics for the HO\pi P calculus, and we prove that in HO\pi P complementary bisimilarity coincides with barbed congruence. In Section 6, we present counter examples that show that Sangiorgi’s notion of normal bisimilarity, which he developed initially for HO\pi , cannot be readily applied to HO\pi P. In Section 7, we show that a form of normal bisimilarity can be defined for a sublanguage of HO\pi P called HOP, which is essentially HO\pi P without restriction, and that normal bisimilarity coincides with barbed congruence in HOP. Section 8 discusses related work. Section 9 concludes the paper and discusses future work. The Appendices Appendix A, Appendix B, and Appendix C gather the proofs of the main theorems presented in the paper.

This paper refines and extends previous papers by the authors [24, 23]. The HO\pi P calculus was first introduced in [24]. The results presented in Section 5 were given in [23] with only proof hints. The results presented in Section 7 and in Section 6.1 were given in [24] with only proof hints. The material in Sections 3, 4, 6.2, and 6.3 is new.

2. The HO\pi and HO\pi P calculi

We recall in this Section previous results on HO\pi , the higher order \pi-calculus. We also introduce HO\pi P, an extension of HO\pi with a passivation operator.

2.1. The Syntax and Contextual Semantics of HO\pi

The higher order \pi-calculus [34] is a variant of the \pi-calculus with higher-order communication: the communication of names of the standard \pi-calculus is replaced by the communication of processes.
\[ P ::= 0 \mid X \mid P \mid P \mid a(X)P \mid \pi(P)P \mid \nu a.P \mid !P \]

Figure 1: Syntax of the Higher Order \(\pi\)-Calculus

We now state some conventions on notations. We let \(a, b, \ldots\) range over names, \(\pi, \bar{\pi}, \ldots\) range over conames, and \(X, Y, \ldots\) range over process variables. We write \(\tilde{x}\) for a set \(\{x_1, \ldots, x_n\}\). Finally, we let \(\gamma\) range over names and conames.

The syntax of the calculus is given in Figure 2.1. Terms include the inactive process \(0\), process variables \(X\), parallel composition of processes \(P \mid P\), input prefixing \(a(X)P\), output prefixing \(a \langle P \rangle P\), name restriction \(\nu a.P\), and process replication \(!P\). The output prefix construction illustrates the higher order aspect of the calculus, as a process (and not a name) is sent.

In process \(a(X)P\), the variable \(X\) is bound. Similarly, in process \(\nu a.P\), the name \(a\) is bound. We write \(fv(P)\) for the free variables of a process \(P\), \(fn(P)\) for its free names, and \(bn(P)\) for its bound names. We write \(P \{Q/X\}\) for the capture-free substitution of \(X\) by \(Q\) in \(P\). For a name \(a\) and a process \(P\), we write \(a.P\) for \(a(X)P\) where \(X\) is not free in \(P\), and \(\alpha.P\) for \(\alpha(0)P\).

Remark 1. As in many other higher order calculi, replication does not have to be built in as it can be encoded using the other constructs. To encode replication in \(HO\pi\) without replication, we first define \(Y\) as \(t(X)(P \mid X \mid \tilde{t}(X)0)\). We then encode \(!P\) by the process \(Q = \nu t.(\tilde{t}(Y)0 \mid Y)\). The process \(Y\) is similar to a copy of \(P\), except that it receives a copy of itself on \(t\) in order to launch a copy of \(P\) and recreate the process \(Q\). Hence the process \(Q\) reduces to \(P \mid Q\), like the process \(!P\).

To encode replication of prefixed processes \(!m.P\), we instead define \(Y\) as \(m.t(X)(P \mid X \mid \tilde{t}(X)0)\). We then encode \(!m.P\) by the process \(Q = \nu t.(\tilde{t}(Y)0 \mid Y)\).

These encoding introduce an extra step to unfold the replication, which raises issues with strong behavioral equivalences. We thus keep replication explicitly in the calculus.

Convention. We identify processes up to \(\alpha\)-conversion of names and variables: processes and agents are always chosen such that their bound names and variables are distinct from free names and variables. In any discussion or proof, we assume that bound names and bound variables of any process or actions under consideration are chosen to be different from the names and variables occurring free in any other entities under consideration. Note that with this convention, we have \(\nu a.(P \mid Q) \equiv P \mid \nu a.Q\) in Figure 2, without qualification on the free names of \(P\).

We now recall structural congruence and the rules of the labelled transition system in Figure 2, omitting the symmetric rules for \(PAR\) and \(HO\). A Process may evolve towards a process (internal actions \(P \xrightarrow{\Delta} P'\)), an abstraction (message input \(P \xrightarrow{\Delta} F = (X)Q\)), or a concretion (message output
Figure 2: Structural Congruence and Contextual Labeled Transition System for HOπ

\[ P \mid (Q \mid R) \equiv (P \mid Q) \mid R \quad P \mid Q \equiv Q \mid P \quad P \mid 0 \equiv P \]

\[ \nu_a.\nu_b.P \equiv \nu_b.\nu_a.P \quad \nu_a.0 \equiv 0 \quad \nu_a.(P \mid Q) \equiv P \mid \nu_a.Q \quad !P \equiv P \mid !P \]

\[ a(X)P \xrightarrow{a} (X)P \quad \text{ABSTR} \quad \pi(Q)P \xrightarrow{\pi} (Q)P \quad \text{CONCR} \]

\[ P \xrightarrow{\alpha} A \quad P \mid Q \xrightarrow{\alpha} A \mid Q \quad \text{PAR} \]

\[ P \xrightarrow{\alpha} A \quad \nu_a.P \xrightarrow{\alpha} \nu_a.A \quad \text{RESTR} \]

\[ P \xrightarrow{\alpha} A \quad \text{REPLIC} \quad P \xrightarrow{\pi} F \quad P \mid Q \xrightarrow{\pi \cdot} F \mid C \quad \text{HO} \]

\[ !P \xrightarrow{\alpha} A \quad \text{REPLIC-HO} \]

\[ P \xrightarrow{\pi} F \quad Q \xrightarrow{\pi} C \]

\[ P \xrightarrow{\pi} F \quad Q \xrightarrow{\pi \cdot} F \mid C \]

\[ \nu_a.(X)P \equiv \nu_b.(R)Q \]. The transition \( P \xrightarrow{a} (X)Q \) means that \( P \) may receive a process \( R \) on \( a \) to continue as \( Q\{R/X\} \). The transition \( P \xrightarrow{\pi} \nu_b.(R)Q \) means that \( P \) may send the process \( R \) on \( a \) and continue as \( Q \), and that the scope of names \( b \), which occur free in \( R \), has to be expanded to encompass the recipient of \( R \). A synchronous higher-order communication takes place when a concretion interacts with an abstraction (rule HO). We define a pseudo-application operator \( \bullet \) between an abstraction \( F = (X)P \) and a concretion \( C = \nu_b.(R)Q \) as follows.

\[ (X)P \bullet \nu_b.(R)Q \triangleq \nu_b.(P\{R/X\} \mid Q) \]

As above, we rely on the convention on bound and free names to avoid name capture. We write \((X)P \circ Q\) for the application of the abstraction \((X)P\) to the process \(Q\), and define it as follows.

\[ (X)P \circ Q \triangleq P\{Q/X\} \]

Let agents, noted \( A \), be the set of processes, abstractions, and concretions. We extend the parallel composition and restriction operators to all agents as follows.
Barbed congruence is the classic reduction-based behavioral equivalence. We define reduction \( \rightarrow \) as \( \equiv \tau \mapsto \equiv \) and weak reduction \( \Rightarrow \) as the reflexive and transitive closure of \( \rightarrow \). Observables \( \gamma \) of a process \( P \), written \( P \downarrow \gamma \), are unrestricted names or conames on which a communication may immediately occur. Contexts \( C \) are terms with a hole \( \Box \). A relation \( R \) is a congruence iff \( P R Q \) implies \( C \{ P \} R C \{ Q \} \) for all \( C \).

**Definition 1.** A symmetric relation on closed processes \( R \) is a strong barbed bisimulation iff \( P R Q \) implies:

- for all \( P \downarrow \gamma \), we have \( Q \downarrow \gamma \);
- for all \( P \rightarrow P' \), there exists \( Q' \) such that \( Q \rightarrow Q' \) and \( P' \ R Q' \).

Two processes \( P, Q \) are strong barbed congruent, written \( P \sim^b Q \), if for all \( C \) there exists a strong barbed bisimulation \( R \) such that \( C \{ P \} R C \{ Q \} \).

**Definition 2.** A symmetric relation on closed processes \( R \) is a weak barbed bisimulation iff \( P R Q \) implies:

- for all \( P \downarrow \gamma \), we have \( Q \Rightarrow \downarrow \gamma \);
- for all \( P \rightarrow P' \), there exists \( Q' \) such that \( Q \Rightarrow Q' \) and \( P' \ R Q' \).

Two processes \( P, Q \) are weak barbed congruent, written \( P \approx^b Q \), if for all \( C \) there exists a weak barbed bisimulation \( R \) such that \( C \{ P \} R C \{ Q \} \).

A relation \( R \) is sound with respect to another relation \( R' \) iff \( R \subseteq R' \); \( R \) is complete with respect to \( R' \) iff \( R' \subseteq R \). If \( R \) is both correct and complete with respect to \( R' \), then it characterizes \( R' \). In the following, we will be interested in relations that are at least correct with respect to strong or weak barbed congruence, and in relations that characterize them.

In [34], Sangiorgi proposed context bisimilarities as alternatives to barbed congruence.

**Definition 3.** Early strong context bisimilarity \( \sim \) is the largest symmetric relation on closed processes \( R \) such that \( P R Q \) implies:

- for all \( P \rightarrow P' \), there exists \( Q' \) such that \( Q \rightarrow Q' \) and \( P' \ R Q' \);
- for all \( P \overset{a.}{\rightarrow} F \), for all \( C \), there exists \( F' \) such that \( Q \overset{a.}{\rightarrow} F' \) and \( (F \bullet C) R (F' \bullet C) \).
• for all $P \xrightarrow{a} C$, for all $F$, there exists $C'$ such that $Q \xrightarrow{a} C'$ and $(F \cdot C) \mathcal{R} (F \cdot C')$.

Note. The late variant of strong context bisimulation can simply be obtained by changing the order of quantifications on concretions and abstractions in the above clauses. Thus the clause for input in late style would be:

for all $P \xrightarrow{a} F$, there exists $F'$ such that $Q \xrightarrow{a} F'$ and for all $C$, we have $(F \cdot C) \mathcal{R} (F' \cdot C)$.

As shown by Sangiorgi [33, 34], strong early context bisimilarity characterizes strong barbed congruence.

Theorem 1. We have $\sim = \sim_b$.

We now proceed to the weak case. We write $\tau = \Rightarrow$ for the reflexive and transitive closure of $\tau \rightarrow$. For every name or coname $\gamma$, we write $\gamma = \Rightarrow$ for $\tau = \Rightarrow \gamma \rightarrow$. As higher order steps result in concretions and abstractions, they may not reduce further; silent steps after this reduction are taken into account in the definition of weak simulation. We define early weak context bisimilarity as:

Definition 4. Early weak context bisimilarity $\approx$ is the largest symmetric relation on closed processes $\mathcal{R}$ such that $P \mathcal{R} Q$ implies:

• for all $P \xrightarrow{\tau} P'$, there exists $Q'$ such that $Q \approx Q'$ and $P' \mathcal{R} Q'$;

• for all $P \xrightarrow{a} F$, for all $C$, there exist $F'$, $Q'$ such that $Q \approx F'$, $F' \cdot C \approx Q'$, and $F \cdot C \mathcal{R} Q'$;

• For all $P \xrightarrow{a} C$, for all $F$, there exist $C'$, $Q'$ such that $Q \approx C'$, $F \cdot C' \approx Q'$ and $F \cdot C \mathcal{R} Q'$.

Sangiorgi proves soundness of $\approx$ in [34].

Theorem 2. We have $\approx \subseteq \approx_b$.

Using the same technique as in the $\pi$-calculus [36], one can prove that $\approx$ is also complete on image-finite processes.

Definition 5. A process $P$ is image finite iff

• the set $\{P', P \xrightarrow{\tau} P'\}$ is finite;

• for all $a, C$, the set $\{P', \exists F, P \xrightarrow{a} F \land (F \cdot C) \xrightarrow{\tau} P'\}$ is finite;

• for all $a, F$, the set $\{P', \exists C, P \xrightarrow{a} C \land (F \cdot C) \xrightarrow{\tau} P'\}$ is finite.

Theorem 3. We have $\approx_b \subseteq \approx$ on image-finite processes.

Context bisimulation may be understood as follows: when two tested processes $P$ and $Q$ perform a partial action, such as sending or receiving a message, the bisimulation considers every context which may complement the action. It is easier to manipulate than barbed congruence, since it features only one test in the internal action case. However, the universal quantification on concretions or abstractions makes the definition still unpractical to use. To address this issue, a simpler behavioral equivalence for HO$\pi$, called normal bisimulation, was invented by Sangiorgi.
2.2. Normal bisimulation

Normal bisimulation is a behavioral equivalence that tries to reduce the number of tests for each pair of processes under consideration. It relies on an encoding of HO\(\pi\) in a first-order \(\pi\)-calculus, leveraging the limited uses of a received process: whether to duplicate or discard it, and when to run or forward the copies. These behaviors can be simulated by replacing the process \(P\) by a name which is used as a trigger to create a copy of \(P\) when needed. Formally, we have the following factorization theorem.

**Theorem 4.** For every agent \(A\), process \(Q\), and name \(m\) with \(m \notin \text{fn}(A, Q)\), the agents \(A\{Q/X\}\) and \(\nu m.(A\{m;0/X\} | !m.Q)\) are weakly late context bisimilar.

The factorization theorem replaces a process \(Q\) by a trigger \(m;0\) that may activate a copy of \(Q\) on demand. This copy is provided by the associated process \(!m.Q\). Normal bisimulation relies on this translation to test equivalences of processes.

**Definition 6.** Normal bisimilarity is the largest symmetric relation on closed processes \(R\) such that \(P R Q\) implies:

- for all \(P \overset{\tau}{\rightarrow} P'\), there exists \(Q'\) such that \(Q \overset{\tau}{\Rightarrow} Q'\) and \(P' R Q'\);
- for all \(P \overset{a}{\rightarrow} F\), there exist \(F', Q'\) and a fresh name \(m\) such that \(Q \overset{a}{\Rightarrow} F'\), \(F' \circ m;0 \overset{\tau}{\Rightarrow} Q'\) and \(F \circ m;0 R Q'\);
- for all \(P \overset{a}{\rightarrow} \nu b.(R)S\), there exist a concretion \(\nu b'.(R')S'\), a process \(Q'\), and a fresh name \(m\) such that \(Q \overset{a}{\Rightarrow} \nu b'.(R')S'\), \(\nu b'.(S' | !m.R') \overset{\tau}{\Rightarrow} Q'\), and \(\nu b.(S | !m.R) R Q'\).

In the message input case, normal bisimilarity tests only a fresh trigger. In the message sending case, normal bisimilarity tests processes where the emitted processes \(R\) and \(R'\) are made available through a name \(m\). Using the factorization theorem and the fact that weak late context bisimulation is a congruence, Sangiorgi proved that normal bisimilarity coincides with weak late context bisimilarity. Cao [6] extended the result to the strong case.

To summarize, context bisimulation is a first step in finding a simple behavioral equivalence: it reduces only slightly the quantifications. Normal bisimulation goes much further as only one test is performed for each transition step of a process pair. We now study such relations for more expressive calculi.

2.3. Syntax and semantics of HO\(\pi\)P

We now study bisimulations in calculi with passivation capabilities as in Homer or Kell. Instead of working in Homer or Kell directly, we define a simpler calculus called HO\(\pi\) with Passivation (HO\(\pi\)P), which extends HO\(\pi\) with a passivation operator. By doing this we avoid the unnecessary features of Homer and Kell (mainly additional control on communication) and we are able to compare more directly bisimulations in HO\(\pi\) and HO\(\pi\)P.
We add localities \(a[P]\), that are passivation units, to the \(\text{HO}\pi\) constructs. With the same notations as for \(\text{HO}\pi\), the syntax of \(\text{HO}\pi P\) is as follows.

\[
P ::= 0 \mid X \mid P \mid a(X)P \mid \pi(P)P \mid \nu a.P \mid !P \mid a[P]
\]

When passivation is not triggered, a locality \(a[P]\) is a transparent evaluation context: process \(P\) may evolve by itself and communicate freely with processes outside of locality \(a\). At any time, passivation may be triggered and the process \(a[P]\) becomes a concretion \(\langle P \rangle 0\). Passivation may thus occur as an internal \(\tau\) step only if there is a receiver on \(a\) ready to receive the contents of the locality.

We extend localities to all agents: if \(F = (X)P\), then \(a[ F ] = (X)a[P]\); if \(C = \nu b.(Q)R\), then \(a[C] = \nu b.(Q)a[R]\). We also add the following rules to the labeled transition system.

\[
\frac{P \xrightarrow{\alpha} A}{a[P] \xrightarrow{\alpha} a[A]} \quad \text{LOC} \quad \frac{a[P] \xrightarrow{\pi} \langle P \rangle 0}{\text{PASSIV}}
\]

Note that rule LOC implies that the scope of restricted names may cross locality boundaries. Scope extrusion outside localities is performed “by need” when a communication takes place. Structural congruence follows the same rules as in \(\text{HO}\pi\) (Figure 2), and as a consequence does not allow the restriction and locality operators to commute freely. If it did, structurally congruent processes would not be contextually bisimilar. For instance, let \(Q = a[\nu b.P] \mid a(X) (X \mid X)\). It reduces to \((\nu b.P) \mid (\nu b.P)\) by triggering the passivation. If we allow the structural extrusion of \(\nu b\) across locality \(a\), we would have \(Q \equiv \nu b.(a[P] \mid a(X)(X \mid X))\), which evolves to \(\nu b.(P \mid P)\). In this case, the name \(b\) is shared by the two instances of \(P\), whereas each instance of \(P\) has its own name \(b\) in the first case. The two obtained processes may have different reductions. For example, assume that \(P = 5.0 \mid b.b.R\).

- In the first case, we have \((\nu b.(5.0 \mid b.b.R)) \mid (\nu b.(5.0 \mid b.b.R))\), which evolves to \((\nu b.b.R) \mid (\nu b.b.R)\). No further reduction is possible.
- In the second case, we get \(\nu b.(5.0 \mid 5.0 \mid b.b.R \mid b.b.R)\), which may evolve to \(\nu b.(R \mid b.b.R)\). All the reductions of \(R\) are possible.

2.4. Context bisimilarity

As in \(\text{HO}\pi\), our goal is to find a simple bisimulation-based characterization of barbed congruence. Observables for \(\text{HO}\pi P\) are unrestricted names or conames on which a communication or a passivation may immediately occur. The definition of strong barbed congruence is identical to Definition 1.

We now define a sound and complete context bisimulation for \(\text{HO}\pi P\) in the strong case. We first notice that the context bisimulation given by Sangiorgi for \(\text{HO}\pi\) (Definition 3) is not sound in our calculus because of passivation. More precisely, the \(\text{HO}\pi\) bisimilarity relates the following processes.

\[
P_0 = \pi(0)!m.0 \quad Q_0 = \pi(m.0)!m.0
\]
The differences between the emitted processes \(0\) and \(m.0\) are shadowed by the process \(!m.0\). More precisely, we have to check that for all \(F\), the processes \((F \cdot 0)!m.0\) and \((F \cdot m.0)!m.0\) are context bisimilar, i.e. for all \(R\), we have \(P' \overset{!}{=} R\{0/X\} \mid !m.0\) in relation with \(Q' \overset{!}{=} R\{m.0/X\} \mid !m.0\). We have three kinds of possible transitions from \(P'\):

- transitions from \(R\) alone: they are matched by the same transitions of \(R\) in \(Q'\);
- synchronizations between \(!m.0\) and \(R\) or \(m\rightarrow\)-transitions from \(!m.0\): they are matched by the same transitions in \(Q'\);
- synchronizations between the copies of the message \(m.0\) and \(R\) or \(m\rightarrow\)-transitions from the message: they are matched by synchronizations between \(!m.0\) and \(R\) or \(m\rightarrow\)-transitions from \(!m.0\) in \(Q'\).

Conversely the transitions of \(Q'\) are matched by \(P'\).

**Remark 2.** This result can be proven formally by considering the symmetric closure of relation \(\{(P\{m.0/X\} \mid !m.0, P\{0/X\} \mid !m.0\}\}\), and showing that this relation is an early strong bisimulation according to definition 3.

However \(P_0\) and \(Q_0\) are not barbed congruent in HO\(\pi\)P. The context \(C = b[\Box] \mid a(X)X \mid b(X)0\) distinguishes them. We have \(C\{P_0\} \rightarrow b[!m.0] \mid 0 \mid b(X)0 = P'\) by a communication on \(a\). This reduction is matched by \(C\{Q_0\} \rightarrow b[!m.0] \mid m.0 \mid b(X)0 = Q'\). By triggering the passivation on \(b\), we have \(P' \rightarrow 0\) and \(Q' \rightarrow m.0\). The two resulting processes are not barbed bisimilar.

In a concretion \(\nu a.(R)S\), the emitted process \(R\) may be sent outside a locality \(b\) while the continuation \(S\) stays in \(b\). If the passivation on \(b\) is triggered, \(S\) may be destroyed (as with \(P_0\) and \(Q_0\)) or put in a different context. Hence the passivation may separate the processes \(R\) and \(S\) and put them in totally different contexts, which is not possible in a calculus without passivation. As in Kell and Homer, we address this issue by testing messages and continuations in different evaluation contexts \(E\). These contexts, when applied to concretions, take into account the fact that a message and its continuation are separated: in the definition of \(a[C]\) for some concretion \(C\), the message part of \(C\) is put outside the locality whereas the continuation part remains inside. The grammar of HO\(\pi\)P evaluation contexts is:

\[E :: = \Box \mid \nu x.E \mid E \mid P \mid P \mid E \mid a[E]\]

We call these contexts used for observational purposes bisimulation contexts.

**Definition 7.** Early strong context bisimilarity \(\sim\) is the largest symmetric relation on closed processes \(R\) such that \(P \overset{a}{=} Q\) implies \(\text{fn}(P) = \text{fn}(Q)\) and:

- transitions from \(R\) alone: they are matched by the same transitions of \(R\) in \(Q\);
- synchronizations between \(!m.0\) and \(R\) or \(m\rightarrow\)-transitions from \(!m.0\): they are matched by the same transitions in \(Q\);
- synchronizations between the copies of the message \(m.0\) and \(R\) or \(m\rightarrow\)-transitions from the message: they are matched by synchronizations between \(!m.0\) and \(R\) or \(m\rightarrow\)-transitions from \(!m.0\) in \(Q\).
• for all \( P \xRightarrow{\tau} P' \), there exists \( Q' \) such that \( Q \xRightarrow{\tau} Q' \) and \( P' \mathcal{R} Q' \);

• for all \( P \xRightarrow{a} F \), for all \( C \), there exists \( F' \) such that \( Q \xRightarrow{a} F' \) and \( (F \cdot C) \mathcal{R} (F' \cdot C) \);

• for all \( P \xRightarrow{\pi} C \), for \( F \), there exists \( C' \) such that \( Q \xRightarrow{\nu b} C' \) and for all \( E \), we have \((F \cdot E\{C\}) \mathcal{R} (F \cdot E\{C'\})\).

This definition is similar to the ones for context bisimilarities in Homer [16] and Kell [38] (except that in Kell, contexts are also added in the abstraction case). The condition \( \text{fn}(P) = \text{fn}(Q) \) has been added because of lazy scope extrusion: two bisimilar processes with different free names may be distinguished. For instance, a process \( P \) which cannot perform any transition but with a free name \( b \) (e.g. \( v.a.b.0 \)) may be distinguished from \( 0 \) by a context \( C = c[vb.\bar{d}(\square)R] \mid d(X)c(Y)(Y \mid Y) \). The process \( C\{P\} \) may reduce to \( \nu b.(R \mid R) \), whereas the process \( C\{0\} \) evolves toward \((\nu b.R) \mid (\nu b.R) \). With an appropriate \( R \), the two processes have different transitions, as illustrated in Section 2.3.

**Example 1.** The tests within contexts \( E \) make the \( HO\pi P \) context bisimilarity \( \sim \) more discriminant than the \( HO\pi \) one. However, the relation \( \sim \) is still bigger than trivial equivalences, such as structural congruence \( \equiv \). For instance, the processes \( mL.0 \mid a[m.0] \mid a[0] \) and \( mL.0 \mid a[m.0] \mid a[0] \) are early context bisimilar but not structural congruent.

**Remark 3.** In the concretion case, one could imagine tests with localities \( F \cdot b[C] \), for a fresh name \( b \), instead of tests with bisimulation contexts \( F \cdot E\{C\} \). The two tests are almost equivalent, except that tests with contexts \( E \) allow capture of free names of \( C \). More precisely, let \( C = c[vb.\bar{d}(\square)R] \mid d(X)c(Y)(Y \mid Y) \). The process \( C\{P\} \) may reduce to \( \nu b.(R \mid R) \), whereas the process \( C\{0\} \) evolves toward \((\nu b.R) \mid (\nu b.R) \). With an appropriate \( R \), the two processes have different transitions, as illustrated in Section 2.3.

The definition of context bisimilarity is similar in the weak case.

**Definition 8.** Early weak context bisimilarity \( \approx \) is the largest symmetric relation on closed processes \( \mathcal{R} \) such that \( P \mathcal{R} Q \) implies:

• for all \( P \xRightarrow{\tau} P' \), there exists \( Q' \) such that \( Q \xRightarrow{\tau} Q' \) and \( P' \mathcal{R} Q' \);

• for all \( P \xRightarrow{a} F \), for all \( C \), there exist \( F',Q' \) such that \( Q \xRightarrow{a} F' \), \( (F' \cdot C) \mathcal{R} Q' \); and \( (F \cdot C) \mathcal{R} Q' \);

• For all \( P \xRightarrow{\pi} C \), for all \( F \), there exists \( C' \) such that \( Q \xRightarrow{\nu b} C' \) and for all \( E \), there exists \( Q' \) such that \( (F \cdot E\{C'\}) \mathcal{R} Q' \) and \( (F \cdot E\{C\}) \mathcal{R} Q' \).

In the following section, we discuss techniques developed for Kell and for Homer that can be used to show that context bisimulation is sound and complete in the strong case. We also explain why these techniques fail in the weak case with early context bisimilarity.
3. Congruence proofs

3.1. Kell soundness proof

As in HO\(\pi\), the soundness proof used for Kell relies on a substitution lemma.

**Lemma 1.** Let \(A\) be an agent and \(P, Q\) be processes; if \(P\) and \(Q\) are strong (respectively weak) context bisimilar, then \(A\{P/X\}\) and \(A\{Q/X\}\) are strong (respectively weak) context bisimilar.

The approach of [34] to prove this lemma in HO\(\pi\) can be summed up by:

- the result is proved for evaluation contexts (parallel composition, replication, and restriction);
- the result is proved for all processes, using the first step.

The distinction is useful since if \(A\) is an evaluation context, the reductions of \(A\{P/X\}\) may come from \(A\) or \(P\), whereas if \(A\) is not an evaluation context, \(P\) cannot be reduced. However, this method fails with HO\(\pi\)P. Unlike HO\(\pi\), an execution context in HO\(\pi\)P may become a non-execution context (a locality may become a message output preventing internal reductions).

More precisely, to show the first step of Sangiorgi’s method in the locality case, we would have to prove that if \(P \sim Q\), then \(a[P] \sim a[Q]\). We would thus have to build a relation \(R\) such that (assuming \(P \sim Q\)):

\[
\begin{array}{c}
a[P] \\
\pi \\
\end{array} \sim \begin{array}{c}
a[Q] \\
\pi \\
\end{array}
\]

\[
\begin{array}{c}
\langle P \rangle X \sim \\
\pi \\
\end{array}
\begin{array}{c}
\langle Q \rangle X \\
\pi \\
\end{array}
\]

and such that \(R\) is a bisimulation. Therefore for all abstractions \((X)R\), we would have \(R\{P/X\} \sim R\{Q/X\}\). To prove a sub-case of the substitution lemma, we would have to consider the relation \(R = \{(R\{P/X\}, R\{Q/X\}), P \sim Q\}\) and show that it is a bisimulation. But this would be the same as proving the substitution lemma directly, making the approach fail.

The method used for the Kell-calculus is the following one. For two finite sets of processes \(\tilde{P} = (P_i)_{i \in I}, \tilde{Q} = (Q_i)_{i \in I}\) of the same size and a relation \(R\), we write \(\tilde{P} \sim R \tilde{Q}\) if we have \(P_i \sim R Q_i\) for all \(i \in I\). We define a relation

\[
R = \{(C\{R[P/\tilde{Y}\}, C\{R[Q/\tilde{Y}\])\}, \text{fv}(R) = \tilde{Y}, \tilde{P} \sim \tilde{Q}\}
\]

and we show that its reflexive and transitive closure is a bisimulation. We assume now that we work with the early definition, but the proof technique works with the late one as well. The candidate relation requires contexts \(C\) in its definition to take into account name capture, which may happen in the message output tests because of bisimulation contexts \(E\).
We first explain why we work with the reflexive and transitive closure instead of the relation itself. To show that $R$ is a bisimulation, we proceed by structural induction on $C$, and we perform a nested induction on the derivation of the transition $C(R\{P/Y\}) \xrightarrow{\alpha} R'$. For any agent $A$ and processes $\tilde{P}$, we write $A\tilde{P}$ for $A\{\tilde{P}/\tilde{Y}\}$. Suppose $C = \square$ and consider the case where $R = R_1 \upharpoonright R_2$, and $R$ evolves by a higher-order communication. We want to close the following diagram

\[
\begin{array}{c}
R_1^1 \mid R_1^2 \xrightarrow{\kappa} R_1^1 \mid R_1^2 \\
\downarrow \tau \\
F_{\tilde{P}} \cdot C_{\tilde{P}^\square}
\end{array}
\]

knowing that $R_1^1 \xrightarrow{\alpha} F_{\tilde{P}}$ and $R_1^2 \xrightarrow{\pi} C_{\tilde{P}^\square}$ for some $\alpha$. By definition we have $R_1^1 \sqsubseteq R_1^1$ so by applying $C_{\tilde{P}^\square}$ to $F_{\tilde{P}}$ (we work with early bisimulation, hence we have to choose the concretion before getting a matching abstraction), we have by induction:

\[
\begin{array}{c}
R_1^1 \xrightarrow{\kappa} R_1^1 \\
\downarrow \alpha \\
\downarrow \alpha \\
F_{\tilde{P}} \xrightarrow{\kappa} F_{\tilde{Q}}
\end{array}
\]

with $F_{\tilde{P}} \cdot C_{\tilde{P}^\square} \sqsubseteq F_{\tilde{Q}} \cdot C_{\tilde{P}^\square}$. We have $R_1^2 \sqsubseteq R_2^2$, so by applying $C_{\tilde{P}^\square}$ to $F_{\tilde{Q}^\square}$ we have:

\[
\begin{array}{c}
R_2^1 \xrightarrow{\kappa} R_2^1 \\
\downarrow \pi \\
\downarrow \pi \\
\downarrow \pi \\
C_{\tilde{P}^\square} \xrightarrow{\kappa} C_{\tilde{Q}^\square}
\end{array}
\]

with $F_{\tilde{Q}^\square} \cdot C_{\tilde{P}^\square} \sqsubseteq F_{\tilde{Q}^\square} \cdot C_{\tilde{Q}^\square}$. From these we can conclude that:

\[
\begin{array}{c}
R_1^1 \mid R_1^2 \xrightarrow{\kappa} R_1^1 \mid R_1^2 \\
\downarrow \tau \\
\downarrow \tau \\
F_{\tilde{P}} \cdot C_{\tilde{P}^\square} \xrightarrow{\kappa} F_{\tilde{Q}^\square} \cdot C_{\tilde{P}^\square} \xrightarrow{\kappa} F_{\tilde{Q}^\square} \cdot C_{\tilde{Q}^\square}
\end{array}
\]

As a result, we have:

\[
\begin{array}{c}
R_{\tilde{P}} \xrightarrow{\kappa} R_{\tilde{Q}} \\
\downarrow \tau \\
\downarrow \tau \\
R'_{\tilde{P} \sqcup \tilde{P}^\square} \xrightarrow{\kappa^2} R'_{\tilde{Q} \sqcup \tilde{Q}^\square}
\end{array}
\]
while we need \( R'_{P \cup P'} R R'_{Q \cup Q'} \). More generally, we prove that \( R \) progresses towards its reflexive and transitive closure \( R^* \), in the sense of \[36\].

**Definition 9.** Let \( R, S \) two relations on processes. The relation \( R \) strongly progresses towards \( S \) in an early style iff \( P R Q \) implies \( \text{fn}(P) = \text{fn}(Q) \) and:

- for all \( P \mapsto P' \), there exists \( Q' \) such that \( Q \mapsto Q' \) and \( P S Q' \);
- for all \( P \xrightarrow{a} F \) and all \( C \), there exists \( F' \) such that \( Q \xrightarrow{a} F' \) and \( F \bullet C S F' \bullet C \);
- for all \( P \xrightarrow{i} F \) and all \( F \), there exists \( C' \) such that \( Q \xrightarrow{i} C' \) and for all \( E \), we have \( F \bullet E \{C\} S F \bullet E \{C'\} \).

Using a diagram, we have:

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
P \xrightarrow{R} Q \\
\downarrow \alpha \\
A \xrightarrow{R^*} A'
\end{array}
\end{array}
\end{array}
\end{array}
\]

In the strong case, it is sufficient to show that \( R^* \) is a bisimulation. Suppose that \( P R^* Q \) and \( P \xrightarrow{a} A' \). There exists \( P_1, \ldots, P_n \) such that \( P R P_1 R \ldots P_n R Q \). We want to close the following diagram:

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
P \xrightarrow{R} P_1 \ldots P_n \xrightarrow{R} Q \\
\downarrow \alpha \\
A
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\]

Since \( R \) progress towards \( R^* \), we build \( A_1 \ldots A_n, A' \) such that \( A R^* A_1 R^* \ldots A_n R^* A' \).

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
P \xrightarrow{R} P_1 \ldots P_n \xrightarrow{R} Q \\
\downarrow a \quad | a \quad | a \quad | a \\
A \xrightarrow{R^*} A_1 \ldots A_n \xrightarrow{R^*} A'
\end{array}
\end{array}
\end{array}
\end{array}
\end{array}
\]

Since \( R^* \) is transitive, we have \( A R^* A' \) as required. The soundness proof using the Kell-calculus technique can be found in \[22\]. This approach fails in the weak case. Suppose now we have \( P R Q \) (where \( R \) is the congruence closure of the weak bisimilarity \( \approx \)) and \( P \xrightarrow{i} P' \). We want to close the following diagram:

\[
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
\begin{array}{c}
P \xrightarrow{R} P_1 \ldots P_n \xrightarrow{R} Q \\
\downarrow i \\
P'
\end{array}
\end{array}
\end{array}
\end{array}
\]
We use the fact that $\mathcal{R}$ progress towards $\mathcal{R}^*$ for $P, P', P_1$. Suppose that for instance, $P_1$ performs at least two internal actions.

$$\begin{array}{c}
P \xrightarrow{\mathcal{R}} P_1 \xrightarrow{\mathcal{R}} P_2 \ldots \xrightarrow{\mathcal{R}} P_n \xrightarrow{\mathcal{R}} Q \\
\downarrow | \tau \\
\downarrow \uparrow P_{12} \\
\tau \downarrow | \tau \\
\downarrow \uparrow P_{13} \\
\downarrow | \uparrow P_n \\
\downarrow \uparrow P' \xrightarrow{\mathcal{R}^*} P_1'
\end{array}$$

We close the sub-diagram $P_1, P_{12}, P_2$:

$$\begin{array}{c}
P \xrightarrow{\mathcal{R}} P_1 \xrightarrow{\mathcal{R}} P_2 \ldots \xrightarrow{\mathcal{R}} P_n \xrightarrow{\mathcal{R}} Q \\
\downarrow | \tau \parallel | \tau \\
\downarrow \uparrow P_{12} \xrightarrow{\mathcal{R}^*} P_{22} \\
\tau \downarrow | \tau \\
\downarrow \uparrow P_{13} \\
\downarrow | \uparrow P_n \\
\downarrow \uparrow P' \xrightarrow{\mathcal{R}^*} P_1'
\end{array}$$

Hence we have $P_{12} \xrightarrow{\mathcal{R}^*} P_{22}$ and $P_{12} \xrightarrow{\tau} P_{13}$. The diagram $P, Q, P'$ we want to close may be smaller than $P_{12}, P_{22}, P_{13}$. The scheme may then recursively and infinitely repeat itself. Knowing that $\mathcal{R}$ progress towards $\mathcal{R}^*$ does not allow to prove that $\mathcal{R}^*$ is a bisimulation in the weak case. This problem is similar to the application of up-to techniques in the weak case [29]. Hence we cannot show that the early bisimulation is a congruence in the weak case with this technique.

**Remark 4.** We have the same results with the late bisimulation: we can prove that the late bisimulation is a congruence in the strong case, but the weak one remains an open problem.

**Remark 5.** On the contrary the method used by Sangiorgi may easily be adapted in the weak case for $\text{HO}_\pi$ without passivation. Transitivity issues are dealt with by using up-to techniques mixing strong and weak bisimilarities. See [34] for further details.
3.2. Howe's Method

Howe’s method [17, 1, 14] is a systematic proof technique to show that a simulation \( R \) is a congruence. The method can be divided in three steps: first, prove some basic properties on the Howe’s closure \( R^* \) of the relation. By construction, \( R^* \) contains \( R \) and is a congruence. Second, prove a simulation-like property for \( R^* \), and finally prove that \( R \) and \( R^* \) coincide on closed processes. Since \( R^* \) is a congruence, conclude that \( R \) is a congruence.

The definition of the Howe’s closure relies on the open extension of \( R \), noted \( R^o \): it extends the definition of the relation \( R \) to open processes, that are processes with free process variables.

**Definition 10.** Let \( P \) and \( Q \) be two open processes. We have \( P \sim R Q \) iff \( P\sigma R Q\sigma \) for all process substitutions \( \sigma \) that close \( P \) and \( Q \).

Howe’s closure is inductively defined as the smallest congruence which contains \( R^o \) and is closed under right composition with \( R^o \).

**Definition 11.** Howe’s closure \( R^* \) of a relation \( R \) is the smallest relation verifying:

- \( R^o \subseteq R^* \);
- \( R^* R^o \subseteq R^* \);
- for all operators \( op \) of the language, if \( \tilde{P} R^* \tilde{Q} \), then \( op(\tilde{P}) R^* op(\tilde{Q}) \).

By definition, \( R^* \) is a congruence, and the composition with \( R^o \) allows some transitivity and gives some additional properties to the relation.

**Remark 6.** In the literature (e.g., [17, 14, 16]) Howe’s closure is usually inductively defined by the following rule for all operators \( op \) in the language:

\[
\frac{P \sim R \quad \tilde{P} R^* \tilde{Q} \quad op(\tilde{P}) R^o Q}{\tilde{P} \sim R^* \tilde{Q} \quad op(P) R^* Q}
\]

Both definitions are equivalent (see [14] for the proof). We believe that Definition 11 is easier to understand and to work with in proofs.

In our case, we want to prove that a bisimilarity \( B \) is a congruence. By definition, we have \( B^o \subseteq B^* \). To have the reverse inclusion, we prove that \( B^* \) is a bisimulation. To this end, we need the following classical properties of the Howe’s closure.

**Lemma 2.** Let \( R \) be a reflexive relation. If \( P \sim R \) and \( R^* \) \( S \), then we have \( P\{R/X\} \sim R^* Q \{S/X\} \).

This lemma is typically used to establish the simulation-like result (second step of the method). We sketch the proof in order to give an idea on why the transitive item \( R^* R^o \subseteq R^* \) is needed in Definition 11. The proof is by induction on the derivation of \( P \sim R \). Suppose we have \( P \sim R^o \). Since \( R \) \( S \) and \( R^* \),
is a congruence, we have $P\{R/X\} R^* P\{S/X\}$. Let $\sigma$ be a substitution that closes $P$ and $Q$ except for $X$; by open extension definition, we have $P\{S/X\} \sigma R Q\{S/X\} \sigma$, i.e., we have $P\{S/X\} R \sigma Q\{S/X\}$. Finally we have $P\{R/X\} R^* R \sigma Q\{S/X\}$, hence we have $P\{R/X\} R^* Q\{S/X\}$. The other cases are easy using the induction hypothesis.

**Remark 7.** One may define Howe’s closure with $R \circ R \subseteq R \circ$ as the transitive item instead of $R \circ R \subseteq R \circ$. However left relation composition with $R \circ$ raises issues when proving weak simulation properties, while right relation composition works in the strong and weak cases.

We cannot prove directly that $B^*$ is symmetric. Instead we use the following lemma.

**Lemma 3.** Let $R$ be an equivalence. Then the reflexive and transitive closure $(R^*)^*$ of $R^*$ is symmetric.

**Proof.** By proving by induction that $P(R^*)^{-1} Q$ implies $P(R^*)^* Q$ for all $P, Q$.

Then one proves that the restriction of $(B^*)^*$ to closed terms is a bisimulation, Consequently we have $B \subseteq B^* \subseteq (B^*)^* \subseteq B$ on closed terms, and we conclude that $B$ is a congruence.

The main difficulty lies in the proof of the simulation-like property for Howe’s closure. In the following subsection, we explain why we cannot directly use Howe’s method with early context bisimilarity (Definitions 3 and 7).

### 3.3. Communication Problem

Proving that a congruence is a simulation raises transitivity issues, as we can see with the Kell proof method (Section 3.1). To avoid this problem, we establish a stronger result. Given a bisimilarity $B$ based on a LTS $P \xrightarrow{\lambda} A$, the simulation-like result follows the pattern below, similar to a higher-order bisimilarity clause, such as the one for Plain CHOCS [40].

Let $P B^* Q$. IF $P \xrightarrow{\lambda} A$, then for all $\lambda B^* \lambda', \text{there exists } B \text{ such that } Q \xrightarrow{\lambda'} B$ and $A B^* B$.

Early bisimulations are those where all the information about a step on one side is known before providing a matching step. In the higher-order setting with concretions and abstractions, it means that when an output occurs, the abstraction that will consume the output is specified before the matching step is given. In fact, the matching step may very well be different for a given output when the abstraction considered is different. Symmetrically, in the case of an input, the matching step is chosen depending on the input and the actual concretion that is provided. In both cases, this amounts to putting the abstraction in the label in the case of an output, and the concretion in the label in case of an input. One is thus lead to prove the following simulation property.
Conjecture 1. If $P \mathcal{R}^* Q$, then:

- for all $P \xrightarrow{\tau} P'$, there exists $Q'$ such that $Q \xrightarrow{\tau} Q'$ and $P' \mathcal{R}^* Q'$;

- for all $P \xrightarrow{a} F$, for all $C \mathcal{R}^* C'$, there exists $F'$ such that $Q \xrightarrow{a} F'$ and $F \cdot C \mathcal{R}^* F' \cdot C'$;

- for all $P \xrightarrow{\pi} C$, for all $F \mathcal{R}^* F'$ there exists $C'$ such that $Q \xrightarrow{\pi} C'$ and for all $E$, we have $F \cdot E \{C\} \mathcal{R}^* F' \cdot E \{C'\}$.

These clauses raise several issues. First, we have to find extensions of Howe’s closure to abstractions and concretions which fit an early style. Even assuming such extensions, there are issues in the inductive proof of conjecture 1 with higher-order communication. The reasoning is by induction on $P \mathcal{R}^* Q$. Suppose we are in the parallel case, i.e., we have $P = P_1 \parallel P_2$ and $Q = Q_1 \parallel Q_2$, with $P_1 \mathcal{R}^* Q_1$ and $P_2 \mathcal{R}^* Q_2$. Suppose that we have $P \xrightarrow{\tau} P'$, and the transition comes from rule HO: we have $P_1 \xrightarrow{a} F$, $P_2 \xrightarrow{\pi} C$ and $P' = F \cdot C$. We want to find $Q'$ such that $Q \xrightarrow{\tau} Q'$ and $P' \mathcal{R}^* Q'$. We also want to use the same rule HO, hence we have to find $F', C'$ such that $Q \xrightarrow{a} F' \cdot C'$. However we cannot use the input clause of the induction hypothesis with $P_1, Q_1$: to have a $F'$ such that $Q_1 \xrightarrow{a} F'$, we have to find first a concretion $C'$ such that $C \mathcal{R}^* C'$. We cannot use the output clause with $P_2, Q_2$ either: to have a $C'$ such that $Q_2 \xrightarrow{\pi} C'$, we have to find first an abstraction $F'$ such that $F \mathcal{R}^* F'$. We cannot bypass this mutual dependency and the inductive proof of conjecture 1 fails.

Remark 8. Note that the reasoning depends more on the bisimilarity than on the calculus: the same problem occurs with early context bisimilarities for $H\mathcal{O}\pi$, Homer, and the Kell calculus.

A simple way to break the mutual dependency between concretions and abstractions is to give up on the early style. An approach, used in [13], is to change the output case to a late style (hence the name, input-early, of their bisimulation): an output is matched by another output independently of the abstraction that receives it. This breaks the symmetry and allows us to proceed forward: first find the matching output $C'$, then for this $C'$ find the matching input using the input-early relation $\sim_{ie}$. Howe’s closure is then extended to concretions $C \sim_{ie} C'$ and a simulation-like property similar to Conjecture 1 is shown, except that the output clause is changed into:

- for all $P \xrightarrow{\pi} C$, there exists $C'$ such that $Q \xrightarrow{\pi} C'$ and $C \sim_{ie} C'$.

However, in the weak case, this input-early approach does not result in a sound and complete characterization of weak barbed congruence. Definition of weak input-early bisimilarity has to be written in the delay style: internal actions are not allowed after a visible action. The delay style is necessary to keep the concretion clause independent from abstractions. It is not satisfactory since delay bisimilarities are generally not complete with respect to weak barbed congruence.
We thus propose a different approach, detailed in Section 4, that works with weak bisimulations defined in the early non-delay style. In our solution, the output clause is not late, just a little less early. More precisely, instead of requiring the abstraction before providing a matching output, we only require the process that will do the reception (that will reduce to the abstraction). This may seem a very small change, yet it is sufficient to break the symmetry. We return to the communication problem where $P_1 | P_2$ is in relation with $Q_1 | Q_2$. The concretion $C'$ from $Q_2$ matching the $P_2 \overset{\lambda}{\rightarrow} C$ step depends only on $P_1$, which is known, and not on some unknown abstraction. We can then obtain the abstraction $F'$ from $Q_2$ that matches the $P_1 \overset{a}{\rightarrow} F$ step. This abstraction depends fully on $C'$, in the usual early style. Technically, we do not use concretions and abstractions anymore. In the LTS, when a communication between $P$ and $Q$ occurs, this becomes a transition from $P$ using $Q$ as a label (rule HO in Fig. 4). Higher in the derivation, the actual output from $P$ is discovered, and we switch to dealing with the input knowing exactly the output (rule OUT in Fig. 5). The proof of the bisimulation property for the candidate relation relies on this serialization of the LTS, which illustrates the break in the symmetry. On the other hand, the gap between a completely early relation and this one is small enough to let us prove that they actually coincide.

### 4. Complementary semantics for HO\(\pi\)

We now propose a new semantics for HO\(\pi\) that coincide with the contextual one yet allow the use of Howe’s method to prove soundness of early bisimilarities.

#### 4.1. Complementary LTS

We define a LTS $P \overset{\lambda}{\rightarrow} P'$ where processes always evolve towards other processes. We have three kinds of transitions: internal actions $P \overset{\lambda}{\rightarrow} P'$, message input $P \overset{a.R}{\rightarrow} P'$, and message output $P \overset{\pi.Q}{\rightarrow} P'$. We call this new LTS
complementary since in the output action, we put the context which complements $P$ in the label $\lambda$ of the transition. For higher-order labels, we define $n(a, R) = n(\pi, R) = a$. Rules of the LTS can be found in Figure 3, except for the symmetric of rules PAR$^\pi$ and HO$^\pi$.

Rules for internal actions $P \xrightarrow{a,R} P'$ are similar to the one for the contextual LTS $P \xrightarrow{\tau} P'$, except for higher-order communication since we change the message output judgement; we detail rule HO$^\pi$ later. Message input $P \xrightarrow{a,R} P'$ means that process $P$ may receive the process $R$ as a message on $a$ and becomes $P'$. In the contextual style, it means that $P \xrightarrow{a} F$ and $P' = F \circ R$ for some $F$; complementary message input is just a contextual message input written in the early style.

The main difference is in how we define output actions. The transition $P \xrightarrow{a,R} P'$ means that $P$ may send a message on $a$, $R$ may receive on $a$, and the communication on $a$ between $P$ and $R$ results in $P'$. It is not the same as writing contextual transition $P \xrightarrow{\pi} C$ in an early style; instead of putting an abstraction $F$ in the label, we put a process $R$. The transition $P \xrightarrow{\pi,R} P'$ means that there exists $F, C$ such that $P \xrightarrow{\pi} C$, $R \xrightarrow{\tau} F$, and $P' = F \bullet C$.

Rules of the LTS (Figure 3) are classic except rules HO$^\pi$ and OUT$^\pi$. In rule HO$^\pi$, the premise $P \xrightarrow{\pi,Q} P'$ means that $P$ and $Q$ can communicate on a name $a$ and the result is $P'$, i.e. $P \parallel \pi \xrightarrow{\tau} P'$ (by communication on $a$), which is exactly what conclusion of the rule states. Rule OUT$^\pi$ has a premise (unlike its equivalent rule CONCR) since in the conclusion we need the result $Q'$ of the input of $R$ on $a$ by $Q$.

The complementary LTS has the same semantics as the contextual LTS, as stated in the following lemma:

**Lemma 4.** Let $P$ be an HO$\pi$ process.

- We have $P \xrightarrow{\tau} P'$ iff $P \xrightarrow{\tau} P'$.
- If $P \xrightarrow{a} F$, then for all $R$ we have $P \xrightarrow{a,R} F \circ R$. If $P \xrightarrow{a,R} P'$, then there exists $F$ such that $P \xrightarrow{a} F$ and $P' = F \circ R$.
- If $P \xrightarrow{\pi} C$, then for all $R$ such that $R \xrightarrow{a} F$, we have $P \xrightarrow{\pi,R} = F \bullet C$. If $P \xrightarrow{\pi,R} P'$, then there exist $F, C$ such that $P \xrightarrow{\pi} C$, $R \xrightarrow{\tau} F$, $P' \equiv F \bullet C$.

The correspondence is up to $\equiv$ because of scope extrusion. The contextual LTS performs scope extrusion iff the name belongs to the free names of the message, while the complementary LTS always performs scope extrusion. For instance, for $P = a(X)X \parallel \nu b.\pi(c.0)b.0$, we have $P \xrightarrow{\tau} c.0 \parallel \nu b.0$ and $P \xrightarrow{\nu b.0} \nu b.0$.

### 4.2. Complementary Bisimilarity

We now define complementary bisimilarity and prove its soundness using Howe’s method. The result in itself, i.e., the definition of a sound bisimilarity in HO$\pi$, is far from being a new one [33, 34]. However, it allows us to explain
why complementary semantics is well suited to apply Howe’s method. Strong complementary bisimilarity for $\text{HO}\pi$ is simply the bisimilarity associated to the complementary LTS.

**Definition 12.** Strong complementary bisimilarity $\sim_m$ is the largest symmetric relation on closed processes $\mathcal{R}$ such that $P \mathcal{R} Q$ and $P \xrightarrow{\lambda} P'$ implies $Q \xrightarrow{\lambda} Q'$ with $P' \mathcal{R} Q'$.

As in context bisimilarity, in the message output case $P \xrightarrow{\lambda} P'$, the matching transition $Q \xrightarrow{\lambda} Q'$ still depends on a receiving entity (here $R$). However, instead of considering a context which directly receives the message (an abstraction $F$), we consider a process $R$ which evolves toward an abstraction. This nuance allows us to use Howe’s method to prove soundness of $\sim_m$. We extend $\sim_m$ to labels $\lambda$: we have $\lambda \sim_m \lambda'$ iff $\lambda = \lambda'$ or $\lambda = (\gamma, R), \lambda' = (\gamma, R')$ with $R \sim_m R'$. We prove the following simulation-like property for $\sim_m$:

**Lemma 5.** Let $P, Q$ be closed processes. If $P \sim_m Q$ and $P \xrightarrow{\lambda} Q'$, then for all $\lambda \sim_m \lambda'$, there exists $Q'$ such that $Q \xrightarrow{\lambda'} Q'$ and $P' \sim_m Q'$.

We do not have the same problem as in Section 3.3 with higher-order communication. We remind that in this case, we have $P_1 | P_2 \sim_m Q_1 | Q_2$ with $P_1 \sim_m Q_1$, $P_2 \sim_m Q_2$ and $P_1 \xrightarrow{\pi_{P_2}} P'$. We can apply directly the message output clause of the induction hypothesis: there exists $Q'$ such that $Q_1 \xrightarrow{\lambda'} Q'$ and $P' \sim_m Q'$. We conclude that $Q_1 | Q_2 \sim_m Q'$ (by rule $\text{HO}\pi$) with $P' \sim_m Q'$ as wished.

**Theorem 5.** Relation $\sim_m$ is a congruence.

Following the correspondence result between the two LTS (Lemma 4), we now prove that the bisimilarities have the same discriminating power. The differences in the message output clauses are covered mainly with Lemma 4. The bisimilarities differ also in how they deal with input actions: complementary bisimilarity tests with a process while context bisimilarity tests with a concretion. Testing with all concretions includes tests with $(P)0$, which are the same as tests with $P$ (up to $\equiv$). Consequently one inclusion is easy to establish:

**Lemma 6.** We have $\sim \subseteq \sim_m$.

The proof is done by showing that $\sim$ is a strong complementary bisimilarity (up to $\equiv$). The reverse inclusion requires the congruence result on $\sim_m$ (Theorem 5).

**Lemma 7.** We have $\sim_m \subseteq \sim$.

We prove the inclusion by showing that $\sim_m$ is an early strong context bisimulation (up to $\equiv$). In the message input case, we have roughly $P'(R/X) \sim_m Q'(R/X)$; by congruence it implies that $\nu\overline{b}.(P'(R/X) | S) \sim_m \nu\overline{b}.(Q'(R/X) | S)$. 

23
\[
\begin{align*}
a(X)P & \xrightarrow{a,R} P\{R/X\} & \quad P \xrightarrow{\mu,} P' & \quad \text{IN}_i^P \\
P & \xrightarrow{\nu} P' & \quad \nu \neq n(\mu) & \quad \text{RESTR}_{i\tau}^P \\
!P & \xrightarrow{\pi,} !P' & \quad \text{REPLIC-HO}^P \\
P & \xrightarrow{\mu} P' & \quad \text{REPLIC}^P_{i\tau} \\
\mu[P] & \xrightarrow{\nu} \mu[P'] & \quad \text{Loc}_{i\tau}^P \\
P & \xrightarrow{\pi,Q,E} P' & \quad \text{HO}^P \\
P & \xrightarrow{\pi,E} P' & \quad \text{PAR}_{i\tau}^P \\
\end{align*}
\]

Figure 4: Complementary LTS for HO\(\pi\)P: Internal and Message Input Actions

\(S\), i.e., \((X)P' \bullet \nu\tilde{b}.\langle R\rangle S \sim_m (X)Q' \bullet \nu\tilde{b}.\langle R\rangle S\). With Theorem 5, tests with processes are as discriminatory as tests with concretions.

We can also define complementary semantics and bisimilarity in the weak case; see [21] for definitions and results. We give more details on the weak case for HO\(\pi\)P (Section 5.2).

5. Application to HO\(\pi\)P

5.1. Complementary LTS

As in Section 4, we define a complementary semantics which considers processes instead of abstractions in the message output case. However, there are two additional issues with HO\(\pi\)P. First, we have to include bisimulation contexts \(E\) since they appear in bisimilarity definitions (Definitions 7 and 8). Second, scope extrusion matters more than in HO\(\pi\), since scope of restricted names may cross locality boundaries by communication but not by structural congruence. We cannot always extrude names and still have an equivalent semantics (up to \(\equiv\)) as in HO\(\pi\).

We let \(\lambda\) range over all complementary labels. Internal actions \(P \xrightarrow{\tau} P'\) and message input \(P \xrightarrow{a,R} P'\) are similar to the HO\(\pi\) complementary transitions, except that we have to add rules for localities. We write \(\xrightarrow{\tau} \cup a,R\) for \(\xrightarrow{\tau} \cup a,R\). Rules can be found in Figure 4 except for the symmetric counterpart of rules \(\text{PAR}_{i\tau}^P\) and HO\(\pi\). Rule HO\(\pi\) relies on message output transitions and is explained later.

In HO\(\pi\)P, context bisimilarities test a message output with an abstraction \(F\) and a bisimulation context \(E\). As in HO\(\pi\), complementary output actions \(P \xrightarrow{\pi,Q,E} P'\) consider a receiving process \(Q\) instead of \(F\). We have to add contexts \(E\) in our labels to keep the same discriminating power, and we also use a set of names \(\tilde{b}\) to deal with scope extrusion. Transition \(P \xrightarrow{\pi,Q,E} P'\) means that \(P\) is put under context \(E\) and emits a message on \(a\), which is received by \(Q\),
\[
\begin{align*}
\text{fn}(R) = \bar{b} & \quad Q \xrightarrow{\pi.R} Q' \quad \text{bn}(E) \cap \bar{b} = \emptyset \quad \text{OUT}_o \\
\pi(R) & \xrightarrow{\pi.Q.E} Q' \mid E\{S\}
\end{align*}
\]

\[
\begin{align*}
\text{fn}(P) = \bar{b} & \quad Q \xrightarrow{b.P} Q' \quad \text{bn}(E) \cap \bar{b} = \emptyset \quad \text{PASSIV}_o \\
\bar{b}[P] & \xrightarrow{\bar{b}.Q.E} Q' \mid E\{0\}
\end{align*}
\]

\[
\begin{align*}
P_1 \xrightarrow{\pi.Q.E(\square[P_2])} \bar{b} P' & \quad \text{PAR}_o \\
P_1 | P_2 & \xrightarrow{\pi.Q.E} \bar{b} P'
\end{align*}
\]

\[
\begin{align*}
P \xrightarrow{\pi.Q.E(\square[F])} \bar{b} P' & \quad \text{LOC}_o \\
\bar{b}[P] & \xrightarrow{\pi.Q.E} \bar{b} P'
\end{align*}
\]

\[
\begin{align*}
P \xrightarrow{\pi.Q.E(\nu.c.F)} \bar{b} P' & \quad \text{RETR}_o \\
\nu.c.P & \xrightarrow{\nu.c.\bar{b}.\nu.c.P'}
\end{align*}
\]

\[
\begin{align*}
P \xrightarrow{\pi.Q.E(\nu.c.F)} \bar{b} P' & \quad \text{CFREE}_o \\
P \xrightarrow{\pi.Q.E(\nu.c.F)} \bar{b} P'
\end{align*}
\]

\[
\begin{align*}
P \xrightarrow{\pi.Q.E(\nu.c.F)} \bar{b} P' & \quad \text{CAPT}_o \\
P \xrightarrow{\pi.Q.E(\nu.c.F)} \bar{b} P'
\end{align*}
\]

Figure 5: Complementary LTS for \(\text{HO}\pi\pi\): Message Output Actions
i.e. we have $E \{ P \} \mid Q \xrightarrow{a} P'$ by communication on $a$. In the contextual style, it means that there exists $F, C$ such that $P \xrightarrow{\pi C} C, Q \xrightarrow{\omega} F$, and $P' = F \cdot E \{ C \}$. Output rules can be found in Figure 5, except for the symmetric of rule $\text{PAR}_o^\circ$.

Scope extrusion may happen in the process under consideration (e.g., $P = \nu c.\pi(S)\mid \nu c.\pi(R)$) or because of the bisimulation context $E$ (e.g., $P = \pi(R)\mid E = d[\nu c.(\Box \mid c.0)]\text{ with } c \in \text{fn}(R)$). We first define auxiliary transitions $P \xrightarrow{\nu c.E} P'$, where we do not allow the latter kind of capture, and we then give rules for general output transitions.

Rule $\text{OUT}^\circ_o$ deals with message output $\pi(R)S \xrightarrow{\pi.Q.E} E\{S\} \mid Q'$. Premise $Q \xrightarrow{a.R} Q'$ checks that $Q$ may receive $R$ on $a$, and the resulting process $Q'$ is run in parallel with the continuation $S$ under context $E$. We check that that $E$ does not capture free names of $R$ with the side-condition $\text{bn}(E) \cap b = \emptyset$. We keep the free names $\tilde{b}$ of $R$ in the label for potential scope extrusion.

For instance, let $P = \pi(R)S$ and $c \in \text{fn}(R)$. Process $\nu c.P$ may emit $R$ on $a$, but the scope of $c$ has to be expanded to encompass the recipient of $R$. First premise of rule $\text{EXTR}^\circ_o$ checks that $P$ may output a message; here we have $\pi(R)S \xrightarrow{\pi.Q.E} E\{S\} \mid Q'$ with $\tilde{b} = \text{fn}(R)$. In conclusion, we have $\nu c.\pi(R)S \xrightarrow{\nu c.\pi(E\{S\})} \nu c.\pi(Q')$. Scope of $c$ includes $Q'$ as wished. For a concretion $C = v\tilde{a}.(P_1)P_2$, the names $\tilde{b}_C$ that may be extruded are the free names of $P_1$ which are not already bound in $\tilde{a}$, i.e. $\tilde{b}_C = \text{fn}(P_1) \setminus \tilde{a}$.

Suppose now that $P = \pi(R)S$ with $c \notin \text{fn}(R)$. Process $\nu c.P$ may emit a message, but the scope of $c$ has to encompass the continuation $S$ only: we want to obtain $\nu c.P \xrightarrow{\pi.Q.E} E\{\nu c.S\} \mid Q'$. To this end, we consider $P \xrightarrow{\pi.Q.E} P'$ as a premise of rule $\text{RESTR}^\circ_o$. In process $P'$, the continuation is put under $E\{\nu c.\Box\}$, hence we obtain $\pi(R)S \xrightarrow{\{\nu c.\Box\}} E\{\nu c.S\} \mid Q' = P'$, as expected and reflected in the conclusion of the rule.

Rule for passivation $\text{PASSIV}^\circ_o$ is similar to rule $\text{OUT}^\circ_o$, while rules $\text{LOC}^\circ_o$, $\text{PAR}^\circ_o$, $\text{REP}^\circ_o$ follow the same pattern as rule $\text{RESTR}^\circ_o$. Rule $\text{CFREE}^\circ_o$ simply means that a transition with a capture-free context is a message output transition. We now explain how to deal with context capture with rule $\text{CAPT}^\circ_o$. Suppose $P = \pi(R)S$ and $E' = d[\nu c.\Box \mid c.0]$ with $c \in \text{fn}(R)$; we want to obtain $P \xrightarrow{\pi.Q.E'} \nu c.d[\Box \mid c.0] \mid Q'$ (with the scope of $c$ extended out of $d$). We first consider the transition $P \xrightarrow{\pi.Q.E\{F\}} P'$ without capture on $c$; in our case we have $P \xrightarrow{\pi.Q.d[\Box]} d[S \mid c.0] \mid Q' = P'$ with $E = d[\Box]$ and $F = \Box \mid c.0$. Using the rule we have $P \xrightarrow{\pi.Q.E} \nu c.P'$, i.e. $P \xrightarrow{\pi.Q.E} \nu c.(d[S \mid c.0] \mid Q')$. The scope of $c$ is extended outside $E$ and includes the recipient of the message as wished.

Premise $P \xrightarrow{\nu c.E} P'$ of rule $\text{HO}^\circ_o$ (Figure 4) means that process $P$ sends a message on $a$ to $Q$ without any context around $P$, and the result is $P'$. Consequently we have $P \xrightarrow{Q} P'$ by communication on $a$, which is the expected
conclusion. Names $\bar{b}$ may no longer be potentially extruded, so we simply forget them.

5.2. Complementary Bisimilarities

We only give definitions and results, and point out the differences with HO\(\pi\) (Section 4.2). Strong complementary bisimilarity is defined as follows.

**Definition 13.** Strong complementary bisimilarity $\sim_m$ is the largest symmetric relation on closed processes $R$ such that $P R Q$ implies $\text{fn}(P) = \text{fn}(Q)$ and for all $P \xrightarrow{\pi.T.E} P'$, there exists $Q \xrightarrow{\pi.T.E} Q'$ such that $P' R Q'$.

To prove the simulation-like result, we have to extend Howe’s closure to bisimulation contexts: we define $E \sim_m F$ as the smallest congruence that contains $\sim_m$ and rule $\sim_m$. Except for this point, Howe’s method is easy to apply.

**Theorem 6.** Relation $\sim_m$ is a congruence and is sound with respect to $\sim_b$.

Correspondence with context bisimilarity is more problematic than in HO\(\pi\). We have two major differences. First, the output clause of complementary bisimilarity requires that transition $P \xrightarrow{\pi.T.E} P'$ has to be matched by a transition $Q \xrightarrow{\pi.T.E} Q'$ with the same set of names $\bar{b}$ which may be extruded. At first glance, we do not have this requirement for the early strong context bisimilarity, hence we have to prove that it is the case. For a concretion $C = \nu b. \langle R \rangle S$, we define $\text{extr}(C) \overset{\Delta}{=} \text{fn}(R) \setminus \bar{b}$.

**Lemma 8.** Let $P \sim Q$. Let $P \xrightarrow{\pi.E} C$, $F$ an abstraction, and $Q \xrightarrow{\pi.E} C'$ such that for all $E$, we have $F \bullet E \{C\} \sim F \bullet E \{C'\}$. Then we have $\text{extr}(C) = \text{extr}(C')$.

In the following proof, for a set of process $(P_i)$, we write $\prod_i P_i$ for $P_1 | \ldots | P_n$.

**Proof.** Let $b, e /\notin \text{fn}(P, Q)$. Given two sets of pairwise distinct names $\bar{c}_i, \bar{d}_i$ with the same number of elements, we define:

$$E_{\bar{c}_i, \bar{d}_i} \overset{\Delta}{=} \nu b.e[\nu \bar{c}_i, e[\square] | e(Y)(\prod_i c_i, 0 | \bar{c}_i, \bar{d}_i, 0)] | b(Z)Z | Z$$

Suppose the scope of a name $c_i$ is extruded outside $b$. After passivation of $e$ and duplication of the content of $b$, it is possible to perform the two synchronizations of $c_i$; the name $d_i$ becomes observable. If $d_i$ becomes observable, then passivation of locality $e$ has been triggered, and a synchronization on $c_i$ is possible. Since passivation of $e$ destroy any possible occurrence of $c_i$ in $e$, the synchronization is possible only if the scope of $c_i$ is extended outside $b$ before duplication of the content of $b$. Finally, the name $d_i$ becomes observable iff name $c_i$ is extruded outside $b$. 

27
Let $\tilde{d}_i$ be a set of pairwise distinct names with the same number of elements as $\text{extr}(C)$, and such that $\tilde{d}_i \cap \text{fn}(P, Q, F) = \emptyset$. Let $P' \overset{\Delta}{=} F \cdot E_{\text{extr}(C), \tilde{d}_i}(C)$. We have $P' \sim \sim Q'$. Let $c_{io} \in \text{extr}(C)$. By definition, $c_{io}$ is extruded outside $b$ in $P'$; hence name $d_{io}$ becomes observable. Since we have $P' \sim Q'$, $d_{io}$ becomes also observable in $Q'$. which is possible only if $c_{io} \in \text{extr}(C')$. Consequently we have $\text{extr}(C) \subseteq \text{extr}(C')$. Conversely let $\tilde{d}_i$ be a set of pairwise distinct names with the same number of elements as $\text{extr}(C')$, and such that $\tilde{d}_i \cap \text{fn}(P, Q, F) = \emptyset$. Let $P' \overset{\Delta}{=} F \cdot E_{\text{extr}(C'), \tilde{d}_i}(C)$. We have $P' \sim P \cdot E_{\text{extr}(C'), \tilde{d}_i}(C') \overset{\Delta}{=} Q'$. With the same reasoning on $Q'$ observables, we can prove similarly $\text{extr}(C') \subseteq \text{extr}(C)$.

\[\square\]

Using Lemma 8, we have the following inclusion.

**Lemma 9.** We have $\sim \subseteq \sim_m$.

The proof is done by showing that $\sim$ is a strong complementary bisimilarity. As a direct consequence, we can deduce that $\sim$ is sound.

**Corollary 1.** We have $\sim \subseteq \sim_b$.

Moreover, if $P \sim_m Q$ and $P \overset{\pi.T,E}{\rightarrow\rightarrow}_b P'$, then the matching transition $Q \overset{\pi.T,E}{\rightarrow\rightarrow}_b Q'$ depends on the context $E$. In the context bisimilarity (Definition 7), the matching transition is independant from $E$; context bisimilarity is late with respect to bisimulation contexts, while complementary bisimilarity is early with respect to these contexts. Proving that $\sim_m \subseteq \sim$ remains an open problem, but we conjecture that this inclusion holds.

**Remark 9.** We can define an early context bisimilarity with respect to contexts by changing the message output clause of Definition 7 into

- for all $P \overset{\pi}{\rightarrow} C$, for $F, E$, there exists $C'$ such that $Q \overset{\pi}{\rightarrow} C'$ and $(F \cdot E(C)) \mathcal{R} (F \cdot E(C'))$.

We can prove that this modified bisimilarity $\sim'$ is sound (using Kell soundness proof method) and complete (with the usual proof scheme). Consequently we have $\sim' = \sim_b$ and $\sim_m = \sim_b$, so we have $\sim_m = \sim'$. However we can prove soundness of $\sim'$ independantly from $\sim_m$ only in the strong case; this reasoning cannot be applied in the weak case.

We extend these results to the weak case. We write $\Rightarrow$ the reflexive and transitive closure of $\rightarrow$. We define $\Rightarrow$ as $\overset{a.R}{\rightarrow\rightarrow} \overset{a.R}{\rightarrow\rightarrow}$. In the weak case, two processes $P$ and $Q$ may evolve independently before interacting with each other. Since a transition $P \overset{\pi.Q.E}{\rightarrow\rightarrow}_b P'$ includes a communication between $P$ and $Q$, we have to authorize $Q$ to perform $\tau$-actions before interacting with $P$ in the weak output transition. We define $P \overset{\pi.Q.E}{\rightarrow\rightarrow}_b P'$ as $P \overset{\pi.Q.E}{\rightarrow\rightarrow} \overset{\pi.Q.E}{\rightarrow\rightarrow}_b P'$ with $Q \sim Q'$. 

28
Definition 14. Weak complementary bisimilarity \( \approx_m \) is the largest symmetric relation on closed processes \( R \) such that \( P \mathrel{R} Q \) implies \( \text{fn}(P) = \text{fn}(Q) \) and for all \( P \vdash P' \), there exists \( Q \vdash Q' \) such that \( P' \mathrel{R} Q' \).

Using the same proof techniques as in the strong case, we have the following results.

Theorem 7. Relation \( \approx_m \) is a congruence.

Theorem 8. We have \( \approx \subseteq \approx_m \).

Bisimilarity \( \approx_m \) coincides with \( \approx_b \) on image-finite processes; a closed process \( P \) is image finite iff for every label \( \lambda \), the set \( \{ P', P \vdash P' \} \) is finite. Using the same proof technique as in [36], we have the following completeness result.

Theorem 9. Let \( P, Q \) be image-finite processes. We have \( P \approx_b Q \) if and only if \( P \approx_m Q \).

Complementary bisimilarity characterizes barbed congruence in the strong and weak cases. However this relation is not completely satisfactory since it tests an infinite number of environments to equates processes, especially in the message output case. The next step is to find a behavioral equivalence with fewer tests, similar to the HO\( \pi \) normal bisimilarity (Section 2.2). In the following section, we give counter-examples which suggest that finding such simpler relations is not possible in HO\( \pi \)P.

6. Abstractions Equivalence in HO\( \pi \)P

In this section, we present counter-examples to show that a simplification similar to HO\( \pi \) normal bisimilarity (Section 2.2) is not possible in HO\( \pi \)P. We prove that testing large sub-classes of HO\( \pi \)P processes (the abstraction-free and the finite processes) is not enough to guarantee bisimilarity of abstraction. We first present a counter-example which relies on the chosen “by need” scope extrusion, and we then give other counter-examples which do not need this mechanism.

6.1. Abstraction-Free Processes

In the following, we omit the trailing zeros to improve readability; in an agent definition, \( m \) stands for \( m.0 \). We also write \( \nu a.b.P \) for \( \nu a.\nu b.P \). Let \( 0_m \triangleq \nu a.a.m \). Process \( 0_m \) cannot perform any transition, like \( 0 \), but it has a free name \( m \). We define the following abstractions:

\[
(X)P \triangleq (X)\nu mb.(b[X | \nu m.\pi(0_m)(m | n | \overline{m}.\overline{n}.p)] | \overline{n}.b(Y)(Y | Y))
\]

\[
(X)Q \triangleq (X)\nu mnb.(b[X | \nu b(0)(m | n | \overline{m}.\overline{n}.p)] | \overline{n}.b(Y)(Y | Y))
\]

The two abstractions differ in the process emitted on \( a \) and in the position of name restriction on \( m \) (inside or outside hidden locality \( b \)). An abstraction-free
process is a process built with the regular HOπP syntax but without message input \( a(X)P \).

We recall that \( \sim \) is the early strong context bisimilarity (Definition 7).

**Lemma 10.** Let \( R \) be an abstraction-free process. We have \((X)P \circ R \sim (X)Q \circ R\).

Since \( R \) is abstraction-free, it cannot receive the message emitted on \( a \); consequently \( R \) cannot interact with \( P \) or \( Q \). Passivation of locality \( b \) and transitions from \( R \) in \((X)P \circ R \) are easily matched by the same transitions in \((X)Q \circ R\).

Let \( P_{m,R} = \nu_m b. (b[R | m \ | n \ | m\cdot m.p] \ | \pi b(Y)(Y | Y)) \), \( F \) be an abstraction, and \( E \) be an evaluation context such that \( m \notin \text{fn}(E,F) \). We now prove that \((X)P \circ R \sim \nu_m (0_m) P_{m,R} \) is matched by \((X)Q \circ R \sim (0) \nu_m P_{m,R} \), i.e., that we have \( \nu_m (F \circ 0_m | E \{P_{m,R}\}) \sim F \circ 0 | E \{ \nu_m P_{m,R} \} \). Since \( m \notin \text{fn}(E,F) \), there is no interaction on \( m \) between \( F,E \), and \( P_{m,R} \), and the inert process \( 0_m \) does not interfere either. Hence the possible transitions from \( \nu_m (F \circ 0_m | E \{P_{m,R}\}) \) are the internal ones from \( F \) and \( E \), interactions between \( F \), \( E \), and \( R \) on names other than \( m \), and internal actions in \( P_{m,R} \). All of them are matched by the same transitions in \( F \circ 0 | E \{ \nu_m P_{m,R} \} \).

Abstractions \((X)P \) and \((X)Q \) may have different behaviors with an argument which may receive on \( a \), like \( a(Z)q \), with \( p \neq q \). By communication on \( a \), we have \((X)Q \circ a(Z)q \overset{\Delta}{\sim} \nu_m b. (b[q \ | m \ | n \ | m\cdot m.p] \ | \pi b(Y)(Y | Y)) \overset{\Delta}{=} Q_1 \). Since \( Q_1 \) may perform a \( q \sim \) transition, it can only be matched by \((X)P \circ a(Z)q \overset{\Delta}{=} Q_1 \). Notice that in \( P_1 \), the restriction on \( m \) remains inside hidden locality \( b \).

After synchronization on \( n \) and passivation/communication on \( b \), we have \( Q_1 (\overset{\Delta}{\sim})^2 \nu m b. (q \ | q \ | m \ | m\cdot m.p \ | m\cdot m.p) \overset{\Delta}{=} Q_2 \) (the process inside \( b \) in \( Q_1 \) is duplicated). After two synchronizations on \( m \), we have \( Q_2 (\overset{\Delta}{\sim})^2 \nu m b. (q \ | p \ | m\cdot m.p) \overset{\Delta}{=} Q_3 \), and \( Q_3 \) may perform a \( \sim \) transition. These transitions cannot be matched by \( P_1 \). Performing the duplication, we have \( P_1 (\overset{\Delta}{\sim})^2 \nu m b. (\nu m (q \ | m \ | m\cdot m.p) \ | \nu m (q \ | m \ | m\cdot m.p)) \overset{\Delta}{=} P_2 \). Each copied sub-process \( q \ | m \ | m\cdot m.p \) of \( P_2 \) has its own private copy of \( m \), and we can no longer perform any transition to have the observable \( p \). More generally, the sequence of transitions \((X)Q \circ a(Z)q \overset{\Delta}{=} P_1 \), consequently \( Q_1 \) and \( P_1 \) (and therefore \((X)Q \circ a(Z)q \) and \((X)P \circ a(Z)q \)) are not bisimilar.

The previous example shows that testing abstractions with abstraction-free processes (such as \( m \cdot 0 \)) is not enough to distinguish them. This example relies heavily on the chosen “by need” scope extrusion (restrictions are extruded outside localities along with messages only when needed), which is also used in Homer or Kell. Using a different definition of scope extrusion, for instance by considering name restriction to be a fresh name generator, is unfortunately not a solution: we present in the next section other counter-examples which do not
rely on scope extrusion yet show that testing a large class of finite processes is not sufficient to derive abstractions equivalence.

6.2. Finite Processes

We define finite processes as follows:

**Definition 15.** A finite process is a \( \text{HO}\pi P \) process built on the following grammar:

\[
P_F ::= 0 \mid P_F \mid P_F \mid \nu a.P_F \mid \pi(P)P_F \mid a(X)P_F \mid a[P_F]
\]

Roughly, finite processes cannot initiate an infinite sequence of transitions. Notice that in a message output, the message does not matter and can be a regular process. We do not allow process variable \( X \) in the syntax, hence finite processes encompass only message inputs \( a(X)P_F \) where either \( X \notin \text{fv}(P_F) \) or where \( X \) appears in emitted messages only (since emitted processes in a message output may be any process). In other words, processes received on input can only be passed around but never activated. With unrestricted message input, we may encode replication (as explained in Section 2.1) and therefore have infinite sequence of transitions.

We extend the definition to all agents in the following way: a concretion \( \nu \tilde{b}.(R)S \) is finite iff \( S \) is finite. An abstraction \( (X)P \) is finite iff \( P \) is finite. We write \( A_F \) the set of finite agents. We give some properties of finite agents:

**Lemma 11.** Let \( F \) be a finite abstraction. For all \( \text{HO}\pi P \) processes \( P \), the process \( F \circ P \) is finite.

Let \( P_F \) be a finite process:

- If \( P_F \xrightarrow{\alpha} A \) for some \( \alpha \), then \( A \) is finite.
- The set \( \{ \alpha \exists A, P_F \xrightarrow{\alpha} A \} \) is finite.
- For all action \( \alpha \), the set \( \{ A \mid P_F \xrightarrow{\alpha} A \} \) is finite.
- There is no infinite sequence of processes \( (P_i) \) such that \( P_0 = P_F \) and for all \( i, P_i \xrightarrow{\tilde{b}} P_{i+1} \) or \( P_i \xrightarrow{\pi} \nu b.(R)P_{i+1} \) or \( P_i \xrightarrow{\alpha} F \) with \( F \circ P = P_{i+1} \) for some \( P \).

The first properties are easy by induction on \( P_F \) or \( F \). The last one means that there is no infinite sequence of transitions started by \( P_F \); the proof can be found in Appendix B. Since the LTS is finitely branching (second and third properties of Lemma 11) and any sequence of transitions initiated by \( P_F \) is finite, we can speak about the length of the longest sequence of transitions initiated by \( P_F \), called depth.

**Definition 16.** We define inductively the depth of a finite agent \( A_F \), written \( d(A_F) \), as:

- \( d(P_F) = 0 \) if there is no transition from \( P_F \).
\[ d(P_F) = 1 + \max \{ d(A) | \exists \alpha, P_F \xrightarrow{\alpha} A \} \text{ otherwise.} \]

- For all finite concretions \( \nu b \langle P \rangle P_F \), we have \( d(\nu b \langle P \rangle P_F) = d(P_F) \).

- For all finite abstractions \( (X) P_F \), we have \( d((X) P_F) = d(P_F) \).

We may think that the depth of an abstraction depends on the interacting process. It is not the case since process variable may occurs in processes emitted in a message output, and the depth of a concretion takes into account the continuation only. Hence we have the following lemma:

**Lemma 12.** Let \( F \) be a finite abstraction. For all HO\( \pi \)P processes \( P \), we have \( d(F \circ P) = d(F) \).

We now use depth to prove that using finite processes to test bisimilarity of abstractions is not sufficient.

### 6.3. Counter-examples

In this section, we give counter-examples to show that testing finite processes is not enough to ensure bisimilarity of abstractions in HO\( \pi \)P (extended with a sum operator). To show this, we define inductively two families of HO\( \pi \)P abstractions \( (F_n) \), \( (G_n) \), such that for any finite process \( P_F \) with \( d(P_F) \leq n \), the processes \( F_n \circ P_F \) and \( G_n \circ P_F \) are context bisimilar, but \( F_n \circ Q_{n+1} \) and \( G_n \circ Q_{n+1} \) (where \( Q_{n+1} \) is a process \( m_{n+1} \ldots m_1.0 \) with \( n+1 \) names) are not context bisimilar.

For \( a \) a higher-order name and \( F = (X) P \) an abstraction, we write \( a.F \) for \( a\langle X \rangle P \). We also define \( \tau.P \Delta = \nu a.(\pi.0 | a.P) \) (with \( a \notin \text{fn}(P) \)). We define:

\[
F_0 \Delta = (X_0)X_0 \\
G_0 \Delta = (X_0)(X_0 | X_0)
\]

and for \( n > 0 \), we define

\[
F_n \Delta = (X_n)(\nu a_n.(a_n[X_n] | a_n.F_{n-1}) + R_n) \\
G_n \Delta = (X_n)(\nu a_n.(a_n[X_n] | a_n.G_{n-1}) + S_n)
\]

with \( R_n = \nu a_n.\tau.G_{n-1} \circ X_n \) and \( S_n = \nu a_n.\tau.F_{n-1} \circ X_n \). Notice that \( R_n \) mimics passivation of locality \( a_n \) in \( G_n \), and \( S_n \) mimics passivation of \( a_n \) in \( F_n \). They have been added to match some particular transitions.

Let \( P_F \) be a finite process such that \( d(P_F) \leq n \). We study first the relation between \( F_n \circ P_F \) and \( G_n \circ P_F \). If \( n = 0 \), which means that \( P_F \) cannot perform any transition, then we have to compare \( P_F \) and \( P_F | P_F \), which are obviously bisimilar. Otherwise, we have three kinds of transitions. We consider first the transition \( F_n \circ P_F \xrightarrow{\tau} \nu a_n.G_{n-1} \circ P_F \), which comes from the sub-process \( R_n \). This transition is easily matched by the passivation of locality \( a_n \) in \( G_n \circ P_F \):

\[ 32 \]
we have \( G_n \circ P_F \xrightarrow{\tau} \nu a_n.G_{n-1} \circ P_F \), the two obtained processes are identical. Similarly, we have \( F_n \circ P_F \xrightarrow{\alpha_i} \nu a_n.F_{n-1} \circ P_F \) by passivation of locality \( a_n; G_n \circ P_F \) matches this transition by the \( \tau \alpha \)-action \( G_n \circ P_F \xrightarrow{\tau} \nu a_n.F_{n-1} \circ P_F \) from the sub-process \( S_n \).

The last kind of evolutions from the process \( F_n \circ P_F \) is the succession of one or several transitions from \( P_F \), followed by passivation of \( a_n \). Roughly we have \( F_n \circ P_F \xrightarrow{\alpha_1} \ldots \xrightarrow{\alpha_k} \nu a_n.(a_n[P'_F] \mid a_n.F_{n-1}) \xrightarrow{\tau} \nu a_n.(F_{n-1} \circ P_F) \), with \( d(P_F) \leq n - 1 \). It can be matched by the same transitions in \( G_n \circ P_F \); we have \( G_n \circ P_F \xrightarrow{\alpha_1} \ldots \xrightarrow{\alpha_k} \nu a_n.(a_n[P'_F] \mid a_n.G_{n-1}) \xrightarrow{\tau} \nu a_n.(G_{n-1} \circ P_F) \). Hence we obtain two processes bisimilar to \( F_{n-1} \circ P_F \) and \( G_{n-1} \circ P_F \) with \( d(P_F) \leq n - 1 \) Consequently, we can prove the following lemma by induction on \( n \):

**Lemma 13.** If \( d(P_F) \leq n \), then \( F_n \circ P_F \sim G_n \circ P_F \).

Now, we consider \((m_k)\) a family of pairwise distinct fresh names which do not occur in any \( F_n \) nor \( G_n \). Let \( Q_1 = m_1.0 \) and \( Q_{k+1} = m_{k+1}.Q_k \) for all \( k > 1 \). We explain why \( F_n \circ Q_{n+1} \) and \( G_n \circ Q_{n+1} \) are not bisimilar. Consider the following sequence of transitions from \( F_n \circ Q_{n+1} \): an \( \xrightarrow{m_{n+1}} \) transition, followed by a passivation of locality \( \nu a_n \); we obtain \( F_n \circ Q_{n+1} \xrightarrow{m_{n+1}} \nu a_n.(a_n[Q_n] \mid a_n.F_{n-1}) \xrightarrow{\tau} \nu a_n.F_{n-1} \circ Q_n \). As this sequence must be matched by \( G_n \circ Q_{n+1} \), in particular the initial \( \xrightarrow{m_{n+1}} \) transition that selects the left process in the choice, we obtain \( F_{n-1} \circ Q_n \) and \( G_{n-1} \circ Q_n \). After repeating this sequence of transitions \( n - 1 \) times, we obtain \( F_0 \circ Q_1 = m_1.0 \) and \( G_0 \circ Q_1 = m_1.0 \mid m_1.0 \), which are clearly not bisimilar. Consequently \( F_n \circ Q_{n+1} \) is not bisimilar to \( G_n \circ Q_{n+1} \).

To summarize, testing a finite process \( P_F \) with depth \( n \) is not enough, since we have \( F_n \circ P_F \sim G_n \circ P_F \), but \( F_n \circ Q_{n+1} \not\sim G_n \circ Q_{n+1} \). Testing a finite set \( P \) of finite processes is not enough either. Since \( P \) is finite, the set \( \{d(P_F) \mid P_F \in P \} \) is finite and has a greatest element \( d \). For all \( P_F \in P \), we have \( F_d \circ P_F \sim G_d \circ P_F \) but \( F_d \circ Q_{d+1} \not\sim G_d \circ Q_{d+1} \). Similarly, testing an infinite set of finite processes with depths bounded by \( d \) is not enough.

Most cases are already covered by the abstraction-free counter-example, however the finite processes counter-examples do not rely on scope extrusion "by need" like the previous one, which means that they may be still valid with other ways to handle scope extrusion. Both counter-examples are not enough to prove that we cannot find a sound and complete behavioral equivalence with finite testing in \( \text{HOLP} \); the problem remains open. We can however define a normal bisimilarity if we remove the restriction operator from \( \text{HOLP} \), as explained in the following section.

7. Normal Bisimilarities in HOP

We now develop a full behavioral theory for HOP, a calculus with passivation but without restriction: we define higher-order and normal bisimilarities which
characterize barbed congruence in both strong and weak cases. HOP (for Higher Order with Passivation) is the calculus obtained by removing restriction from HO\(\pi\)P and adding a sum operator (to obtain the characterization result, since + is needed to show the completeness of HO bisimilarity and requires restriction to be faithfully encoded). The LTS contextual rules for HOP are the same as the HO\(\pi\)P ones, with the addition of the rule

\[
\frac{P \xrightarrow{\alpha} A}{P + Q \xrightarrow{\alpha} A}
\]

and of its symmetric rule. The structural congruence rules for HOP, also written \(\equiv\), is the smallest congruence that verifies the following laws.

\[
\begin{align*}
P | (Q | R) & \equiv (P | Q) | R \\
P | Q & \equiv Q | P \\
P | 0 & \equiv P
\end{align*}
\]

\[
P + (Q + R) \equiv (P + Q) + R \\
P + Q & \equiv Q + P \\
P + 0 & \equiv P \\
!P & \equiv P !!P
\]

Even without restriction, HOP remains quite expressive since it is an extension of the Turing-complete HOcore calculus defined in [20].

7.1. HO Bisimulation

We first give an LTS-based characterization of strong barbed congruence (Definition 1). As pointed out in Section 2.4, a message and its continuation may be put in different contexts because of passivation. Moreover, they are completely independent since they no longer share private names, as there is no restriction. Instead of keeping them together, we can now study them separately and still have a sound and complete bisimilarity. We propose the following bisimulation, called HO bisimulation, similar to the higher-order bisimulation given by Thomsen for Plain CHOCS [40].

**Definition 17.** Early strong HO bisimilarity, written \(\sim\), is the largest symmetric relation \(R\) such that \(P \sim Q\) implies:

- for all \(P \xrightarrow{\tau} P'\), there exists \(Q'\) such that \(Q \xrightarrow{\tau} Q'\) and \(P' \sim Q'\).
- for all \(P \xrightarrow{\alpha} F\), for all closed processes \(R\), there exists \(F'\) such that \(Q \xrightarrow{\alpha} F'\) and \(F \circ R \sim F' \circ R\).
- for all \(P \xrightarrow{\pi} (R)S\), there exists \(R', S'\) such that \(Q \xrightarrow{\pi} (R')S'\), \(R \sim R'\), and \(S \sim S'\).

In the following we also use the late counterpart of HO bisimilarity, written \(\sim_l\), which is obtained by replacing the input case by:

- For all \(P \xrightarrow{\alpha} F\), there exists \(F'\) such that \(Q \xrightarrow{\alpha} F'\) and for all closed processes \(R\), \(F \circ R \sim F' \circ R\).

We show later that early and late HO bisimilarities coincide (as in HO\(\pi\)). Howe’s method works with \(\sim_l\); there is no need to define a complementary semantics.
Theorem 10. We have $P \approx_Q Q$ iff $P$ and $Q$ are strong barbed congruent.

We define early weak (non-delay) HO bisimulation as:

Definition 18. Early weak HO bisimilarity, written $\approx e$, is the largest symmetric relation on closed processes $R$ such that $P R Q$ implies:

• for all $\tau \rightarrow P$, there exists $Q'$ such that $Q \mathbin{\Rightarrow} Q'$ and $P' R Q'$.
  
• for all $\alpha \rightarrow F$, for all closed processes $R$, there exist $F', Q'$ such that $Q \mathbin{\Rightarrow} F'$, $F' \circ R \mathbin{\Rightarrow} Q'$, and $F \circ R \mathbin{\Rightarrow} Q'$.

• for all $\tau \rightarrow (\tau)S$, there exist $R', S''$, $S'$ such that $Q \mathbin{\Rightarrow}_e (\tau)S''$, $S'' \mathbin{\Rightarrow}_e S'$, $R \mathbin{\Rightarrow}_e R'$, and $S \mathbin{\Rightarrow}_e S'$.

We define late weak HO bisimilarity, written $\approx l$, by replacing the input clause by:

• for all $\alpha \rightarrow F$, there exists $F'$ such that $Q \mathbin{\Rightarrow} F'$ and for all closed processes $R$, there exists $Q'$ such that $F' \circ R \mathbin{\Rightarrow} Q'$ and $F \circ R \mathbin{\Rightarrow} Q'$.

As in the strong case, we prove soundness of $\approx e$ using Howe's method.

Theorem 11. If $P \approx Q$, then $P$ and $Q$ are weak barbed congruent.

We prove completeness on image-finite processes. A HOP process $P$ is image finite iff for all $\alpha$, the set $\{ A | P \mathbin{\Rightarrow} A \}$ is finite.

Theorem 12. Let $P, Q$ be image finite processes. If $P, Q$ are weak barbed congruent, then they are early weak HO bisimilar.

We note that the definitions of higher-order bisimulations are easier to use since there is no universal quantification in the concretion case. In the following subsection, we show that the one in the abstraction case is not necessary.

7.2. Normal Bisimulation

In this section, we define a sound and complete bisimulation for the strong and weak cases without any universal quantification, similar to HO$\pi$ normal bisimulation [34]. Sangiorgi first defined it in the weak case, and then Cao extended it to the strong case [6].

In the message input case, HO$\pi$ normal bisimulation tests abstractions with only one trigger $m.0$, where $m$ is a fresh name. This testing is not sufficient in HOP. Consider the following processes:

$$P_1 \equiv \alpha[X] \mid \beta[0] \quad Q_1 \equiv X \mid P_1$$

Let $P_m \equiv P_1\{m.0/X\}$, $Q_m \equiv Q_1\{m.0/X\}$, $P_{m,n} \equiv P_1\{m.n.0/X\}$, and $Q_{m,n} \equiv Q_1\{m.n.0/X\}$, where $m, n$ do not occur in $P_1, Q_1$. 

35
We first prove that $P_m \xrightarrow{\sim} Q_m$. Since the other transitions are easily matched, we consider only the move $Q_m \overset{m}{\rightarrow} 0 | P_m$. It can only be matched by a replicated locality $a[m.0]$; we have $P_m \overset{m}{\rightarrow} a[0] | P_m$. The two resulting processes $0 | P_m$ and $a[0] | P_m$ are immediately bisimilar, due to the presence of $!a[0]$ in $P_m$. Consequently we have $P_m \xrightarrow{\sim} Q_m$.

However we have $P_{m,n} \not\xrightarrow{\sim} Q_{m,n}$. Indeed, the transition $Q_{m,n} \overset{m}{\rightarrow} n.0 | P_{m,n} \overset{a}{\rightarrow} Q'_{m,n}$ can only be matched by $P_{m,n} \overset{m}{\rightarrow} a[n.0] | P_{m,n} \overset{a}{\rightarrow} P'_{m,n}$. Processes $P_{m,n}$ and $Q_{m,n}$ are not HO bisimilar: by passivation of locality $a[n.0]$, we have $P'_{m,n} \not\xrightarrow{\sim} Q_{m,n}$. The emitted processes are not pairwise HO bisimilar, consequently we have $P'_{m,n} \not\xrightarrow{\sim} Q'_{m,n}$.

One could argue that the weakness of the distinguishing power of the trigger $m.0$ is due to the fact that localities are completely transparent, thus the presence of a message may not be directly observed. However, the existence of localities around a message has indirect effects, when passivation transforms an evaluation context (the locality) into a message that may be discarded. Triggers of the form $m.n.0$ allow the observation of an evaluation context (there is an emission on $m$) that disappears (there is no further emission on $n$), thus the presence of enclosing localities.

We now generalize this idea to show that it may be used to pinpoint the position of a process variable in the locality tree. Suppose we have $P\{m.n.0/X\}$ bisimilar to $Q\{m.n.0/X\}$, with $m,n$ not occurring in $P,Q$. Suppose further that $P \overset{m}{\rightarrow} P'$ is matched by $Q \overset{m}{\rightarrow} Q'$. The processes $P',Q'$ may now perform one and only one $\overset{\sim}{\rightarrow}$ transition from the single process $n.0$. Now suppose that $n.0$ is in a locality $a$ in $P'$. Passivation of this locality results in a concretion whose message $R$ is such that $R \overset{\sim}{\rightarrow}$. The process $Q'$ has to match these transitions with $Q' \overset{\sim}{\rightarrow} (R')S'$ such that $R \overset{\sim}{\rightarrow} R'$. Since $R \overset{m}{\rightarrow}$, we have $R' \overset{n}{\rightarrow}$; it is possible if and only if the single occurrence of $n.0$ in $Q'$ was in a locality $a$. With the same argument on $R,R'$, we prove that the locality hierarchies around $n.0$ in $P'$ and $Q'$ are the same. This result is formalized by the following lemma:

**Lemma 14.** Let $P,Q$ such that $\text{fe}(P,Q) \subseteq \{X\}$ and $m,n$ two names which do not occur in $P,Q$. Suppose we have $P\{m.n.0/X\} \overset{\sim}{\rightarrow} Q\{m.n.0/X\}$ and $P'\{m.n.0/X\} \overset{m}{\rightarrow} P'\{m.n.0/X\} \{n.0/Y\} \overset{a}{\rightarrow} Q'\{m.n.0/X\} \{n.0/Y\} \overset{a}{\rightarrow} Q_n$ with $P_n \overset{a}{\rightarrow} Q_n$. There exist $k \geq 0$, $a_1, \ldots, a_k$, $P_1 \ldots P_{k+1}, Q_1 \ldots Q_{k+1}$ such that either $P_n \equiv a_1[\ldots a_{k-1}[a_k[n.0 | P_{k+1}] | P_k] | P_{k-1} \ldots] | P_1 \quad Q_n \equiv a_1[\ldots a_{k-1}[a_k[n.0 | Q_{k+1}] | Q_k] | Q_{k-1} \ldots] | Q_1$

and for all $1 \leq j \leq k+1$, $P_j \overset{a}{\rightarrow} Q_j$.

The lemma allows us to decompose $P_n, Q_n$ in bisimilar sub-processes. For instance, if we have $P_n \equiv a[b[n.0 | P_3] | P_2] | P_1$ with $P_n \overset{a}{\rightarrow} Q_n$, then $Q_n \equiv
Let \( P, Q \) two processes such that \( \text{fv}(P, Q) \subseteq \{X\} \) and \( m, n \) two names which do not occur in \( P, Q \). If \( P\{m.n.0/X\} \sim I Q\{m.n.0/X\} \), then for all closed processes \( R \), we have \( P\{R/X\} \sim I Q\{R/X\} \).

We sketch the proof of Theorem 13 to explain how Lemma 14 is used.

**Sketch.** We show that the symmetric closure of the relation

\[
R \triangleq \{(P\{R/X\}, Q\{R/X\}) \mid P\{m.n.0/X\} \sim I Q\{m.n.0/X\}, m, n \text{ not in } P, Q\}
\]

is a late HO bisimulation. It is done by case analysis on the transition performed by \( P\{R/X\} \). Suppose we have \( P\{R/X\} \xrightarrow{R/P'} \{R'/X_i\}\{R/X\} \), i.e., a copy of \( R \) (at position \( X_i \)) performs a transition \( R \xrightarrow{R} R' \). Occurrence \( X_i \) is in an evaluation context, so we have \( P\{m.n.0/X\} \xrightarrow{m} P'\{n.0/X_j\}\{m.n.0/X\} = P' \), matched by \( Q\{m.n.0/X\} \xrightarrow{Q'} Q'\{n.0/X_j\}\{m.n.0/X\} = Q' \) with \( P' \sim I Q' \). As \( X_i \) is also in an evaluation context, we have \( Q\{R/X\} \xrightarrow{R} Q'\{R'/X_j\}\{R/X\} \).

We now have to prove that \( P'\{R'/X_i\}\{m.n.0/X\} \sim I Q'\{R'/X_j\}\{m.n.0/X\} \).

Lemma 14 allows us to write \( P' \equiv a_1[\ldots a_k[n.0 | P_{k+1}] \mid P_k \ldots] \mid P_1 \) and \( Q' \equiv a_1[\ldots a_k[n.0 | Q_{k+1}] \mid Q_k \ldots] \mid Q_1 \) with \((P_r), (Q_r)\) pairwise bisimilar processes for \( r \in \{1 \ldots k + 1\} \). Since \( P_{k+1} \sim I Q_{k+1} \) and \( \sim I \) is sound, we have \( a_k[R' | P_{k+1}] \sim I a_k[R' | Q_{k+1}] \). By induction on \( r \in \{k \ldots 1\} \), we prove that \( a_r[\ldots a_k[R' | P_{k+1}] \mid P_k \ldots] \mid P_j \sim I a_r[\ldots a_k[R' | Q_{k+1}] \mid Q_k \ldots] \mid Q_j \), obtaining \( P'\{R'/X_i\}\{m.n.0/X\} \sim I Q'\{R'/X_j\}\{m.n.0/X\} \) (for \( r = 1 \)) as needed.

Using this result we define a normal bisimulation for HOP:

**Definition 19.** Strong normal bisimilarity \( \sim_n \) is the largest symmetric relation on closed processes \( R \) such that \( P \sim R Q \) implies:

- for all \( P \xrightarrow{R} P' \), there exists \( Q' \) such that \( Q \xrightarrow{R} Q' \) and \( P' \sim R Q' \).
- for all \( P \xrightarrow{a} F \), there exists \( F' \) such that \( Q \xrightarrow{a} F' \) and for two names \( m, n \) which do not occur in processes \( P, Q \), we have \( F \circ m.n.0 \sim R F' \circ m.n.0 \).
- for all \( P \xrightarrow{R} (R)S \), there exists \( R', S' \) such that \( Q \xrightarrow{R} (R)'S' \), \( R \sim R' \) and \( S \sim S' \).

As a corollary of Theorem 13, we have

**Corollary 2.** \( \sim I = \sim_n = \sim \).

By definition, we have \( \sim I \subseteq \sim \subseteq \sim_n \). The inclusion \( \sim_n \subseteq \sim I \) is a consequence of Theorem 13.

Weak normal bisimilarity that coincides with weak HO bisimilarity may also be defined.
Definition 20. Weak normal bisimilarity $\approx_n$ is the largest symmetric relation on closed processes $\mathcal{R}$ such that $P \mathcal{R} Q$ implies:

- for all $P \xrightarrow{\tau} P'$, there exists $Q'$ such that $Q \xrightarrow{\tau} Q'$ and $P' \mathcal{R} Q'$.
- for all $P \xrightarrow{a} F$, there exists $G$ such that $Q \xrightarrow{a} F'$ and for two names $m, n$ which do not occur in processes $P, Q$, there exists $Q'$ such that $F' \circ m.n.0 \xrightarrow{Q'}$ and $F \circ m.n.0 \mathcal{R} Q'$.
- for all $P \xrightarrow{a} \langle R \rangle S$, there exists $R', S'', S'$ such that $Q \xrightarrow{a} \langle R' \rangle S''$, $S'' \xrightarrow{\tau} S'$, $R \mathcal{R} R'$ and $S \mathcal{R} S'$.

Theorem 14. $\approx_n = \approx_i$

The proof technique is similar to the strong case and relies on weak versions of Theorem 13 and Lemma 14. Hence in a calculus with passivation and without restriction, we can define a suitable bisimulation without any universal quantification in the strong and weak cases.

8. Related work

Behavioral equivalences in higher-order calculi. Very few higher-order calculi feature a coinductive characterization of weak barbed congruence, let alone one with finite testing, similar to normal bisimilarity. It is the case in HO$\pi$ (discussed in Section 2.1), and in a fragment of concurrent ML with local names [18]. In both calculi, normal bisimilarity comes from a triggered semantics, where triggers are passed instead of processes, which equates the “regular” semantics in the weak case. Cao [6] has extended HO$\pi$ normal bisimilarity to the strong case.

HOcore [20] is a minimal higher-order calculus (without any restriction or replication constructors), with various characterizations of strong barbed congruence, including higher-order and normal ones. Lanese et al. also give an axiomatization for bisimilarity, which shows that a behavioral equivalence in HOcore is in fact very discriminating. The authors do not know if their results holds in the weak case or when replication is added to the calculus.

Mobile Ambients [8] is a calculus with hierarchical localities and subjective linear process mobility. Localities, called ambients, may move by themselves in the locality hierarchy, without any acknowledgement from their environment, but they cannot be duplicated. Contextual characterizations of weak barbed congruence have been defined for Mobile Ambients [27] and its variant NBA [5]. A normal characterization has yet to be found in both calculi.

Difficulties arise in more expressive process calculi. The Seal calculus [10] is a calculus with objective process mobility which allows more flexibility than Mobile Ambients; in particular localities may be stopped, and duplicated. Process mobility requires synchronization between three processes (a process sending a name $a$, a receiving process, and a locality named $a$). The authors define a weak delay context bisimilarity in [10] called Hoe bisimilarity for the Seal calculus and prove its soundness. The authors point out that Hoe bisimilarity is
not complete, not only because of the delay style, but also because of the labels introduced for partial synchronization which are most likely not all observable.

The Kell-calculus [38] and Homer [16] are two higher-order calculi featuring a more general process mobility called passivation or active mobility. The two calculi differ in how they handle communication; in particular, Kell-calculus allows join patterns while Homer does not. Sound and complete contexts bisimilarities have been defined for both calculi in the strong case. As stated before, a weak delay input-early bisimilarity has been proven sound in Homer using Howe’s method.

**Congruence proof method.** In [26], Li and Liu propose a labelled transition system and a strong bisimilarity similar to the complementary semantics for HO\(\pi\) (Section 4). However they do not use the Howe’s method to prove congruence of the bisimilarity; instead they use an ad hoc method which relies on the factorization theorem (Theorem 4). This proof method cannot be applied to a different calculus.

Howe’s method has been originally used to prove congruence in a lazy functional programming language [17]. Baldamus and Frauenstein [2] are the first to adapt the method to process calculi for variants of Plain CHOCS [40]. They prove congruence of a late delay context bisimilarity in a calculus with static scoping, and then use it for late and early delay higher-order bisimilarities in a calculus with dynamic scoping, where emitted messages may escape the scope of their restricted names. Hildebrandt and Godskesen adapt Howe’s method for their calculus Homer [16]. As already explained through this paper, they prove congruence for late delay [16] and input-early delay [13] context bisimilarities.

In [35], Sangiorgi et al. propose *environmental bisimilarity* for several higher-order languages, including HO\(\pi\). The idea is to compare \(P\) and \(Q\) using an environment \(E\), which represents the knowledge that an observer has about these processes. This environment contains for instance the processes emitted by \(P\) and \(Q\). The observer uses the environment to challenge \(P\) and \(Q\). For instance, the observer is able to compare inputs from \(P\) and \(Q\) with any messages built from the processes inside \(E\). Environmental bisimilarity characterizes barbed congruence in HO\(\pi\). We do not know if it is possible to obtain a characterization result using an environmental bisimilarity in a calculus with passivation.

Instead of proving directly congruence of the bisimilarity, it is possible to design the LTS so that the associated bisimilarity is automatically a congruence. We briefly mention three methods which rely on this principle. A first method is to respect some LTS *rule format* that guarantees that the corresponding bisimilarity is a congruence. Checking that a LTS follows a given format is usually simpler than proving congruence directly. For higher-order calculi, Mousavi et al. [30] propose the Promoted and Higher-Order PANTH formats. The Promoted PANTH format guarantees that the regular bisimilarity (where an action is matched by exactly the same action) associated to the LTS is a congruence, and the Higher-Order PANTH format guarantees that the higher-order bisimilarity (where a higher-order action is matched by a bisimilar one, as in Section 7.1) is a congruence. However, these formats can be used for strong
bisimilarities only. Furthermore, they exclude side-conditions on names (such as $a \in \text{fn}(R)$), making lazy scope extrusion (as in HO$\pi$P) impossible to write.

In [31, 32], the LTS rules are automatically derived from the reduction rules and observable so that the associated bisimilarity is a congruence. Reduction rules are decomposed in order to identify the reacting sub-term and the context the environment has to provide to trigger the reduction. The method has been applied to the $\pi$-calculus [31], HO$\pi$ [31], and the Ambients [32], but only to prove congruence of strong bisimilarities. We do not know if the method works for weak ones.

Process calculi can be viewed as reactive systems, where transitions from a term $C\{P\}$ to $P'$ are written $P \xrightarrow{C} P'$. The main goal is then to find the minimal context $C$ such that an interaction with $P$ is possible. Bonchi et al. [4] propose a LTS derived from reactive systems for the Ambients, and use barbed bisimilarities to characterize strong and weak barbed congruence. We do not know if it is possible to encode calculi with passivation as reactive systems.

9. Conclusions and Future Work

Behavioral theory in calculi with passivation (like the Kell calculus or Homer) is less developed than the HO$\pi$ one. They are equipped with a sound and complete context bisimulation in the strong case only, which features additional tests on contexts in the message output case. Using HO$\pi$P, a higher-order calculus with passivation, we explain why usual congruence proof methods fail in the weak case in calculi with passivation. In particular, we explain that Howe’s method cannot be applied to early context bisimilarities because of the interdependency between the message input and message output clauses. To overcome this difficulty, we define a complementary labelled transition system where message outputs do not depend on an abstraction, but on a process which evolves to an abstraction. This modification allows to use the Howe’s method to prove congruence in the strong and weak cases.

We define a complementary semantics for HO$\pi$ and HO$\pi$P (and also for the Seal [10] in [21]). In HO$\pi$, the complementary semantics is sound and complete, and coincides with early context bisimilarity. We obtain similar results in HO$\pi$P, except we only have one inclusion instead of equality between the relations; we conjecture that they are indeed equal. We also define a complementary semantics for the Kell in (Lenglet’s PhD dissertation)$^5$, with mixed results. The main issue is dealing with join patterns. To complement an emitting process $P$, we need a receiving process $Q$, but also other emitting processes $R$ to match the receiving pattern of $Q$. We cope with this difficulty by progressively instantiating the pattern of $Q$: to receive $n$ messages, we use $n$ transitions instead of one. To apply the Howe’s method, we have to consider the bisimilarity which relates partially instanciated inputs. As a result, we obtain a sound but not complete

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$^5$The defense of the PhD is scheduled for January 2010; the document will be available shortly after. We will include the proper citation as soon as the document is available.
bisimilarity in the weak case. Nevertheless, we believe it is possible to define a sound and complete complementary bisimilarity in a Kell variant without join patterns or in Homer.

The crucial step in defining a complementary semantics for a given calculus is the definition of the transition rules, especially the message output ones. If these rules are written under some restrictions, the congruence proof of the associated bisimilarity is straightforward. A future work would be to explicit these restrictions. For instance, the classical rule for replication

\[ P \parallel P \xrightarrow{\alpha} A \]

\[ \parallel P \xrightarrow{\alpha} A \]

makes inductive proofs of the Howe’s method fails, because \( P \parallel P \) in the premise is not a subterm of the process \( \parallel P \) in the conclusion. Identifying all these constraints can lead to the definition of a rule format which guarantees the soundness of the associated complementary bisimilarity, similar to the Promoted or Higher-Order PANTH format for higher-order calculi [30].

We also plan to study complementary bisimilarities defined with the regular contextual semantics. As mentioned before, in the Kell (and more generally, in calculi with join-patterns), it is not possible to define a satisfactory complementary semantics; the associated bisimilarity is not complete. We want to come back to contextual semantics in order to fix this issue. It means that we change the message output clause of the early context bisimilarity such that the matching transition depends on a process, and not on an abstraction. For instance in \( \text{HO}^{\pi} \), we have to consider is the following clause:

- If \( P \xrightarrow{\tau} C \), then for all process \( R \), there exists \( C' \) such that \( Q \xrightarrow{\tau} C' \), and for all \( F \) such that \( R \xrightarrow{\tau} F \), we have \( F \cdot C \xrightarrow{\cdot} F \cdot C' \).

The corresponding relation is not completely early, because the matching transition does not depend on an abstraction \( F \), but it is not late either, because the transition depends on a process \( R \). We believe we can prove directly soundness of this “between late and early” bisimilarity with the Howe’s method, and we hope we can use this technique to obtain a characterization result in the weak case for the Kell.

Complementary and context bisimilarities are not completely satisfactory as substitutes for barbed congruence, since they reduce only slightly the quantifications. The following step is to find a characterization with fewer quantifications, similar to normal bisimilarity in \( \text{HO}^{\pi} \). We give counterexamples which suggest that it is not possible to find such relations in \( \text{HO}^{\pi} \). We conjecture that in a calculus featuring passivation and name restriction, we cannot define a sound and complete strong bisimilarity with fewer tests than in Definition 7. We are however able to define such relation in \( \text{HOP} \), a calculus with passivation but without restriction. In the case of \( \text{HO}^{\pi} \), normal bisimulation comes from an encoding of higher-order processes into first-order ones, which is not possible
in HOP. Instead, normal bisimulation in HOP relies on some means (a process \( m.n.0 \)) to observe locality hierarchies and to decompose abstractions in bisimilar sub-processes. We wonder if we can go further, and define an axiomatisation of barbed congruence in HOP. We plan to study a minimal calculus with passivation (simpler than HOP) to see if we can obtain an axiomatisation result similar to the HOcore one [20].

Finally, we obtain very different characterization results in HO\(\pi\)P and HOP, the two calculi with passivation we have studied in this paper. We summarize our results and compare them to results in similar calculi in Figure 6. Passivation in itself is not a problem when defining behavioral equivalences; the additional complexity previously observed in calculi such as Homer or Kell comes from the interaction between passivation and restriction.


Appendix A. Weak Complementary Semantics in HO\(\pi\)P

Appendix A.1. Correspondence Lemmas

Lemma 15. If \( P \xrightarrow{a} F \), then for all \( R \) we have \( P \xrightarrow{a,R} F \circ R \). If \( P \xrightarrow{a,R} P' \), then there exists \( F \) such that \( P \xrightarrow{a} F \) and \( P' = F \circ R \).

Proof. We proceed by structural induction on \( P \).

- If \( P = a(X)P' \), then by rule ABSTR we have \( P \xrightarrow{a} F = (X)P' \), and by rule Inv\(i\) we have \( P \xrightarrow{a,R} P'[R/X] = F \circ R \) for all \( R \), hence the result holds.


• Let $P = P_1 | P_2$. Suppose we have $P \xrightarrow{a} F$, which is possible only by rule PAR (and its symmetric, which is handled similarly). Consequently we have $P_1 \xrightarrow{a} F'$ and $F = F' | P_2$. By induction we have $P_1 \xrightarrow{a,R} F' \circ R$ for all $R$, hence by rule PAR$^P_{tr}$ we have $P \xrightarrow{a,R} F' \circ R | P_2 = F \circ R$, as required. Suppose we have $P \xrightarrow{a,R} P'$, which is possible only by rule PAR$^P_{tr}$ (and its symmetric, which is handled similarly). Consequently we have $P_1 \xrightarrow{a,R} P'_1$ and $P' = P'_1 | P_2$. By induction there exists $F$ such that $P_1 \xrightarrow{a} F$ and $P'_1 = F \circ R$. Consequently by rule PAR we have $P \xrightarrow{a} F | P_2$ with $P' = (F | P_2) \circ R$, as required.

• The locality, restriction, and replication cases are similar to the parallel case.

\[
\square \]

For a concretion $C = \nu b. (R)S$, we remind that $extr(C) \triangleq \text{fn}(R) \setminus b$.

**Lemma 16.** Let $P$ be an HOPnP process.

Suppose $P \xrightarrow{\pi} C$. For all $Q$ such that $Q \xrightarrow{a} F$ and for all $E$ such that $\text{bn}(E) \cap extr(C) = \emptyset$, we have $P \xrightarrow{\pi, Q, E}_{extr(C)} F \bullet E\{C\}$. If $P \xrightarrow{\pi, Q, E}_{\bar{b}} P'$, then there exists $F, C$ such that $P \xrightarrow{\pi} C, Q \xrightarrow{a} F, \bar{b} = extr(C)$, and $P' = F \bullet E\{C\}$.

**Proof.** We proceed by structural induction induction on $P$.

• Let $P = \pi\langle P_1 \rangle P_2$. We have $P \xrightarrow{\pi} \langle P_1 \rangle P_2 = C$. Let $Q$ such that $Q \xrightarrow{a} F$ and $E$ such that $\text{bn}(E) \cap \bar{b} = \emptyset$. We have $F \bullet E\{C\} = F \circ P_1 | E\{P_2\}$. By Lemma 15, we have $Q \xrightarrow{a,P_1} F \circ P_1$. Let $\bar{b} = \text{fn}(P_1)$; by rule Out$^P_{\bar{b}}$, we have $P \xrightarrow{\pi, Q, E}_{\bar{b}} F \bullet E\{C\}$ with $\bar{b} = \text{fn}(P_1) = extr(C)$ as wished.

We now prove the reverse implication. We have $P \xrightarrow{\pi, Q, E}_{\bar{b}} Q' | E\{P_2\}$ with $Q \xrightarrow{a,P_1} Q'$ and $\bar{b} = \text{fn}(P_1)$. By Lemma 15, there exists $F$ such that $Q \xrightarrow{a} F$ and $Q' = F \circ P_1$. Let $C = \langle P_1 \rangle P_2$. We have $P \xrightarrow{\pi} C, P' = F \bullet E\{C\}$ and $\bar{b} = \text{fn}(P_1) = extr(C)$, as required.

• Let $P = P_1 | P_2$. Suppose we have $P \xrightarrow{\pi} C$, which is possible by rule PAR or its symmetric. In the case of rule PAR, we have $P_1 \xrightarrow{\pi} C'$ and $C = C' | P_2$. Let $Q \xrightarrow{a} F$ and $E$ be an evaluation context. By induction we have $P_1 \xrightarrow{\pi, Q, E\{P_2\}} F \bullet E\{C' | P_2\}$ with $\bar{b} = extr(C')$. By rule PAR$^P_{\bar{b}}$ we have $P \xrightarrow{\pi, Q, E}_{\bar{b}} F \bullet E\{C\}$, and we have $\bar{b} = extr(C') = extr(C)$, as required.

Suppose we have $P \xrightarrow{\pi, Q, E}_{\bar{b}} P'$, which is possible by rule PAR$^P_{\bar{b}}$ or its symmetric. In the case of rule PAR$^P_{\bar{b}}$, we have $P_1 \xrightarrow{\pi, Q, E\{P_2\}} P'$. By
Lemma 17. Let $P$ be an HOπP process.

induction there exists $F, C$ such that $P_1 \xrightarrow{\pi} C$, $Q \xrightarrow{a} F$, $\bar{b} = extr(C)$ and $P' = F \bullet E\{C \mid P_2\}$. Consequently by rule PAR we have $P \xrightarrow{\pi} C \mid P_2 = C'$ with $P' = F \bullet E\{C'\}$ and $\bar{b} = extr(C) = extr(C')$, as required.

- The locality case is similar to the parallel one for the evaluation rules (Loc and $\text{Loc}_b$), and to the message output one for the passivation rules (Passiv and $\text{Passiv}_b$).

- The replication case is similar to the parallel one.

- Let $P = \nu c. P_1$. Suppose first we have $P \xrightarrow{\pi} C$. By rule RESTRICT we have $P_1 \xrightarrow{\pi} C'$ and $C = \nu b. C'$. Let $Q \xrightarrow{a} F$ and $E$ be an evaluation context. We distinguish two cases:

  - If $c \in extr(C')$, then we have $F \bullet E\{\nu c. C'\} = \nu c. (F \bullet E\{C'\})$. By induction we have $P_1 \xleftarrow{\pi, Q, E} \bar{b} P'_1$ with $\bar{b} = extr(C')$ and $P'_1 = F \cdot E\{C'\}$. We have $c \in \bar{b}$, so by rule $\text{EXTR}^\nu_b$ we have $P \xleftarrow{\pi, Q, E} \bar{b}\\{c\} \nu c. P'_1 = F \bullet E\{\nu c. C'\}$. We have $extr(C) = extr(C')\cup\{c\} = \bar{b}\\{c\}$, hence the result holds.

  - If $c \notin extr(C')$, then by induction we have $P_1 \xrightarrow{\pi, Q, E[\nu b, \cdot]} \bar{b} P'_1$ with $\bar{b} = extr(C')$ and $P'_1 = F \cdot E\{\nu c. C'\} = F \cdot E\{C\}$. By rule $\text{RESTRICT}^\nu_b$ we have $P \xrightarrow{\pi, Q, E} \bar{b} F \cdot E\{C\}$, and we have $\bar{b} = extr(C') = extr(C)$, as required.

Suppose now that $P \xrightarrow{\pi, Q, E} \bar{b} P'$. We have two cases:

- Rule $\text{RESTRICT}^\nu_b$: we have $P_1 \xrightarrow{\pi, Q, E[\nu b, \cdot]} \bar{b} P'$ with $c \notin \bar{b}$. By induction there exists $F, C$ such that $P_1 \xrightarrow{\pi} C$, $Q \xrightarrow{a} F$, $\bar{b} = extr(C)$ and $P' = F \cdot E\{\nu c. C\}$. By rule RESTRICT we have $P \xrightarrow{\pi} \nu c. C = C'$, and $extr(C') = extr(C) = \bar{b}$ since $c \notin \bar{b}$. We have $P' = F \cdot E\{C'\}$, as required.

- Rule $\text{EXTR}^\nu_b$: we have $P_1 \xrightarrow{\pi, Q, E[\nu b, \cdot]} \bar{b} \cup\{c\} P'_1$ with $P' = \nu c. P'_1$. By induction there exists $F, C$ such that $P_1 \xrightarrow{\pi} C$, $Q \xrightarrow{a} F$, $\bar{b} \cup\{c\} = extr(C)$, and $P'_1 = F \cdot E\{C\}$. By rule RESTRICT we have $P \xrightarrow{\pi} \nu c. C = C'$. Since $\bar{b} \cup\{c\} = extr(C)$, $c$ is free in the message of $C$, consequently we have $F \cdot E\{C'\} = \nu c. (F \cdot E\{C\}) = P'$. We also have $\bar{b} = extr(C) = extr(C')$, as required.

\qed
If $P \xrightarrow{\pi} C$, then for all $Q$ such that $Q \xrightarrow{\alpha} F$ and for all $E$, we have $P \xrightarrow{\pi,Q,E}_{\text{extr}(C)} F \cdot E \{C\}$.

If $P \xrightarrow{\pi,Q,E}_{\text{extr}(C)} P'$, then there exists $F,C$ such that $P \xrightarrow{\pi} C$, $Q \xrightarrow{\alpha} F$, $\bar{b} = \text{fn}(o(C)) \setminus \text{bn}(C)$, and $P' = F \cdot E \{C\}$.

**Proof.** Let $P \xrightarrow{\pi} C$, $Q \xrightarrow{\alpha} F$, and $E$ an evaluation context. We prove the first result by induction on the number of captures by $E$, i.e. on the size of the set $\text{bn}(E) \cap \text{extr}(C)$. If $\text{bn}(E) \cap \bar{b} = \emptyset$, then by Lemma 16 we have $P \xrightarrow{\pi,Q,E}_{\text{extr}(C)} F \cdot E \{C\}$. By rule $\text{CFREE}^p_E$, we have the required result.

Otherwise, there exists $c, E_1, E_2$ such that $E = E_1 \{\nu c.E_2\}$. The context $E_1 \{E_2\}$ is performing less capture than $E$, so by induction we have $P \xrightarrow{\pi,Q,E_1(E_2)} F \cdot E_1 \{E_2\} \{C\}$. By rule $\text{CAPT}^p_E$, we have $P \xrightarrow{\pi,Q,E}_{\text{extr}(C)} \nu c.(F \cdot E_1 \{E_2\} \{C\}) = F \cdot E \{C\}$, as required.

We prove the reverse implication by induction on the derivation of $P \xrightarrow{\pi,Q,E}_{\text{extr}(C)} P'$. If the transition comes from rule $\text{CFREE}^p_E$, we have $\text{bn}(E) \cap \bar{b} = \emptyset$, and we can use Lemma 16. Otherwise, by rule $\text{CAPT}^p_E$ there exists $c, E_1, E_2, P''$ such that $E = E_1 \{\nu c.E_2\}$, $P \xrightarrow{\pi,Q,E_1(E_2)} P''$ with $P'' = \nu c.P''$, and $c \in \bar{b}$. By induction there exists $F,C$ such that $P \xrightarrow{\pi} C$, $Q \xrightarrow{\alpha} F$, $P'' = F \cdot E_1 \{E_2\} \{C\}$, and $\text{extr}(C) = \bar{b}$. Since $c \in \bar{b} = \text{extr}(C)$, we have $F \cdot E \{C\} = \nu c.(F \cdot E_1 \{E_2\} \{C\}) = \nu c.P'' = P'$, as required.

**Lemma 18.** Let $P$ be an $\text{HO} \pi P$ process. We have $P \xrightarrow{\pi} P'$ iff $P \xrightarrow{\pi} P'$.

**Proof.** We proceed by structural induction on $P$.

Let $P = P_1 \mid P_2$. By case analysis on the rule used to derive $P \xrightarrow{\pi} P'$:

- **PAR**: in this case we have $P_1 \xrightarrow{\pi} P'_1$ and $P'' = P'_1 \mid P_2$. By induction we have $P_1 \xrightarrow{\pi} P'_1$, hence by rule $\text{PAR}^p_{\pi}$ we have $P \xrightarrow{\pi} P'$, as required.

- **HO**: in this case, we have $P_1 \xrightarrow{\alpha} F$, $P_2 \xrightarrow{\pi} C$, and $P' = F \cdot C$. By induction we have $P_2 \xrightarrow{\pi,P_1,\square} F \cdot C$, so by rule $\text{HO}^p_E$ we have $P \xrightarrow{\pi} P'$, as required.

We now prove the reverse implication.

- **PAR$^p_{\pi}$**: we have $P_1 \xrightarrow{\pi} P'_1$ and $P'' = P'_1 \mid P_2$. By induction we have $P_1 \xrightarrow{\pi} P'_1$, hence we have $P \xrightarrow{\pi} P'_1$ by rule PAR.

- **HO$^p_{\pi}$**: we have $P_1 \xrightarrow{\pi,P_2,\square} P'$. By induction there exists $F,C$ such that $P_1 \xrightarrow{\pi} C$, $P_2 \xrightarrow{\alpha} F$ and $P'' = F \cdot C$. By rule HO, we have $P \xrightarrow{\pi} P'$, as required.

The locality, restriction, and replication cases are similar. □
Lemma 19. Let $P$ be an $HO\pi P$ process.

- We have $P \xrightarrow{\cdot} P'$ iff $P \xrightarrow{\sim} P'$.

- Let $R$ be a closed process. If $P \xrightarrow{a} F$ and $F \circ R \xrightarrow{\cdot} P'$ then we have $P \xrightarrow{aR} F \circ R$. If $P \xrightarrow{aR} P'$, then there exists $F$ such that $P \xrightarrow{a} F$ and $F \circ R \xrightarrow{\cdot} P'$.

- If $P \xrightarrow{a} C$, then for all $Q, E$ such that $Q \xrightarrow{a} F$ and $F \cdot E \{C\} \xrightarrow{\cdot} P'$, we have $P \xrightarrow{\pi Q \cdot E}_b P'$ with $b = extr(C)$. If $P \xrightarrow{\pi Q \cdot E}_b P'$, then there exists $F, C$ such that $P \xrightarrow{\pi C} C$, $Q \xrightarrow{\cdot} F$, $b = extr(C)$, and $F \cdot E\{C\} \xrightarrow{\cdot} P'$.

Proof. By Lemma 18 we have $\sim = \approx$, so we have $\sim = \approx$.

If $P \xrightarrow{\cdot} P'' \xrightarrow{a} F$ and $F \circ R \xrightarrow{\cdot} P'$, then we have $P \xrightarrow{aR} P''$ and $F \circ R \xrightarrow{\cdot} P'$ by the first result. By Lemma 15 we have $P'' \xrightarrow{aR} F \circ R$, consequently we have $P \xrightarrow{aR} P'$. If $P \xrightarrow{\cdot} P_1 \xrightarrow{aR} P_2 \xrightarrow{\cdot} P'$, then we have $P \xrightarrow{\cdot} P_1$ and $P_2 \xrightarrow{\cdot} P'$. By Lemma 15 there exists $F$ such that $P_1 \xrightarrow{\cdot} F$ and $F \circ R = P_2$. Consequently we have $P \xrightarrow{\cdot} F$ and $F \circ R \xrightarrow{\cdot} P'$ as wished.

Let $P \xrightarrow{r} P'' \xrightarrow{\pi} C$, $Q \xrightarrow{r} Q'' \xrightarrow{a} F$, and $F \cdot E \{C\} \xrightarrow{\cdot} P'$. We have $P \xrightarrow{r} P''$, $Q \xrightarrow{r} Q''$, and $F \cdot E\{C\} \xrightarrow{\cdot} P'$ by the first result. By Lemma 17 we have $P'' \xrightarrow{\pi Q'' \cdot E}_b F \cdot E\{C\}$ with $b = extr(C)$, so we have $P \xrightarrow{\pi Q'' \cdot E}_b P'$. Consequently we have $P \xrightarrow{\pi Q \cdot E}_b P'$, as required. If $P \xrightarrow{\pi Q \cdot E}_b P'$, then we have $P \xrightarrow{\cdot} P_1 \xrightarrow{\pi Q \cdot E}_b P_2 \xrightarrow{\cdot} P'$ with $Q \xrightarrow{\cdot} Q'$. We have $P \xrightarrow{\cdot} P_1$, $P_2 \xrightarrow{\cdot} P'$, and $Q \xrightarrow{\cdot} Q'$ by the first result. By Lemma 17 there exists $F, C$ such that $P_1 \xrightarrow{\cdot} C$, $Q' \xrightarrow{\cdot} F$, $b = extr(C)$, and $P_2 = F \cdot E\{C\}$. Consequently we have $P \xrightarrow{\cdot} C$, $Q \xrightarrow{\cdot} F$, and $F \cdot E\{C\} \xrightarrow{\cdot} P'$, as required.

We now prove the correspondence between $\approx$ and $\approx_m$. The correspondence proof for $\sim$ and $\sim_m$ is similar.

Lemma 20. If $P \xrightarrow{\pi} C$ then we have $fn(C) \subseteq fn(P)$.

Proof. By induction on $P \xrightarrow{\pi} C$. 

Lemma 21. Let $P \approx Q$. Let $P \xrightarrow{\pi} C$, $F$ an abstraction, and $Q \xrightarrow{\pi} C'$ such that for all $E$, there exists $Q'$ such that $F \cdot E\{C\} \xrightarrow{\cdot} Q'$ and $F \cdot E\{C\} \approx Q'$. Then we have $extr(C) = extr(C')$.

Proof. Let $b, e \notin fn(P, Q)$. Given two sets of pairwise distinct names $\tilde{c}_i, \tilde{d}_i$ with the same number of elements, we define:

$$E_{\tilde{c}_i, \tilde{d}_i} \xrightarrow{\cdot} vbe.b[v\tilde{c}_i, e[\square] | e(Y)(\prod_i c_i, 0 | \tilde{c}_i, \tilde{d}_i, 0)] | b(Z)Z | Z$$

Suppose the scope of a name $c_i$ is extruded outside $b$. After passivation of $e$ and duplication of the content of $b$, it is possible to perform the two synchronizations of $c_i$; the name $d_i$ becomes observable. If $d_i$ becomes observable,
then passivation of locality $e$ has been triggered, and a synchronization on $c_{i_0}$ is possible. Since passivation of $e$ destroys any possible occurrence of $c_{i_0}$ in $e$, the synchronization is possible only if the scope of $c_{i_0}$ is extended outside $b$ before duplication of the content of $b$. Finally, the name $d_{i_0}$ becomes observable iff name $c_{i_0}$ is extruded outside $b$.

Let $\bar{d}_i$ be a set of pairwise distinct names with the same number of elements as $\text{extr}(C)$, and such that $\bar{d}_i \cap \text{fn}(P, Q, F) = \emptyset$. Let $P' \overset{\Delta}{=} F \bullet \text{E}_{\text{extr}(C), \bar{d}_i}(C)$. There exists $Q'$ such that $F \bullet \text{E}_{\text{extr}(C), \bar{d}_i}(C) \overset{\Rightarrow}{=} Q'$ and $P' \approx Q'$. Let $c_{i_0} \in \text{extr}(C)$. By definition, $c_{i_0}$ is extruded outside $b$ in $P'$, hence name $d_{i_0}$ becomes observable. Since we have $P' \approx Q'$, $d_{i_0}$ becomes also observable in $Q'$, which is possible only if $c_{i_0} \in \text{extr}(C')$. Consequently we have $\text{extr}(C) \subseteq \text{extr}(C')$. Conversely let $\bar{d}_i$ be a set of pairwise distinct names with the same number of elements as $\text{extr}(C)$, and such that $\bar{d}_i \cap \text{fn}(P, Q, F) = \emptyset$. Let $P' \overset{\Delta}{=} F \bullet \text{E}_{\text{extr}(C'), \bar{d}_i}(C)$. There exists $Q'$ such that $F \bullet \text{E}_{\text{extr}(C'), \bar{d}_i}(C) \overset{\Rightarrow}{=} Q'$ and $P' \approx Q'$. With the same reasoning on $Q'$ observables, we can prove similarly $\text{extr}(C') \subseteq \text{extr}(C)$.

\[\square\]

**Lemma 22.** If $P \approx Q$ then $P \overset{m}{=} Q$.

**Proof.** We prove that $\approx$ is a weak complementary bisimulation. Let $P \approx Q$. We have $\text{fn}(P) = \text{fn}(Q)$ by definition.

- If $P \overset{\widetilde{\rightarrow}}{\longrightarrow} P'$ then by Lemma 18 we have $P \overset{\rightarrow}{\longrightarrow} P'$. By definition there exists $Q'$ such that $Q \overset{\Rightarrow}{\Rightarrow} Q'$ and $P' \approx Q'$. By Lemma 19 we have $Q \overset{\Rightarrow}{\Rightarrow} Q'$, and we have $P' \approx Q'$ as wished.

- If $P \overset{\alpha \circ R}{\longrightarrow} P'$, then by Lemma 15 there exists $F$ such that $P \overset{\alpha}{\longrightarrow} F$ and $P' = F \circ R$. By definition there exists $G, Q'$ such that $Q \overset{\alpha}{\Rightarrow} G$, $G \bullet \langle R \rangle 0 \overset{\Rightarrow}{\Rightarrow} Q'$ and $Q' \approx F \bullet \langle R \rangle 0$. We have $G \bullet \langle R \rangle 0 \equiv G \circ R$ so by Lemma 19 we have $Q \overset{\alpha \circ R}{\Rightarrow} Q' \approx F \bullet \langle R \rangle 0 \equiv P'$ as wished.

- If $P \overset{\pi.T.E}{\longrightarrow}_b P'$, then by Lemma 17 there exists $F, C$ such that $T \overset{\alpha}{\rightarrow} F$, $P \overset{\pi}{\Rightarrow} C$, $b = \text{fn}(o(C)) \setminus \text{bn}(C)$ and $P' = F \bullet E \{ C \}$. By definition there exists $D, Q'$ such that $Q \overset{\pi}{\Rightarrow} D$, $F \bullet E \{ D \} \overset{\Rightarrow}{=} Q'$ and $F \bullet E \{ C \} \approx Q'$. By Lemma 21 we have $\text{extr}(D) = \text{extr}(C) = \bar{b}$. By Lemma 19 we have $Q \overset{\pi.T.E}{\Rightarrow}_{\bar{b}} Q'$, and we have $P' \approx Q'$ as required.

\[\square\]

**Appendix A.2. Howe’s Method**

We first prove a result we extensively use in the following. We let $\overset{\lambda}{\rightarrow}$ range over $\overset{\rightarrow}{\rightarrow}$, $\overset{\rightarrow}{\rightleftharpoons}$, and $\overset{\pi.T.E}{\rightleftharpoons}$, and we write $\overset{\Rightarrow}{\Rightarrow}$ for the weak counterpart.
Lemma 23. If $P \approx_m Q$ and $P \xrightarrow{\pi.T.E} P'$, then there exists $T', Q'$ such that $T \xrightarrow{\tau} T'$, $Q \xrightarrow{\pi.T'.E} Q'$, and $P' \approx_m Q'$.

Proof. Since we have $P \xrightarrow{\pi.T.E} P'$, we have $P \xrightarrow{\pi.T.E} P'$ by rule CFREE$^0$. By bisimilarity, there exists $Q'$ such that $Q \xrightarrow{\pi.T.E} b \xrightarrow{\tau} Q'$, and $P' \approx_m Q'$. By definition there exists $T'$ such that $Q \xrightarrow{\tau} \pi.T'.E \xrightarrow{b} \tau Q'$. Context $E$ is capture-free w.r.t. to $b$, so the output transition comes from rule CFREE$^0$. Consequently we have $Q \xrightarrow{\tau} \pi.T'.E \xrightarrow{b} \tau Q'$ as wished.

Lemma 24. Let $P \approx Q$.

- If $P \xrightarrow{\tau} P'$ then there exists $Q'$ such that $Q \xrightarrow{\tau} Q'$ and $P' \approx Q'$.

- If there exists $T'$ such that $T \xrightarrow{\tau} T'$ and $P \xrightarrow{\tau} \pi.T'.E \xrightarrow{b} \tau P'$, then there exists $T''$, $Q'$ such that $T \xrightarrow{\tau} T''$, $Q \xrightarrow{\pi.T'.E} b \xrightarrow{\tau} Q'$, and $P' \approx_m Q'$.

Proof. If $P \xrightarrow{\tau} P'$, we proceed by induction on the number of $\tau$-steps. For 0 step, the result holds (chose $Q' = Q$). Suppose the result holds for $n$. If $P(\ldots)^nP_n \xrightarrow{\tau} P'$, then by induction there exists $Q'_n$ such that $Q \xrightarrow{\tau} Q'_n$ and $P'_n \approx Q'_n$. By bisimulation definition, there exists $Q'$ such that $Q'_n \xrightarrow{\tau} Q'$ and $P' \approx Q'$. Since we have $Q \xrightarrow{\tau} Q'$, we have the required result.

If $P \xrightarrow{\tau} P_1 \xrightarrow{a.R} P_2 \xrightarrow{\tau} P'$, then by the first result there exists $Q'_1$ such that $Q \xrightarrow{\tau} Q'_1$ and $P_1 \approx Q'_1$. By bisimulation definition there exists $Q'_2$ such that $P \xrightarrow{a.R} Q'_2$ and $P_2 \approx Q'_2$. By the first result there exists $Q'_2 \xrightarrow{\tau} Q'$ and $P' \approx Q'$. We have $Q \xrightarrow{\tau} Q'$ hence the result holds.

If $P \xrightarrow{\tau} P_1 \xrightarrow{\pi.T.E} b P_2 \xrightarrow{\tau} P'$ with $T \xrightarrow{\tau} T'$, then by the first result there exists $Q'_1$ such that $Q \xrightarrow{\tau} Q'_1$ and $P_1 \approx Q'_1$. By bisimulation definition there exists $Q'_2$ such that $Q'_1 \xrightarrow{\pi.T.E} b Q'_2$ and $P_2 \approx Q'_2$. By the first result there exists $Q'$ such that $Q'_2 \xrightarrow{\tau} Q'$ and $P' \approx Q'$. We have $Q \xrightarrow{\pi.T.E} b Q'$ as wished.

If $P \xrightarrow{\tau} P_1 \xrightarrow{\pi.T.E} b P_2 \xrightarrow{\tau} P'$ with $T \xrightarrow{\tau} T'$, then by the first result there exists $Q'_1$ such that $Q \xrightarrow{\tau} Q'_1$ and $P_1 \approx Q'_1$. By Lemma 23 there exists $T''$, $Q'_2$ such that $T' \xrightarrow{\tau} T''$, $Q'_1 \xrightarrow{\pi.T'.E} b \xrightarrow{\tau} Q'_2$ and $P_2 \approx Q'_2$. By the first result there exists $Q'$ such that $Q'_2 \xrightarrow{\tau} Q'$ and $P' \approx Q'$. We have $Q \xrightarrow{\pi.T'.E} b Q'$ with $T \xrightarrow{\tau} T''$, as wished.

\[ \square \]
We recall the definitions of open extension and Howe’s closure of weak bisimilarity $\approx_m$.

**Definition 21.** Let $P$ and $Q$ be two open processes. We have $P \approx_m^o Q$ iff $P\sigma \approx_m Q\sigma$ for all substitutions that close $P$ and $Q$.

**Definition 22.** The Howe’s closure $\approx^\bullet_m$ is the smallest relation verifying:

- $\approx^o \subseteq \approx^\bullet_m$.
- $\approx^\bullet_m \subseteq \approx^o \subseteq \approx^\bullet_m$.
- For all operators $op$ of the language, if $\bar{P} \approx^\bullet Q$, then $op(\bar{P}) \approx^\bullet op(\bar{Q})$.

**Lemma 25.** $\approx^\bullet_m$ is reflexive.

**Proof.** Because $\approx^o_m$ is reflexive. \qed

**Lemma 26.** If $P \approx^\bullet_m Q$, then $fn(P) = fn(Q)$.

**Proof.** By induction on the derivation of $P \approx^\bullet_m Q$.

- If $P \approx^o_m Q$, then we have $fn(P) = fn(Q)$ by definition.
- If $P \approx^\bullet_m T \approx^o_m Q$, then we have $fn(P) = fn(T) = fn(Q)$ by bisimulation definition. Consequently we have $fn(P) = fn(Q)$.
- If $\bar{P} \approx^\bullet Q$, we have $fn(P_i) = fn(Q_i)$ for each item on the list by induction, hence using definition of free names we have $fn(op(\bar{P})) = fn(op(\bar{Q}))$.

**Lemma 27.** If $R \approx^\bullet_m R'$, then $P\{R/X\} \approx^\bullet_m P\{R'/X\}$.

If $P \overset{a,R}{\rightarrow} P'$ and $R \approx^\bullet_m R'$, then there exists $P''$ such that $P \overset{a,R'}{\rightarrow} P''$ and $P' \approx^\bullet_m P''$.

**Proof.** The first item is done by structural induction on $P$:

- $P = 0$: the result holds.
- $P = X$: $P\{R/X\} = R \approx^\bullet_m R' = P\{R'/X\}$, hence the result holds.
- $P = Y \neq X$: the result holds.
- $P = P_1 | P_2$: by induction we have $P_1\{R/X\} \approx^\bullet_m P_1\{R'/X\}$ and $P_2\{R/X\} \approx^\bullet_m P_2\{R'/X\}$. Since $\approx^\bullet_m$ is a congruence we have $P\{R/X\} = P_1\{R/X\} | P_2\{R'/X\} \approx^\bullet_m P_1\{R'/X\} | P_2\{R'/X\} = P\{R'/X\}$, as required.
- The locality, message input, message output, and replication cases are similar to the case above.
Proof. By induction on the derivation of $P \xrightarrow{α,R} P'$:

- Rule $\text{In}_{i}^{R}$: we have $P = a(X)P_{1} \xrightarrow{a,R} P_{1}\{R/X\}$. Using first item we have $P_{1}\{R/X\} \approx_{m}^{*} P_{1}\{R'/X\}$, and by rule $\text{In}_{i}^{R}$ we have $P \xrightarrow{α,R'} P_{1}\{R'/X\}$, as required.

- Rule $\text{Par}^{R}_{\tau}$: we have $P = P_{1} \parallel P_{2}$ with $P_{1} \xrightarrow{a,R'} P'_{1}$ and $P' = P'_{1} \parallel P_{2}$. By induction there exists $P''_{1}$ such that $P_{1} \xrightarrow{a,R'} P''_{1}$ and $P'_{1} \approx_{m}^{*} P''_{1}$. By rule $\text{Par}^{R}_{\tau}$, we have $P \xrightarrow{a,R'} P''_{1} \parallel P_{2} = P''$, and since $\approx_{m}^{*}$ is a congruence, we have $P' \approx_{m}^{*} P''$, as required.

- Rules $\text{Loc}^{R}_{\tau}$ and $\text{Replic}^{R}_{\tau}$: similar to the case above.

- Rule $\text{Rest}^{R}_{\tau}$: we have $P = \nu b.P_{1}$ with $P_{1} \xrightarrow{a,R} P'_{1}$, $b \neq a$, and $P' = \nu b.P'_{1}$. By induction there exists $P''_{1}$ such that $P_{1} \xrightarrow{a,R'} P''_{1}$ and $P'_{1} \approx_{m}^{*} P''_{1}$. By rule $\text{Rest}^{R}_{\tau}$ we have $P \xrightarrow{a,R'} \nu b.P''_{1} = P''$, and since $\approx_{m}^{*}$ is a congruence we have $P' \approx_{m}^{*} P''$, as required.

\[\square\]

Lemma 28. For all $P \approx_{m}^{*} Q$ and all $R \approx_{m}^{*} R'$, we have $P\{R/X\} \approx_{m}^{*} Q\{R'/X\}$.

Proof. By induction on the derivation of $P \approx_{m}^{*} Q$.

- $P \approx_{m}^{*} Q$: by Lemma 27, we have $P\{R/X\} \approx_{m}^{*} P\{R'/X\}$. Let $σ$ be a substitution which closes $R'$ and $P, Q$ except for $X$. By open extension definition we have $P\{R'σ/X\}σ \approx_{m} Q\{R'σ/X\}σ$, i.e. we have $P\{R'/X\}σ \approx_{m} Q\{R'/X\}$. Consequently we have $P\{R'/X\} \approx_{m} Q\{R'/X\}$, so we have $P\{R/X\} \approx_{m} Q\{R'/X\}$, i.e. $P\{R/X\} \approx_{m} Q\{R'/X\}$, as required.

- $P \approx_{m}^{*} T \approx_{m}^{*} Q$: by induction we have $P\{R/X\} \approx_{m}^{*} T\{R'/X\}$, and using the same technique as in the first case we have $T\{R'/X\} \approx_{m}^{*} Q\{R'/X\}$, hence we have $P\{R/X\} \approx_{m}^{*} Q\{R'/X\}$, as required.

- $op(\widetilde{P}) \approx_{m}^{*} op(\widetilde{P})$ with $\widetilde{P} \approx_{m}^{*} \widetilde{Q}$. By induction we have $P\{R/X\} \approx_{m}^{*} Q\{R'/X\}$, hence we have $op(P\{R'/X\}) \approx_{m}^{*} op(Q\{R'/X\})$ since $\approx_{m}^{*}$ is congruence. Consequently we have $P\{R/X\} \approx_{m}^{*} Q\{R'/X\}$, as required.

\[\square\]

We write $(\approx_{m})^{c}$ the restriction of $\approx_{m}$ to closed processes.
Lemma 29. Let \( P \sim_m^* Q \). For every substitution \( \sigma \), we have \( P\sigma \sim_m^* Q\sigma \) using a derivation of the same size.

Proof. By induction on \( P \sim_m^* Q \). Most cases are immediate by induction. The base case is \( P \sim_m^0 Q \). We show that \( P\sigma \sim_m^0 Q\sigma \). Let \( \sigma' \) a substitution that closes \( P\sigma \) and \( Q\sigma \), then \( \sigma\sigma' \) closes \( P \) and \( Q \), thus \( P\sigma\sigma' \sim_m Q\sigma\sigma' \).

Lemma 30. Let \( \langle \approx_m \rangle_c^* Q \). If \( P \xrightarrow{a.R} P' \), then for all \( R' \) such that \( R \approx_m^* P' \), there exists \( Q' \) such that \( Q \xrightarrow{a.R} Q' \) and \( P' \approx_m^* Q' \).

Proof. By induction on the size of the derivation of \( P \approx_m^* Q \).

- \( P \approx_m^0 Q \). Since \( P, R \) are closed, \( P' \) is closed. By Lemma 27 there exists \( P'' \) such that \( P \xrightarrow{a.R'} P'' \) and \( P' \approx_m^* P'' \). Since \( P, Q \) are closed, we have \( \approx_m P \); by bisimulation definition there exists \( Q' \) such that \( Q \xrightarrow{a.R'} Q' \) and \( P'' \approx_m Q' \). Let \( \sigma \) a substitution that closes \( P'' \). Since \( Q, R' \) are closed, \( Q' \) is closed and we have \( P''\sigma \approx_m Q' \) by Lemma 28. Consequently, we have \( P' \approx_m^* Q' \), and since \( P', Q' \) are closed, we have \( P' \approx_m^* Q' \), as required.

- \( P \approx_m^* T \approx_m^0 Q \). Let \( \sigma \) a substitution that closes \( T \); since \( P \) is closed and by lemma 29, we have \( P \approx_m^* T \sigma \). By induction there exists \( T' \) such that \( T\sigma \xrightarrow{a.R'} T' \) and \( P' \approx_m^* T' \). By open extension definition and since \( Q \) is closed, we have \( T\sigma \approx_m Q \). By Lemma 24 there exists \( Q' \) such that \( Q \xrightarrow{a.R'} Q' \) and \( T' \approx_m Q' \). Consequently we have \( P' \approx_m^* Q' \), and since \( P', Q' \) are closed too. Finally we have \( P' \approx_m^* Q' \), as required.

- \( \text{op}(\bar{P}) \approx_m^* \text{op}(\bar{Q}) \) with \( \bar{P} \approx_m^* \bar{Q} \). By case analysis on \( \text{op} \).

- \( P = P_1 | P_2 \) and \( Q = Q_1 | Q_2 \) with \( P_1 \xrightarrow{a.R} P'_1 \). By induction there exists \( Q'_1 \) such that \( Q_1 \xrightarrow{a.R'} Q'_1 \) and \( P'_1 \approx_m Q_1 \). Using rules \( \text{PAR}_\tau \) for \( \tau \)-actions and \( \text{PAR}_\tau^p \) for the observable action, we have \( Q \xrightarrow{a.R'} Q'_1 | Q_2 \). Since \( \approx_m \) is a congruence, we have \( P'_1 | P_2 \approx_m Q'_1 | Q_2 \). Since \( P, Q, R, R' \) are closed, all the involved processes are closed and we have \( P'_1 | P_2 \approx_m Q'_1 | Q_2 \), as required.

- Locality, replication: similar to the case above.

- \( P = a(X)P_1, Q = a(X)Q_1 \) with \( P \xrightarrow{a.R} P_1 \{R/X\} \). By Lemma 28, we have \( P_1 \{R/X\} \approx_m Q_1 \{R'/X\} \). Using rule \( \text{IN}_p \), we have \( Q \xrightarrow{a.R} Q_1 \{R'/X\} \). Since the involved processes are closed, we have \( P_1 \{R/X\} \approx_m Q_1 \{R/X\} \) as required.

- \( P = \nu b. P_1 \) and \( Q = \nu b. Q_1 \). Similar to the parallel case.
We inductively define $E \approx^*_{m} F$ as:

- $\Box \approx^*_{m} \Box$
- If $E \approx^*_{m} F$ and $P \approx^*_{m} Q$ then $E \mid P \approx^*_{m} F \mid Q$.
- If $E \approx^*_{m} F$ then $\nu a.E \approx^*_{m} \nu a.F$.
- If $E \approx^*_{m} F$ then $a[E] \approx^*_{m} a[F]$.

**Lemma 31.** If $E \approx^*_{m} F$, $P \approx^*_{m} Q$, and $E' \approx^*_{m} F'$ then $E \{P\} \approx^*_{m} F\{Q\}$ and $E \{E'\} \approx^*_{m} F\{F'\}$.

**Proof.** By induction on $E \approx^*_{m} F$.

- $\Box \approx^*_{m} \Box$: the result holds.
- $E_1 \mid P_1 \approx^*_{m} F_1 \mid Q_1$ by induction we have $E_1 \{P\} \approx^*_{m} F_1 \{Q\}$ and $E_1 \{E'\} \approx^*_{m} F_1 \{F'\}$. By congruence we have $E_1 \{P\} \mid P_1 \approx^*_{m} F_1 \{Q\} \mid Q_1$ and $E_1 \{E'\} \mid P_1 \approx^*_{m} F_1 \{F'\} \mid Q_1$, hence the result holds.
- Restriction, locality: similar to the parallel case.

We define $\text{fn}(E) = \text{fn}(E \{\Box\})$.

**Lemma 32.** If $E \approx^*_{m} F$ then $\text{fn}(E) = \text{fn}(F)$.

**Proof.** By induction on the derivation of $E \approx^*_{m} F$.

**Corollary 3.** Let $E \approx^*_{m} F$ and $P \approx^*_{m} Q$. We have $E \{\Box \mid P\} \approx^*_{m} F\{\Box \mid Q\}$, $E \{\nu a.\Box\} \approx^*_{m} F\{\nu a.\Box\}$, and $E \{a[\Box]\} \approx^*_{m} F\{a[\Box]\}$.

**Lemma 33.** If $E \approx^*_{m} F$ and $E = E_1\{\nu c.E_2\}$, then there exists $F_1, F_2$ such that $E_1 \approx^*_{m} F_1, E_2 \approx^*_{m} F_2,$ and $F = F_1\{\nu c.F_2\}$.

**Proof.** By induction on $E \approx^*_{m} F$

- If $E = E' \mid P$, $F = F' \mid Q$ with $E' \approx^*_{m} F'$ and $P \approx^*_{m} Q$. There exists $E'_1$ such that $E' = E'_1\{\nu c.E_2\}$ and $E_1 = E'_1 \mid P$. By induction there exists $F'_1, F'_2$ such that $F' = F'_1\{\nu c.F'_2\}$, $F'_1 \approx^*_{m} E'_1$, and $F'_2 \approx^*_{m} E_2$. We have $F = F_1\{\nu c.F_2\} \mid Q$ with $F'_1 \mid Q \approx^*_{m} E'_1 \mid P$ by congruence, hence the result holds.
- $E = \nu a.E'$, $F = \nu a.F'$ with $E' \approx^*_{m} F'$. If $c = a$, then we have $E_1 = \Box$ and $E_2 = E'$. We define $F_1 = \Box$ and $F_2 = F'$. We have the required result. If $c \neq a$, we use the same scheme as in the parallel case.
- Locality: similar to the parallel case.
Lemma 34. Let $P \xrightarrow{\pi,T,E} b P', T \approx_m T'$, and $E \approx_m F$; then there exists $T'', P''$ such that $T' \xrightarrow{\tau} T'', P \xrightarrow{\pi,T'', F} \approx \pi,T,E \xrightarrow{\tau} b P''$ and $P' \approx_m P''$.

Proof. By induction on the size of the derivation of $P \xrightarrow{\pi,T,E} b P'$.

- $P = \pi(R)S$ with $f_\pi(R) = \bar{b}, T_1 \xrightarrow{a,R} T_0$ and $P' = T_0 | E \{ S \}$. By Lemma 30 there exists $T''$ such that $T' \xrightarrow{a,R} T''$ and $T_0 \approx_m T''$. There exists $T_1, T_2$ such that $T_0 \xrightarrow{\tau} T_1 \xrightarrow{a,R} T_2 \xrightarrow{\tau} T''$. By rule Out$^\pi$, we have $P \xrightarrow{\pi,T_1,F} b T_2 \xrightarrow{\tau} F \{ S \}$. With $T_2 \xrightarrow{\tau} T''$, we have $T_2 \xrightarrow{F \{ S \} \tau} b T'' \xrightarrow{F \{ S \} \tau} P''$ by rule PAR$^\pi$, so finally we have $P \xrightarrow{\pi,T_1,F} b T'' \xrightarrow{F \{ S \} \tau} P''$ with $T'' \xrightarrow{\tau} T_1$. Since $\approx_m$ is a congruence, we have $P' \approx_m P''$, as required.

- $P = b[P_1]$ and passivation occurs: similar to the case above.

- $P = b[P_1]$ with $P_1 \xrightarrow{\pi,T,E \{ \bar{b} \{ \bar{d} \} \} \bar{b}} b P'_1$. By induction there exists $T'', P''$ such that $P_1 \xrightarrow{\tau} P''$ with $T' \xrightarrow{\tau} T''$, and $P'_1 \approx_m P''$. By rules Loc$^\pi$ and Loc$^b$, we have $P \xrightarrow{\tau} P''$ with $P'_1 \approx_m P''$ as wished.

- Parallel, replication: similar to the case above.

- $P = \nu_c P_1$ with $P_1 \xrightarrow{\pi,T,E \{ \nu_c, \bar{b} \{ \bar{d} \} \} \bar{b}} b P'_1$, $y \notin \bar{b}$. Similar to the case above.

- $P = \nu_c P_1$ with $P_1 \xrightarrow{\pi,T,E \{ \nu_y, \bar{d} \} \bar{b}} b P'_1$. By induction there exists $T'', P''$ such that $P_1 \xrightarrow{\tau} P''$ with $T' \xrightarrow{\tau} T''$, and $P'_1 \approx_m P''$. Using RESTR$^\nu$ for silent actions and EXTR$^b$, we have $P \xrightarrow{\tau} b P'' \xrightarrow{\tau} P'_1 \approx_m \nu_c P''$. Since $\approx_m$ is a congruence, we have $\nu_c P'_1 \approx_m \nu_c P''$, as required.

Lemma 35. Let $P \approx_m Q$. If $P \xrightarrow{\pi,T,E \{ \bar{d} \} b} P', T \approx_m T'$, and $E \approx_m F$, then there exists $T'', Q'$ such that $T' \xrightarrow{\tau} T'', Q \xrightarrow{\pi,T''} Q'$ and $P' \approx_m Q'$. Proof. We proceed by induction on the size of the derivation of $P \approx_m Q$.

- Suppose $P \approx_m Q$. Since $P, Q$ are closed, we have $P \approx_m Q$. By Lemma 34, there exists $T'', P''$ such that $T' \xrightarrow{\tau} T'', P \xrightarrow{\pi,T'', F} \approx \pi,T,E \xrightarrow{\tau} b P''$ and $P' \approx_m P''$. By Lemma 24, there exists $Q'$ such that $Q \xrightarrow{\pi,T''} Q'$ and $P'' \approx_m Q'$. Since the involved processes are closed, $P''$ is closed, so we have $P' \approx_m Q'$, and since the involved processes are closed, we have $P' \approx_m Q'$ as required.
• Suppose \( P \approx_{m}^{*} R \approx_{m}^{*} Q \). Let \( \sigma \) be a substitution that closes \( R \). Since \( P \) is closed, we have \( P \approx_{m}^{*} R \sigma \) by Lemma 29. Since \( Q \) is closed, we have \( R \sigma \approx_{m}^{*} Q \) by open extension definition. By induction, there exists \( T'' \), \( R' \) such that \( T'' \xrightarrow{\tau} T', \ R \sigma \xrightarrow{\pi_{T''},f_{\nu_{c}}} \tilde{b} \xrightarrow{\tau} R' \) and \( P' \approx_{m}^{*} R' \). By Lemma 24, there exists \( Q' \) such that \( Q \xrightarrow{\pi_{T''},f_{\nu_{c}}} \tilde{b} \xrightarrow{\tau} Q' \) and \( P' \approx_{m}^{*} Q' \). Since \( R', Q' \) are closed, we have \( R' \approx_{m}^{*} Q' \), consequently we have \( P' \approx_{m}^{*} Q' \). The involved processes are closed, hence we have \( P' \approx_{m}^{*} Q' \) as wished.

• If \( P = \text{op}(\tilde{P}) \) and \( Q = \text{op}(\tilde{Q}) \) with \( \tilde{P} \approx_{m}^{*} \tilde{Q} \).

\(- P = \pi(P_{1})P_{2} \) and \( Q = \pi(Q_{1})Q_{2} \) with \( T \xrightarrow{\pi_{P_{1}}} U, \ b = \text{fn}(P_{1}) \), and \( P' = U \mid E\{P_{2}\} \). Since \( P_{1} \approx_{m}^{*} Q_{1} \), we also have \( \text{fn}(Q_{1}) = \tilde{b} \). By Lemma 30 there exists \( U' \) such that \( T' \xrightarrow{\pi_{Q_{1}}} U' \) and \( U \approx_{m}^{*} U' \). There exists \( U_{1}, U_{2} \) such that \( T' \xrightarrow{\pi_{Q_{1}}} U_{1} \xrightarrow{\pi_{U_{1},f_{\nu_{c}}} \tilde{b}} U' \). Consequently we have \( Q \xrightarrow{\pi_{U_{1},f_{\nu_{c}}} \tilde{b}} U_{1} \mid E\{Q_{2}\} \). We have \( T' \xrightarrow{\pi_{Q_{1}}} U_{1} \) and \( Q \xrightarrow{\pi_{Q_{1}}} \tilde{b} \xrightarrow{\tau} U' \mid F\{Q_{2}\} = Q' \). We have \( P_{2} \approx_{m}^{*} Q_{2} \) and \( E \approx_{m}^{*} F_{\nu_{c}} \), so we have \( E\{P_{2}\} \approx_{m}^{*} F\{Q_{2}\} \) by Lemma 31, hence we have \( P' \approx_{m}^{*} Q' \), as required.

\(- P = b[P_{1}] \) with passivation: similar to the case above.

\(- P = P_{1} \mid P_{2} \) with \( P_{1} \xrightarrow{\pi_{T,E}\Box P_{2}} P' \). Since \( P_{2} \approx_{m}^{*} Q_{2} \) we have \( E\{\Box \mid P_{2}\} \approx_{m}^{*} F\{\Box \mid Q_{2}\} \). By induction there exists \( T' \), \( Q' \) such that \( T' \xrightarrow{\pi_{Q_{1}}} U_{1} \xrightarrow{\pi_{U_{1},f_{\nu_{c}}} \tilde{b}} U' \mid F\{Q_{2}\} = Q' \). By rules \( \text{PAR}^{P}_{T} \) and \( \text{PAR}^{Q}_{F} \) we have \( Q \xrightarrow{\pi_{U_{1},f_{\nu_{c}}} \tilde{b}} U' \mid F\{Q_{2}\} = Q' \), as required.

\(- P = \nu_{c}P_{1} \) with \( P_{1} \xrightarrow{\pi_{T,E}\Box \nu_{c}F} P' \). Similar to the case above above.

\(- P = \nu_{c}P_{1} \) with \( P_{1} \xrightarrow{\pi_{T,E}\Box c,\tilde{b}} P' \). By induction there exists \( T', Q'_{1} \) such that \( T' \xrightarrow{\tau} T', Q_{1} \xrightarrow{\pi_{T',f_{\nu_{c}}} \tilde{b}} Q'_{1} \) and \( P'_{1} \approx_{m}^{*} Q'_{1} \). By rules \( \text{PAR}^{P}_{T} \) and \( \text{EXTR}^{F}_{\nu_{c}} \) we have \( Q \xrightarrow{\pi_{T',f_{\nu_{c}}} \tilde{b}} \nu_{c}Q'_{1} \). Since \( \approx_{m}^{*} \) is a congruence and the involved processes are closed, we have \( \nu_{c}P'_{1} \approx_{m}^{*} \nu_{c}Q'_{1} \), as required.

\( \square \)

**Lemma 36.** Let \( P \approx_{m}^{*} Q \). If \( P \xrightarrow{\pi_{T,E}\tilde{b}} P', T \approx_{m}^{*} T', \ E \approx_{m}^{*} F \), then there exists \( Q' \) such that \( Q \xrightarrow{\pi_{T',F}\tilde{b}} Q' \) and \( P' \approx_{m}^{*} Q' \).
Proof. We proceed by induction on the number of names in $\tilde{b} \cap \text{bn}(E)$.

If this number is zero, the transition $P \xrightarrow{\pi.T.E} P'$ comes from rule $\text{CFree}_0^\pi$, we have $P \xrightarrow{\pi.T.E} P'$. By Lemma 35, there exists $T', Q'$ such that $T' \xrightarrow{\tau} T''$, $Q \xrightarrow{\pi.T.E} \tilde{b} Q'$, and $P' (\approx_m)_c Q'$. Using rule $\text{CFree}_0^\pi$ we have $Q \xrightarrow{\tilde{b} \pi.T.E} Q'$, so we have $Q \xrightarrow{\tilde{b} \pi.T.E} Q'$, as wished.

Otherwise, the derivation comes from rule $\text{Capt}_0^\pi$: we have $E = E_1 \{\nu.c.E_2\}$, $c \in \tilde{b}$, $P' = \nu.c.P'_1$ and $P \xrightarrow{\pi.T.E.(E_2)} P'_1$. By Lemma 33 there exists $F_1, F_2$ such that $F = F_1 \{\nu.c.F_2\}$, $F_1 \approx_m E_1$, and $F_2 \approx_m E_2$. By induction there exists $Q'_1$ such that $Q \xrightarrow{F} Q'_1$ and $P'_1 \approx_m Q'_1$. By rule $\text{Capt}_0^\pi$ we have $Q \xrightarrow{\tau} \nu.c.Q'_1 = Q'$. By congruence, we have $P' (\approx_m)_c Q'$, as wished.

\[\square\]

Lemma 37. Let $P (\approx_m)_c Q$. If $P \xrightarrow{E} P'$ then there exists $Q'$ such that $Q \xrightarrow{E} Q'$ and $P' (\approx_m)_c Q'$.

Proof. We proceed by induction on the size of the derivation of $P (\approx_m)_c Q$.

- Suppose $P \approx_m Q$. Since $P, Q$ are closed, we have $P \approx_m Q$. The result holds by bisimilarity definition (and since the processes are closed).

- Suppose $P \approx_m R \approx_m Q$. Let $\sigma$ be a substitution that closes $R$. Since $P$ is closed, we have $P \approx_m R \sigma$ by Lemma 29. Since $Q$ is closed, we have $R \sigma \approx_m Q$ by open extension definition. By induction, there exists $R'$ such that $R \sigma \xrightarrow{E} R'$ and $P' (\approx_m)_c R'$. By Lemma 24, there exists $Q'$ such that $Q \xrightarrow{E} Q'$ and $R' \approx_m Q'$. Since $R', Q'$ are closed, we have $R' \approx_m Q'$, consequently we have $P' \approx_m Q'$. The involved processes are closed, hence we have $P' (\approx_m)_c Q'$ as wished.

- If $P = op(\tilde{P})$ and $Q = op(\tilde{Q})$ with $\tilde{P} (\approx_m)_c \tilde{Q}$.

  - $P = P_1 | P_2$ with $P_1 \xrightarrow{E} P'_1$. By induction there exists $Q'_1$ such that $Q_1 \xrightarrow{E} Q'_1$ and $P'_1 (\approx_m)_c Q'_1$. Using rule $\text{Par}_r^\pi$, we have $Q \xrightarrow{E} Q'_1 | Q_2$ and since $\approx_m$ is a congruence and the involved processes are closed, we have $P'_1 | P_2 (\approx_m)_c Q'_1 | Q_2$ as required.

  - Locality, restriction, replication without communication: similar to the case above.

  - Communication: $P = P_1 | P_2$ with $P_1 \xrightarrow{\pi.P_2}{b} P'$. Since $P_2 (\approx_m)_c Q_2$, by Lemma 36 there exists $Q'$ such that $Q_1 \xrightarrow{\pi.Q_2}{b} Q'$ and $P' (\approx_m)_c Q'$. We have $Q_1 \xrightarrow{E} Q'_1 \pi.Q_2 \xrightarrow{b} Q'$ and $Q_2 \xrightarrow{E} Q'_2$. By $\text{Par}_r^\pi$, we have $Q \xrightarrow{E} Q'_1 | Q'_2$: by $\text{HO}_r^\pi$ and $\text{Par}_r^\pi$, we have $Q'_1 | Q'_2 \xrightarrow{E} Q'$. Hence we have $Q \xrightarrow{E} Q'$ and $P' (\approx_m)_c Q'$, as required.

58
− Replication with communication: similar to the case above.

Notice that Lemmas 37, 36, and 30 show that \((\approx_m)_c^\ast\) is a weak complementary simulation.

**Lemma 38.** If \(P (\approx_m)_c^\ast Q\) and \(P \xrightarrow{\lambda} P'\), there exists \(Q'\) such that \(Q \xrightarrow{\lambda} Q'\) and \(P' (\approx_m)_c^\ast Q'\).

**Proof.** Similar to the one of Lemma 24, using Lemmas 37, 36, and 30.

**Lemma 39.** Let \((\approx_m)^\ast\) be the reflexive and transitive closure of \((\approx_m)^\ast\).

- \((\approx_m)^\ast\) is symmetric.
- \((\approx_m)^\ast \) is a weak complementary bisimulation.

**Proof.** We prove that \((\approx_m)^\ast\) is symmetric, it is enough to prove that \((\approx_m)^\ast\) is a weak complementary simulation. Let \(P (\approx_m)^\ast Q\); there exists \(k\) such that \(P (\approx_m)^\ast Q\). We proceed by induction on \(k\). The result holds for \(k = 0\), suppose it holds for \(l \leq k\), we prove for \(k + 1\). Let \(P((\approx_m)^\ast P_k (\approx_m)^\ast Q\).

- \(\text{fn}(P) = \text{fn}(P_k) = \text{fn}(Q)\)

- If \(P \xrightarrow{\lambda} P'\), then by induction there exists a process \(P_k\) such that \(P_k \xrightarrow{\lambda} P'_k\) and \(P'((\approx_m)_c^\ast P'_k\). By Lemma 38, there exists \(Q'\) such that \(Q \xrightarrow{\lambda} Q'\) and \(P_k (\approx_m)_c^\ast Q'\). The result then holds by transitivity.

**Theorem 15.** \(\approx_m\) is a congruence.

**Proof.** We have \(\approx_m \subseteq ((\approx_m)_c^\ast \subseteq \approx_m\), hence \((\approx_m)_c^\ast = \approx_m\), and \((\approx_m)^\ast\) is a congruence.
Appendix B. Abstraction Equivalence in $\text{HO}\pi P$

We remind the definition of finite processes:

**Definition 23.** A finite process is a HO$\pi P$ process built on the following grammar:

$$P_F ::= 0 \mid P_F \mid P_F \mid \nu a.P_F \mid a(P)P_F \mid a[P_F]$$

A concretion $\nu a.(R)S$ is finite iff $S$ is finite. An abstraction $(X)P$ is finite iff $P$ is finite. We write $A_F$ the set of finite agents.

We first prove some properties on finite processes:

**Lemma 40.** Let $F$ be a finite abstraction. For all process $P$, $F \circ P$ is finite.

*Proof.* We have $F = (X)P_F$ for some finite process $P_F$, hence $F \circ P = P_F[P/X]$. Since $P_F$ is finite, $X$ appears in messages only, hence after substitution, $P$ appears in messages only. Since any processes are allowed as messages, $F \circ P$ is a finite process.

**Lemma 41.** Let $P_F$ be a finite process. If $P_F \xrightarrow{\alpha} A$ for some $\alpha$, then $A$ is finite.

*Proof.* By induction on $P_F$:

- $P_F = 0$: no available transition.

- $P_F = P_1 \mid P_2$, where $P_1$ and $P_2$ are finite processes. The possible transitions come from rules PAR, HO, and their symmetric. In the PAR case, we have $P_1 \xrightarrow{\alpha} A$, and $P_F \xrightarrow{\alpha} A \mid P_2$. By induction, $A$ is finite, hence $A \mid P_2$ is finite. The proof is similar for the symmetric rule.

In the HO case, we have $P_1 \xrightarrow{\nu a} C = \nu x.(R)S$, $P_2 \xrightarrow{\nu a} F$, and $P_F \xrightarrow{\alpha} F \bullet C$. By induction, $F$ and $S$ are finite. By lemma 40, $F \circ R$ is finite, hence $\nu x.(F \circ R \mid S) = F \bullet C$ is finite.

- $P_F = \nu a.P'$, where $P'$ is finite. The possible transitions come from rule RESTR: we have $P_F \xrightarrow{\nu a} \nu a.A$ with $P' \xrightarrow{\nu a} A$. By induction, $A$ is finite, hence $\nu a.A$ is finite.

- $P_F = \pi(R)S$, where $S$ is finite. The possible transition comes from rule CONCR $P_F \xrightarrow{\pi} \langle R \rangle S$. Since $S$ is finite, $\langle R \rangle S$ is finite.

- $P_F = a(X)P'$, where $P'$ is finite. The possible transition comes from rule ABSTR $P_F \xrightarrow{a} (X)P'$. Since $P'$ is finite, $(X)P'$ is finite.

- $P_F = a[P']$, where $P'$ is finite. The possible transitions come from rules PASSIV and LOC. In the PASSIV case, we have $P_F \xrightarrow{P'} \langle P' \rangle 0$. Since $0$ is finite, $\langle P' \rangle 0$ is finite. In the LOC case, we have $P' \xrightarrow{\alpha} A$ and $P_F \xrightarrow{\alpha} a[A]$. By induction, $A$ is finite, hence $a[A]$ is finite.
Lemma 42. Let \( P_F \) be a finite process.

- The set \( \{ \alpha | \exists A, P_F \xrightarrow{\alpha} A \} \) is finite.
- For all action \( \alpha \), the set \( \{ A | P_F \xrightarrow{\alpha} A \} \) is finite.

Proof. Easy by induction on \( P_F \).

We now prove that a finite process “terminates”. To this end, we introduce the size of a finite abstraction \( F \) only on the size of the continuation, we have

\[
s(0) = 0 \quad s(P_1 | P_2) = s(P_1) + s(P_2) \quad s(\nu a.P) = s(P)
\]

\[
s(\pi(R)S) = 1 + s(S) \quad s(a(X)P) = 1 + s(P) \quad s(a[P]) = 1 + s(P)
\]

\[
\text{The size of a finite concretion } \nu \bar{x}.(R)P_F \text{ is defined by } s(C) = s(P_F), \text{ and the size of a finite abstraction } F = (X)P_F \text{ is defined by } s(F) = s(P_F). \text{ By definition, the size of an agent is a non-negative integer.}
\]

Lemma 43. Let \( F \) be a finite abstraction. For all processes \( P \), we have \( s(F \circ P) = s(F) \).

Proof. We have \( F = (X)P_F \) for some finite process \( P_F \), hence \( F \circ P = P_F \{ P/X \} \). Since \( P_F \) is finite, \( X \) appears in messages only, hence after substitution, \( P \) appears in messages only. Since the size of a message output depends only on the size of the continuation, we have \( s(F \circ P) = s(F) \).

Lemma 44. Let \( P_F \) be a finite process. If \( P_F \xrightarrow{\alpha} A \), then \( s(P_F) > s(A) \).

Proof. By induction on \( P_F \):

- \( P_F = 0 \): no available transition.
- \( P_F = P_1 | P_2 \), where \( P_1 \) and \( P_2 \) are finite processes. The possible transitions come from rules \( \text{PAR}, \text{HO}, \) and their symmetric. In the \( \text{PAR} \) case, we have \( P_1 \xrightarrow{\alpha} A \), and \( P_F \xrightarrow{\alpha} A | P_2 \). By induction, we have \( s(A) < s(P_1) \), hence we have \( s(A | P_2) = s(A) + s(P_2) < s(P_1) + s(P_2) = s(P_F) \) as required.

In the \( \text{HO} \) case, we have \( P_1 \xrightarrow{\pi} C = \nu \bar{x}.(R)S, P_2 \xrightarrow{\alpha} F, \) and \( P_F \xrightarrow{\tau} F \bullet C \). By induction, \( s(F) < s(P_1) \) and \( s(C) < s(P_2) \). By lemma 43, we have \( s(F \circ R) = s(F) \), hence we have \( s(F \bullet C) = s(F \circ R | S) = s(F \circ R | S + s(S) = s(F) + s(S) < s(P_1) + s(P_2) = s(P_F) \) as required.
- \( P_F = \nu a.P' \), where \( P' \) is finite. The possible transitions come from rule \( \text{RESTR} \) : we have \( P_F \xrightarrow{\alpha} \nu a.A \) with \( P' \xrightarrow{\alpha} A \). By induction, we have \( s(A) < s(P') \) hence we have \( s(\nu a.A) = s(A) < s(P') = s(P_F) \) as required.
• $P_F = \pi(R)S$, where $S$ is finite. The possible transition comes from rule
Concr $P_F \xrightarrow{\pi} \langle R \rangle S$. We have $s(P_F) = 1 + s(S) > s(S) = s(\langle R \rangle S)$ as required.

• $P_F = a(X)P'$, where $P'$ is finite. The possible transition comes from rule
Abstr $P_F \xrightarrow{a} (X)P'$. We have $s(P_F) = 1 + s(P') > s(P') = s((X)P')$ as required.

• $P_F = a[P']$, where $P'$ is finite. The possible transitions comes from rules
Passiv and Loc. In the Passiv case, we have $P_F \xrightarrow{\pi} \langle P' \rangle 0$. We have
$s(P_F) = 1 + s(P') > 0 = s(\langle P' \rangle 0)$ as required.
In the Loc case, we have $P' \xrightarrow{a} A$ and $P_F \xrightarrow{a} a[A]$. We have $s(A) < s(P')$, hence we have $s(a[A]) = 1 + s(A) < 1 + s(P') = s(P_F)$ as required.

Lemma 45. Let $P_F$ be a finite process. There is no infinite sequence of processes
$(P_i)$, such that $P_0 = P_F$ and for all $i$, $P_i \xrightarrow{i} P_{i+1}$ or $P_i \xrightarrow{\nu \bar{x}. \langle R \rangle P_{i+1}}$ or $P_i \xrightarrow{a} F$
with $F \circ P = P_{i+1}$ for some $P$.

Proof. Suppose we have a sequence of processes $(P_i)_i$ as defined in the lemma.
In this case, the sequence $(s(P_i))_i$ is an infinite sequence of strictly decreasing
non-negative integers by lemma 44, which is not possible.

All these properties allow us to define the depth of a finite process:

Definition 24. We define inductively the depth of a finite agent $A_F$, written
$d(A_F)$, as:

• $d(P_F) = 0$ if there is no transition from $P_F$.
• $d(P_F) = 1 + \max \{d(A) \mid \exists \alpha, P_F \xrightarrow{\alpha} A\}$ otherwise.
• For all finite concretions $\nu \bar{x}. \langle P \rangle P_F$, we have $d(\nu \bar{x}. \langle P \rangle P_F) = d(P_F)$.
• For all finite abstractions $(X)P_F$, we have $d(F) = d(P_F)$.

We now prove that testing a finite process is not enough. We suppose that internal choice is added to the calculus, and we define:

$$F_0 \triangleq (X_0)X_0, G_0 \triangleq (X_0)(X_0 \mid X_0)$$

and for all $n > 0$

$$F_n \triangleq (X_n)R^1_n + R^2_n$$
$$G_n \triangleq (X_n)S^1_n + S^2_n$$

62
with
\[
\begin{align*}
R_n^1 & \triangleq \nu a_n.(a_n[X_n] \mid a_n \cdot F_{n-1}) \\
R_n^2 & \triangleq \nu a_n.\tau. (G_{n-1} \circ X_n) \\
S_n^1 & \triangleq \nu a_n.(a_n[X_n] \mid a_n \cdot G_{n-1}) \\
S_n^2 & \triangleq \nu a_n.\tau. (F_{n-1} \circ X_n)
\end{align*}
\]

In the following proofs, for all processes \(P, R\) and abstraction \(F\) such that \(\text{fv}(P) = \{X\}\) and \(X\) appears exactly once in \(P\), we define \(P \circ R \triangleq P\{R/X\}\) and \(P \circ F \triangleq P\{F/X\}\).

**Lemma 46.** For all \(P_F\) such that \(d(P_F) = 0\), we have \(F_0 \circ P_F \sim G_0 \circ P_F\).

**Proof.** Since \(d(P_F) = 0\), \(P_F\) and \(P_F \mid P_F\) cannot perform any transition. We have \(\text{fn}(P_F) = \text{fn}(P_F \mid P_F)\), so we have \(F_0 \circ P_F \sim G_0 \circ P_F\).

**Lemma 47.** Let \(n > 0\). For all \(P_F\) such that \(d(P_F) \leq n\), we have \(F_n \circ P_F \sim G_n \circ P_F\).

**Proof.** We prove that relation
\[
\mathcal{R}_n \triangleq \{(\{P\{F_k \circ P_F^k, F_{k_1} \circ P_F^{k_1}, F_{k_2} \circ P_F^{k_2}, \ldots \}/X\}, \{P\{G_k \circ P_F^k, S_{k_1} \circ P_F^{k_1}, S_{k_2} \circ P_F^{k_2}, \ldots \}/X\}) \mid d(P_F^k) \leq k \leq n \land d(P_F^l) \leq l - 1 \leq n\}
\]
is a strong early context simulation.

Let \(P_1 \mathcal{R}_n P_2\). We proceed by case analysis on the transition initiated by \(P_1\).

If the transition comes from \(P\) or \(C\) without any interaction with the processes \(F_k \circ P_F^k\), then \(P_2\) matches with the same transition.

The transition comes from a process \(F_{k_0} \circ P_F^{k_0}\), in which passivation of locality \(a_{k_0}\) has been triggered. We have
\[
P_1 \xrightarrow{\tau} \mathcal{C}\{P\{\nu a_{k_0}.(F_{k_0-1} \circ P_F^{k_0})/X_{k_0}\}/(F_{k_0} \circ P_F^{k_0}, R_{k_1} \circ P_F^{k_1}/(X \setminus X_{k_0}))\} \triangleq P'_1.
\]

We distinguish two cases; suppose first that we have \(d(P_F^{k_0}) \leq k_0 - 1\). Process \(P_2\) matches with passivation of \(a_{k_0}\) in \(G_{k_0} \circ P_F^{k_0}\), i.e.
\[
P_2 \xrightarrow{\tau} \mathcal{C}\{P\{\nu a_{k_0}.(G_{k_0-1} \circ P_F^{k_0})/X_{k_0}\}/(G_{k_0} \circ P_F^{k_0}, S_{k_1} \circ P_F^{k_1}/(X \setminus X_{k_0}))\} \triangleq P'_2.
\]

Let \(P' \triangleq P\{\nu a_{k_0}.X_{k_0}/X_{k_0}\}\). Processes \(P'_1\) and \(P'_2\) can be written
\[
\begin{align*}
P'_1 &= \mathcal{C}\{P'\{F_k \circ P_F^k, F_{k_0-1} \circ P_F^{k_0}, R_{k_1} \circ P_F^{k_1}/X\}\} \\
P'_2 &= \mathcal{C}\{P'\{G_k \circ P_F^k, G_{k_0-1} \circ P_F^{k_0}, S_{k_1} \circ P_F^{k_1}/X\}\}
\end{align*}
\]

63
and since we have $d(P_{F_k}^{k_0}) \leq k_0 - 1 \leq n$, we have $P'_1 \mathcal{R}_n P'_2$, as required.

In the case $d(P_{F_k}^{k_0}) = k_0$, process $P_2$ matches with the $\tau$-action in the sub-process $S_{k_0}$ of $G_{k_0} \circ P_{F_k}^{k_0}$. We have then

$$P_2 \xrightarrow{\tau} \mathcal{C}\{P\{ \nu_{a_{k_0}} (F_{k_0-1} \circ P_{F_k}^{k_0}))/X_{k_0})\{G_k \circ P_{F_k}^{k_0}, S_1 \circ P_{F_k}^{k_0} / (\bar{X} \setminus X_{k_0})\}\} \triangleq P'_2$$

Let $P' \triangleq P\{ \nu_{a_{k_0}} (F_{k_0-1} \circ P_{F_k}^{k_0})/X_{k_0}\}; P'_1$ and $P'_2$ can be written

$$P'_1 = \mathcal{C}\{P'\{ F_k \circ P_{F_k}^{k_0}, R_1 \circ P_{F_k}^{k_0} / (\bar{X} \setminus X_{k_0})\}\}$$
$$P'_2 = \mathcal{C}\{P'\{ G_k \circ P_{F_k}^{k_0}, S_1 \circ P_{F_k}^{k_0} / (\bar{X} \setminus X_{k_0})\}\}$$

Hence we have $P'_1 \mathcal{R}_n P'_2$ as required.

The transition from $P_1$ comes from a process $R_{k_0}^1 \circ P_{F_k}^{k_0}$, in which passivation of locality $a_{k_0}$ is triggered. By definition, we have $d(P_{F_k}^{k_0}) \leq l_0 - 1$, hence this case is similar to first sub-case of the previous case.

Suppose that the transition from $P_1$ comes from the $\tau$-action of a process $R_{k_0}$ inside a process $F_{k_0} \circ P_{F_k}^{k_0}$. We have then

$$P_1 \xrightarrow{\tau} \mathcal{C}\{P\{ \nu_{a_{k_0}} (G_{k_0-1} \circ P_{F_k}^{k_0}))/X_{k_0})\{F_k \circ P_{F_k}^{k_0}, R_1 \circ P_{F_k}^{k_0} / (\bar{X} \setminus X_{k_0})\}\} \triangleq P'_1.$$  

Process $P_2$ matches with passivation of $a_{k_0}$ inside process $G_{k_0} \circ P_{F_k}^{k_0}$, i.e.

$$P_2 \xrightarrow{\tau} \mathcal{C}\{P\{ \nu_{a_{k_0}} (G_{k_0-1} \circ P_{F_k}^{k_0}))/X_{k_0})\{G_k \circ P_{F_k}^{k_0}, S_1 \circ P_{F_k}^{k_0} / (\bar{X} \setminus X_{k_0})\}\} \triangleq P'_2$$

Let $P' \triangleq P\{ \nu_{a_{k_0}} (G_{k_0-1} \circ P_{F_k}^{k_0})/X_{k_0}\}; we rewrite $P'_1$ and $P'_2$ in

$$P'_1 = \mathcal{C}\{P'\{ F_k \circ P_{F_k}^{k_0}, R_1 \circ P_{F_k}^{k_0} / (\bar{X} \setminus X_{k_0})\}\}$$
$$P'_2 = \mathcal{C}\{P'\{ G_k \circ P_{F_k}^{k_0}, S_1 \circ P_{F_k}^{k_0} / (\bar{X} \setminus X_{k_0})\}\}$$

hence we have $P'_1 \mathcal{R}_n P'_2$.

The transition comes from a process $F_k \circ P_{F_k}^{k_0}$, in which $P_{F_k}^{k_0}$ performs an action $P_{F_k}^{k_0} \xrightarrow{\tau} P_{F_k}^{k_0}$. We have then

$$P_1 \xrightarrow{\tau} \mathcal{C}\{P\{ R_{k_0}^1 \circ P_{F_k}^{k_0}/X_{k_0})\{F_k \circ P_{F_k}^{k_0}, R_1 \circ P_{F_k}^{k_0} / (\bar{X} \setminus X_{k_0})\}\} \triangleq P'_1.$$  

Process $P_2$ matches with a similar transition

$$P_2 \xrightarrow{\tau} \mathcal{C}\{P\{ S_{k_0} \circ P_{F_k}^{k_0}/X_{k_0})\{G_k \circ P_{F_k}^{k_0}, S_1 \circ P_{F_k}^{k_0} / (\bar{X} \setminus X_{k_0})\}\} \triangleq P'_2.$$  

By the definition of depth, we have $d(P_{F_k}^{k_0}) \leq d(P_{F_k}^{k_0}) - 1 \leq k_0 - 1 \leq n$, hence we have $P'_1 \mathcal{R}_n P'_2$.  

64
The transition from $P_1$ comes from a process $F_{k_0} \circ P^{k_0}_F$, in which $P^{k_0}_F$ performs an input $\nu F' \xrightarrow{E} F$. We have then

$$P_1 \xrightarrow{\nu b} C\{P\{R^1_{k_0} \circ F/X_{k_0}\}{F_k \circ P^k_F, R^l_1 \circ P^l_F/(\tilde{X} \setminus X_{k_0})}\} \overset{\Delta}{=} F_1.$$  

Let $C = \nu\tilde{b}.(T)$. Process $P_2$ matches with a similar transition

$$P_2 \xrightarrow{\nu b} C\{P\{S^1_{k_0} \circ F/X_{k_0}\}{G_k \circ P^k_F, S^l_1 \circ P^l_F/(\tilde{X} \setminus X_{k_0})}\} \overset{\Delta}{=} F_2.$$  

We have

$$F_1 \cdot C = \nu\tilde{b}.(C | U); F_1 \cdot C \text{ and } F_2 \cdot C \text{ can be written as}$$
$$F_1 \cdot C = C'\{P\{R^1_{k_0} \circ (F \circ T)/X_{k_0}\}{F_k \circ P^k_F, R^l_1 \circ P^l_F/(\tilde{X} \setminus X_{k_0})}\}$$
$$F_2 \cdot C = C'\{P\{S^1_{k_0} \circ (F \circ T)/X_{k_0}\}{G_k \circ P^k_F, S^l_1 \circ P^l_F/(\tilde{X} \setminus X_{k_0})}\}$$

By definition of depth, we have $d(F \circ T) = d(F) \leq d(P^{k_0}_F) - 1 \leq k_0 - 1 \leq n$, hence we have $F_1 \cdot C \overset{\Delta}{=} F_2 \cdot C$.

The transition from $P_1$ comes from a process $F_{k_0} \circ P^{k_0}_F$, in which $P^{k_0}_F$ performs an output $\nu F' \xrightarrow{E} C = \nu b.(T)U$. We have

$$P_1 \xrightarrow{\nu b, \tilde{b}'.(T)C'} P\{R^1_{k_0} \circ U/X_{k_0}\}{F_k \circ P^k_F, R^l_1 \circ P^l_F/(\tilde{X} \setminus X_{k_0})}\} \overset{\Delta}{=} C_1$$

where $\tilde{b}'$ is the set of names captured by $C$, and $C'$ is the context resulting from $C$ after removing the name restrictions on $\tilde{b}'$. Let $F$ be an abstraction and $E$ be an evaluation context. Process $P_2$ matches with the transition

$$P_2 \xrightarrow{\nu b, \tilde{b}'.(T)C'} P\{S^1_{k_0} \circ U/X_{k_0}\}{G_k \circ P^k_F, S^l_1 \circ P^l_F/(\tilde{X} \setminus X_{k_0})}\} \overset{\Delta}{=} C_2.$$  

We have

$$F \cdot E\{C_1\} =$$
$$\nu b, \tilde{b}', \tilde{b}'.(F \circ T | E'\{C'\{P\{R^1_{k_0} \circ U/X_{k_0}\}{F_k \circ P^k_F, R^l_1 \circ P^l_F/(\tilde{X} \setminus X_{k_0})}\}\})$$

$$F \cdot E\{C_2\} =$$
$$\nu b, \tilde{b}', \tilde{b}'.(F \circ T | E'\{C'\{P\{S^1_{k_0} \circ U/X_{k_0}\}{G_k \circ P^k_F, S^l_1 \circ P^l_F/(\tilde{X} \setminus X_{k_0})}\}\})$$
where \( \tilde{b} \) and \( E' \) are defined the same way as \( b' \) and \( C' \).

Let \( C^n \triangleq \nu b, b', \tilde{b}.(F \circ \tau | E'(C')) \); \( F \cdot E(C_1) \) and \( F \cdot E(C_2) \) can be written

\[
F \cdot E(C_1) = C^n \{ P[R_k \circ U/X_k] \{ F_k \circ P_{F}, R_{k_1} \circ P_{F}/(X \setminus X_k) \} \}
\]

\[
F \cdot E(C_2) = C^n \{ P[S_k \circ U/X_k] \{ G_k \circ P_{F}, R_{k_1} \circ P_{F}/(X \setminus X_k) \} \}
\]

By definition of depth we have \( d(U) = d(C) \leq d(P_{F}^{k_0}) - 1 \leq k_0 - 1 \leq n \), hence we have \( F \cdot E(C_1) \mathrel{\triangleleft} R_n F \cdot E(C_2) \).

The transition from \( P_1 \) comes from the communication between two finite processes, between a finite process and \( P \), or between a finite process and \( C \). We only deal with communication between finite processes, the other cases are similar. Suppose we have \( P_{F}^{k_0} \mathrel{\triangleleft} F \) and \( P_{F}^{k_1} \mathrel{\triangleleft} C = \nu b.(T)U \). Then we have

\[
P_1 \xrightarrow{\nu a} C \{ P'[F_k \circ P_{F}, R_{k_1} \circ P_{F}, R_{k_0} \circ (F \circ T), R_{k_1} \circ U/X_k, X_k] \} \mathrel{\triangleleft} P_1'
\]

where \( P' \) is obtained from \( P \) by scope extrusion of names \( \tilde{b} \). Process \( P_2 \) matches with the following transition:

\[
P_2 \xrightarrow{\nu a} C \{ P'[G_k \circ P_{F}, S_{k_1} \circ P_{F}, S_{k_0} \circ (F \circ T), S_{k_1} \circ U/X_k, X_k, X_k] \} \mathrel{\triangleleft} P_2'
\]

We have \( d(F \circ T) = d(F) \leq d(P_{F}^{k_0}) - 1 \leq k_0 - 1 \leq n \) and \( d(S) = d(C) \leq d(P_{F}^{k_1}) - 1 \leq k_1 - 1 \leq n \), hence we have \( P_1' \mathrel{\triangleleft} P_2' \), as required.

Similarly, we can prove that \( R_n \) is a strong early context bisimilarity.

\( \square \)

Let \( (m_k) \) be a sequence of pairwise distinct fresh names. Let \( Q_1 \mathrel{\triangleleft} m_1,0 \) and \( Q_{k+1} \mathrel{\triangleleft} m_{k+1},Q_k \) for all \( k > 1 \).

**Lemma 48.** For all \( n \), we have \( F_n \circ Q_{n+1} \sim G_n \circ Q_{n+1} \).

**Proof.** We proceed by induction on \( n \). For \( n = 0 \), we have \( F_0 \circ m_1,0 = m_1,0 \sim m_1,0 | m_1,0 = G_0 \circ m_1,0 \) as wished.

Let \( n > 0 \). We have

\[
F_n \circ Q_{n+1} \xrightarrow{m_{n+1}} \nu a_n.(a_n[Q_n] | a_n.F_{n-1}) \mathrel{\triangleleft} P_1,
\]

and \( G_n \circ Q_{n+1} \) can match only with transition

\[
G_n \circ Q_{n+1} \xrightarrow{m_{n+1}} \nu a_n.(a_n[Q_n] | a_n.G_{n-1}) \mathrel{\triangleleft} P_2.
\]

After passivation of locality \( a_n \), we have

\[
P_1 \xrightarrow{\nu a_n.(F_{n-1} \circ Q_n)},
\]

66
and $P_2$ can match only with

$$P_2 \xrightarrow{\nu a_n} (G_{n-1} \circ Q_n).$$

Since we have $a_n \notin \text{fn}(F_{n-1} \circ Q_n)$ (respectively $a_n \notin \text{fn}(G_{n-1} \circ Q_n)$), we have $\nu a_n . (F_{n-1} \circ Q_n) \sim F_{n-1} \circ Q_n$ (respectively $\nu a_n . (G_{n-1} \circ Q_n) \sim G_{n-1} \circ Q_n$).

By induction, we have $F_{n-1} \circ Q_n \sim G_{n-1} \circ Q_n$, hence we have $F_n \circ Q_{n+1} \sim G_n \circ Q_{n+1}$.

**Appendix C. Normal Bisimilarity in HOP**

**Lemma 49.** Let $E$ be an evolution context and $P \xrightarrow{e} A$. Then $E\{P\} \xrightarrow{e} E\{A\}$ and the hole in $E$ is not under a replication or choice operator.

**Proof.** Immediate by induction on $E$, and considering the rules PAR, LOC, REPLIC, SUM.

**Lemma 50.** Let $P, Q$ such that $fs(P, Q) \subseteq \{X\}$ and $m, n$ two names which do not occur in $P, Q$. Suppose we have $P\{m.n.0/X\} \sim \nu Q\{m.n.0/X\}$ and $P\{m.n.0/X\} \xrightarrow{m} P'\{m.n.0/X\} \{n.0/Y\} = P_n$ matched by $Q\{m.n.0/X\} \xrightarrow{m} Q'\{m.n.0/X\} \{n.0/Y\} = Q_n$. One of the following holds:

- There exists $P_1, Q_1$ such that $P_n = n.0 \mid P_1, Q_n = n.0 \mid Q_1$ with $P_1 \sim \nu Q_1$.
- There exists $a_1, \ldots, a_k, P_1 \ldots P_{k+1}, Q_1 \ldots Q_{k+1}$ such that

  $$P_n = a_1[\ldots a_{k-1}[a_k[n.0 \mid P_{k+1}] \mid P_k] \mid P_{k-1} \ldots ] \mid P_1$$

  and

  $$Q_n = a_1[\ldots a_{k-1}[a_k[n.0 \mid Q_{k+1}] \mid Q_k] \mid Q_{k-1} \ldots ] \mid Q_1$$

  and for all $1 \leq j \leq k + 1$, $P_j \sim \nu Q_j$.

**Proof.** Since $P_n$ can only perform one $\xrightarrow{e}$ transition, we can detect if $n.0$ is in a locality or not: if there exists a transition $P_n \xrightarrow{\alpha} (R'n)S'_n$ for some $a$ such that $R'_n$ may perform a transition $\xrightarrow{e}$, then the transition is a passivation and the process $n.0$ is in a locality in $P_n$. Otherwise, $n.0$ is not in a locality.

By lemma 49, $n.0$ is only under localities and parallel compositions in $P_n$ and $Q_n$.

We show that if $n.0$ is not under a locality in $P_n$, it is also not under a locality in $Q_n$. Suppose $n.0$ is not in a locality in $P_n$ and is in a locality in $Q_n$. We have $Q_n \xrightarrow{e} (E\{n.0\})Q''$ for some $a, E, Q''$. These transitions can only be matched by a passivation of $n.0$ in $P_n$, which is impossible by hypothesis, hence a contradiction. We have the same reasoning if $n.0$ is in a locality in $P_n$ and not in a locality in $Q_n$. Therefore if $n.0$ is not in a locality in $P_n$, it is not in a locality in $Q_n$. Consequently in this case, there exists $P_1, Q_1$ such that
$P_n = n \cdot 0 \mid P_1$ and $Q_n = n \cdot 0 \mid Q_1$. Hence we have $P_n \Rightarrow P_1$, which can only be matched by $Q_n \Rightarrow Q_1$, so we have $P_1 \sim_1 Q_1$.

We suppose now that $n \cdot 0$ is under a locality in $P_n$ and $Q_n$. We prove that $n \cdot 0$ is under the same hierarchy of localities in $P_n, Q_n$, and the existence of the pairwise bisimilar processes defined in the lemma. Suppose $n \cdot 0$ is under $k$ localities $a_1, \ldots, a_k$ in $P_n$ and under $l$ localities $b_1, \ldots, b_l$ in $Q_n$, with $k > l$. We have $P_n \Rightarrow \langle P_1 \{ (n \cdot 0/X_i) \} \rangle P_1$, so there exists $Q_1, Q_1'$ such that $Q_n \Rightarrow \langle Q_1 \{ (n \cdot 0/X_j) \} \rangle Q_1$ with $a_1 = b_1$ and $P_1 \{ (n \cdot 0/X_j) \} \sim_1 Q_1' \{ (n \cdot 0/X_j) \}$. The process is under $k - 1$ localities in $P_1'$ and under $l - i$ localities in $Q_1'$, with $i \geq 1$. After $l$ passivation, we have $P_1'$ such that the process $n \cdot 0$ is under $k - l$ localities, and a process $Q_1'$ such that the process $n \cdot 0$ is not under a locality and with $P_1' \sim_1 Q_1'$, which is not possible (same proof as in the first case). If $k < l$, we have a similar contradiction by reasoning on $Q_n$, consequently we have $k = l$.

Therefore there exists $a_1 \ldots a_k$, $P_1 \ldots P_k$, $Q_1 \ldots Q_k$, such that $P_n = a_1 \ldots a_{k-1} [a_k \{ n \cdot 0 \mid P_{k+1} \} \mid P_k \mid P_{k-1} \ldots] \mid P_1$ and $Q_n = a_1 \ldots a_{k-1} [a_k \{ n \cdot 0 \mid Q_{k+1} \} \mid Q_k \mid Q_{k-1} \ldots] \mid Q_1$. Let $P_1'$ (resp $Q_1'$) be the process inside the locality $a_i$ in $P_n$ (resp $Q_n$). We have $P_n \Rightarrow \langle P_1' \rangle P_1$, with $P_1' \Rightarrow$, which is matched by a passivation $Q_n \Rightarrow \langle Q_1' \rangle Q_1$ such that $P_1 \sim_1 Q_1$, $P_1' \sim_1 Q_1'$ and $Q_1' \Rightarrow$. If $i \neq 1$, we have the process under $k - 1$ localities in $P_1'$ and in $k - i < k - 1$ localities in $Q_1'$, with $P_1' \sim_1 Q_1$: contradiction. Hence we have $i = 1$, $P_1 \sim_1 Q_1$ and $P_1' = n \cdot 0 \mid P_{k+1} \sim_1 n \cdot 0 \mid Q_{k+1} = Q_k'. $ Since the reduction $P_k' \Rightarrow P_{k+1}$ can only be matched $Q_k' \Rightarrow Q_{k+1} \Rightarrow Q_{k+1}$, we have $P_{k+1} \sim_1 Q_{k+1}$, consequently we have the required result.

In the following, we write $X_i$ the $i$-th occurrence of $X$ in a process $P$.

**Lemma 51.** Let $P, Q$ two open processes such that fr$(P, Q) \subseteq \{ X \}$ and $m, n$ two names which do not occur in $P, Q$. Let $R, R'$ two closed processes such that $R \sim_1 R'$. Suppose we have $P \{ m.n.0/X \} \Rightarrow Q \{ m.n.0/X \}$ and $P \{ m.n.0/X \} \Rightarrow Q \{ m.n.0/X \} \Rightarrow$ $P' \{ m.n.0/X \} \{ n.0/X_i \} = P_n$ is matched by the transition $Q \{ m.n.0/X \} \Rightarrow Q' \{ m.n.0/X \} \{ n.0/X_j \} = Q_n$ (with $P_n \sim_1 Q_n$). Then we have the relation $P' \{ m.n.0/X \} \{ R/X_i \} \sim_1 Q' \{ m.n.0/X \} \{ R'/X_j \}.$

**Proof.** By lemma 50, we have two cases to consider:

- Suppose we have $P_n = n \cdot 0 \mid P_1$, $Q_n = n \cdot 0 \mid Q_1$ with $P_1 \sim_1 Q_1$. Since $P_1 \sim_1 Q_1$, $R \sim_1 R'$ and $\sim_1$ is a congruence we have $R | P_1 \sim_1 R' | Q_1$ by transitivity, consequently the result holds.

- Suppose we have $P_n = a_1 \ldots a_{k-1} [a_k \{ n \cdot 0 \mid P_{k+1} \} \mid P_k \mid P_{k-1} \ldots] \mid P_1$ and $Q_n = a_1 \ldots a_{k-1} [a_k \{ n \cdot 0 \mid Q_{k+1} \} \mid Q_k \mid Q_{k-1} \ldots] \mid Q_1$ and for all $1 \leq j < k$, $P_j \sim_1 Q_j$. Since $P_{k+1} \sim_1 Q_{k+1}, R \sim_1 R'$, $\sim_1$ is a congruence and is transitive, we have $R | P_{k+1} \sim_1 R' | Q_{k+1}$. So we have $a_k[R \mid P_{k+1}] | P_k \sim_1 a_k[R' \mid Q_{k+1} | Q_k.$ By induction on $1 \leq j \leq k$, we
have \( a_j \ldots a_k[R \mid P_{k+1} \mid P_k \ldots ] \mid P_j \sim_1 a_j \ldots a_k[R' \mid Q_{k+1} \mid Q_k \ldots ] \mid Q_j \), so we have the required result with \( j = 1 \).

\[ \square \]

**Theorem 16.** Let \( P, Q \) two open processes such that \( \text{fv}(P, Q) \subseteq \{X\} \) and \( m, n \) two names which do not occur in \( P, Q \). If \( P\{m.n.0/X\} \sim_1 Q\{m.n.0/X\} \), then for all closed processes \( R \), we have \( P\{R/X\} \sim_1 Q\{R/X\} \).

**Proof.** We show that the relation \( R = \{ (P\{R/X\}, Q\{R/X\}), P\{m.n.0/X\} \sim_1 Q\{m.n.0/X\}, m, n \text{ not in } P, Q \} \) is a strong bisimulation. Since the relation is symmetrical, it is enough to prove that it is a simulation. We make a case analysis on the transition from \( P\{R/X\} \):

- **Process case** \( P' \). Since \( P\{m.n.0/X\} \sim_1 Q\{m.n.0/X\} \), there exists \( Q' \) such that \( Q\{m.n.0/X\} \overset{α}{\rightarrow} Q' \) and \( P''\{m.n.0/X\} \sim_1 Q' \). Since \( m \) does not occur in \( P, Q \), we have \( α \neq m \), so the transition \( Q\{m.n.0/X\} \overset{α}{\rightarrow} Q' \) comes only from \( Q \). Therefore \( Q' \) can be written \( Q' = Q''\{m.n.0/X\} \) for some \( Q'' \), and we have \( Q\{R/X\} \overset{α}{\rightarrow} Q''\{R/X\} \). We have \( P''\{R/X\} R Q''\{R/X\} \), hence the result holds.

- **Abstraction case** \( F \). Since \( P\{m.n.0/X\} \sim_1 Q\{m.n.0/X\} \), there exists \( F' \) such that \( Q\{m.n.0/X\} \overset{α}{\rightarrow} F' \) and \( (F\{m.n.0/X\}) \overset{α}{\rightarrow} F' \) for all processes \( T \). Since the transition is on a higher-order name, we have \( α \neq m \), so the transition \( Q\{m.n.0/X\} \overset{α}{\rightarrow} F' \) comes only from \( Q \). Therefore \( F' \) can be written \( F''\{m.n.0/X\} \) for some \( F'' \), and we have \( Q\{R/X\} \overset{α}{\rightarrow} F''\{R/X\} \). Since \( T \) is a closed process, we have \( (F\{R/X\}) \overset{α}{\rightarrow} F''\{R/X\} \). Hence \( (F \circ T)\{R/X\} \overset{α}{\rightarrow} F''\{R/X\} \circ T \overset{α}{\rightarrow} (F'' \circ T)\{R/X\} = (F''\{R/X\}) \circ T \), hence the result holds.

- **Concretion case** \( C = \langle T \rangle.S \). Since \( P\{m.n.0/X\} \sim_1 Q\{m.n.0/X\} \), there exists \( C' = \langle T' \rangle.S' \) such that \( Q\{m.n.0/X\} \overset{α}{\rightarrow} C', T\{m.n.0/X\} \sim_1 T' \) and \( S\{m.n.0/X\} \sim_1 S' \). We have \( α \neq m \), so the transition \( Q\{m.n.0/X\} \overset{α}{\rightarrow} C' \) comes only from \( Q \). Therefore \( T', S' \) can be written \( T''\{m.n.0/X\} \) and \( S''\{m.n.0/X\} \) for some \( T'', S'' \), and we have \( Q\{R/X\} \overset{α}{\rightarrow} \langle (T'')\{S''\}\{R/X\} \rangle \). We have \( T\{R/X\} \overset{α}{\rightarrow} T'\{R/X\} \) and \( S\{R/X\} \overset{α}{\rightarrow} S''\{R/X\} \), hence the result holds.

The transition comes only from \( R \). A copy of \( R \) is in an evaluation context and performs a transition. We write \( X_i \) the occurrence of \( X \) where the copy of \( R \) performs the transition. We have \( P\{R/X\} \overset{α}{\rightarrow} P''\{R/X\}\{A/X_i\} \) with \( R \overset{α}{\rightarrow} A \). Since \( X_i \) is in an evaluation context, we have \( P\{m.n.0/X\} \overset{α}{\rightarrow} P''\{m.n.0/X\}\{n.0/X_i\} \). Since we have \( P\{m.n.0/X\} \sim_1 Q\{m.n.0/X\} \), there
exists a transition \( Q\{m.n.0/X\} \xrightarrow{m} Q'\{m.n.0/X\}\{n.0/X_j\} \) (an occurrence of \( X \), noted \( X_j \), is in an evaluation context in \( Q \)) with \( P'\{m.n.0/X\}\{n.0/X_i\} \sim_i Q'\{m.n.0/X\}\{n.0/X_j\} \). Consequently we have \( Q\{R/X\} \xrightarrow{a} Q'\{R/X\}\{A/X_j\} \).

We distinguish three cases for \( A \):

- **Process case \( R' \).** We have \( P'\{m.n.0/X\}\{R'/X_i\} \sim_i Q'\{m.n.0/X\}\{R'/X_j\} \) by lemma 51, so we have \( P'\{R/X\}\{R'/X_i\} \mathcal{R} Q'\{R/X\}\{R'/X_j\} \) as required.

- **Abstraction case \( F \).** By lemma 51, we have \( P'\{m.n.0/X\}\{F \circ T/X_i\} \sim_i Q'\{m.n.0/X\}\{F \circ T/X_j\} \) for all \( T \). We have \( (P'\{R/X\}\{F/X_i\}) \circ T = P'\{R/X\}\{F \circ T/X_i\} \mathcal{R} Q'\{R/X\}\{F \circ T/X_j\} = (Q'\{R/X\}\{F/X_j\}) \circ T \) as required.

- **Concretion case \( \langle S \rangle T \).** By lemma 51, we have \( P'\{m.n.0/X\}\{T/X_i\} \sim_l Q'\{m.n.0/X\}\{T/X_j\} \), so we have \( P'\{R/X\}\{T/X_i\} \mathcal{R} Q'\{R/X\}\{T/X_j\} \). Moreover we have \( S \sim_l S \), and since \( \sim_l \subseteq \mathcal{R} \) (with \( P,Q \) closed processes), we have \( S \mathcal{R} S \) and \( P'\{R/X\}\{T/X_i\} \mathcal{R} Q'\{R/X\}\{T/X_j\} \) as required.

**A higher-order communication takes place between \( R \) and \( P \).** A copy of \( R \) is in an evaluation context and communicate with a sub-process \( P' \) of \( P \). We have two cases to consider.

The first possibility is \( R \xrightarrow{a} F \) and \( P' \not\xrightarrow{\pi} \langle T\{R/X\}\rangle S\{R/X\} \) for some \( a \). We have the transition

\[
P\{R/X\} \xrightarrow{\tau} E_1.R\{E_2.R\{F \circ (T\{R/X\})\} \mid E_3.R\{S\{R/X\}\}}
\]

for some evaluation contexts \( E_1.R, E_2.R, E_3.R \) (the subscript \( R \) means that occurrences of \( X \) in the context are filled with \( R \)). We have

\[
P\{m.n.0/X\} \xrightarrow{m} P'\{m.n.0/X\}\{E_1.m.n.0\{E_2.m.n.0\{n.0\} \mid E_3.m.n.0\{S\{m.n.0/X\}\}}\}
\]

so by bisimilarity hypothesis, there exists \( T',E' \) such that we have

\[
Q\{m.n.0/X\} \xrightarrow{m} T'\{m.n.0/X\}\{E'\{m.n.0\{n.0\}\}
\]

and the messages and continuations are bisimilar, i.e. we have

\[
T\{m.n.0/X\} \sim_l T'\{m.n.0/X\}
\]

and

\[
E_1.m.n.0\{E_2.m.n.0\{n.0\} \mid E_3.m.n.0\{S\{m.n.0/X\}\}\} \sim_l E'\{m.n.0\{n.0\}
\]

From the relation on messages, we have

\[
F \circ (T\{m.n.0/X\}) \sim_l F \circ (T'\{m.n.0/X\})
\]
Hence by lemma 51 and the relation on continuations, we have

\[ E_{1,m,n,0} \{ E_{2,m,n,0} \{ F \circ (T\{m.n.0/X\}) \} \} \sim_i E_{1,m,n,0} \{ E_{3,m,n,0} \{ S\{m.n.0/X\} \} \} \]

\[ \sim_i E_{1,m,n,0} \{ E_{3,m,n,0} \{ S\{m.n.0/X\} \} \} \]

We have \( Q\{R/X\} \xrightarrow{i} E'_{R}\{F \circ (T'\{R/X\})\} \) and

\[ E_{1,R}\{E_{2,R}\{F \circ (T\{R/X\})\} \} \sim E_{1,R}\{E_{2,R}\{F \circ (T\{R/X\})\} \} \]

hence the result holds.

The second possibility is \( R \xrightarrow{a} (T)S \) and \( P' \xrightarrow{a} F\{R/X\} \) for some \( a \). We have the transition

\[ P\{R/X\} \xrightarrow{m.a} E_{1,R}\{E_{2,R}\{S\} \} \sim E_{1,R}\{E_{2,R}\{S\} \} \]

for some evaluation contexts \( E_{1,R}, E_{2,R}, E_{3,R} \). We have the transitions

\[ P\{m.n.0/X\} \xrightarrow{m.a} E_{1,m,n,0}\{E_{2,m,n,0}\{n.0\} \} \sim E_{1,m,n,0}\{E_{2,m,n,0}\{n.0\} \} \]

so there exists \( F' \) such that

\[ Q\{m.n.0/X\} \xrightarrow{m.a} E_{1,m,n,0}\{E_{2,m,n,0}\{n.0\} \} \sim E_{1,m,n,0}\{E_{2,m,n,0}\{n.0\} \} \]

for some contexts and we have

\[ E_{1,m,n,0}\{E_{2,m,n,0}\{n.0\} \} \sim E_{1,m,n,0}\{E_{2,m,n,0}\{n.0\} \} \]

By lemma 51, we have the relation

\[ E_{1,m,n,0}\{E_{2,m,n,0}\{S\} \} \sim E_{1,m,n,0}\{E_{2,m,n,0}\{S\} \} \]

We have \( Q\{R/X\} \xrightarrow{i} E'_{1,R}\{E_{2,R}\{S\} \} \sim E'_{1,R}\{E_{2,R}\{S\} \} \)

hence the result holds.

A higher-order communication takes place between two copies of \( R \). Two copies of \( R \) are in evaluation contexts and communicate. There exists \( F, (T)S \) such that \( R \xrightarrow{a} F \) and \( R \xrightarrow{T} (T)S \) for some \( a \). We note \( X_i, X_j \) the two occurrences of \( X \) in \( P \) where the transitions are performed: the transition can be written \( P\{R/X\} \xrightarrow{i} P''\{R/X\} \{F \circ T/X_i\} \{S/X_j\} \).
We have $P\{R/X\} \xrightarrow{a} P'\{R/X\}\{F/X_i\}$. Since $X_i$ is in an evaluation context, we have $P\{m.n.0/X\} \xrightarrow{m} P'\{m.n.0/X\}\{n.0/X_i\}$, so there exists $Q'$ such that $Q\{m.n.0/X\} \xrightarrow{m} Q'\{m.n.0/X\}\{n.0/X_k\}$ and $P'\{m.n.0/X\}\{n.0/X_i\} \sim_l Q'\{m.n.0/X\}\{n.0/X_k\}$. Since $F \circ T \sim_l F \circ T$, we have $P'\{m.n.0/X\}\{F \circ T/X_i\} \sim_l Q'\{m.n.0/X\}\{F \circ T/X_k\}$ by lemma 51.

Since $X_j$ is in an execution context, we have $P'\{m.n.0/X\}\{F \circ T/X_i\} \xrightarrow{m} P''\{m.n.0/X\}\{F \circ T/X_j\} \{n.0/X_j\}$. Consequently by the previous equivalence there exists $Q''$ such that $Q'\{m.n.0/X\}\{F \circ T/X_k\} \xrightarrow{m} Q''\{m.n.0/X\}\{F \circ T/X_k\}\{n.0/X_j\}$ and $P''\{m.n.0/X\}\{F \circ T/X_i\}\{n.0/X_j\} \sim_l Q''\{m.n.0/X\}\{F \circ T/X_k\}\{n.0/X_j\}$. Since $S \sim_l S$, by lemma 51 we have $P''\{m.n.0/X\}\{F \circ T/X_i\}\{S/X_j\} \sim_l Q''\{m.n.0/X\}\{F \circ T/X_k\}\{S/X_j\}$. We have $Q\{R/X\} \xrightarrow{R} Q''\{R/X\}\{F \circ T/X_k\}\{S/X_j\}$ and the relation $P''\{R/X\}\{F \circ T/X_i\}\{S/X_j\} \sim_l Q''\{R/X\}\{F \circ T/X_k\}\{S/X_j\}$, hence the result holds.

□