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Effective Constraints of Loop Quantum Gravity

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Outline

1. Motivation

2. Effective approximation. Overview.

3. Anomaly issue.

4. Effective constraints.

5. Summary

Motivation.

Test semi-classical limit of LQG.

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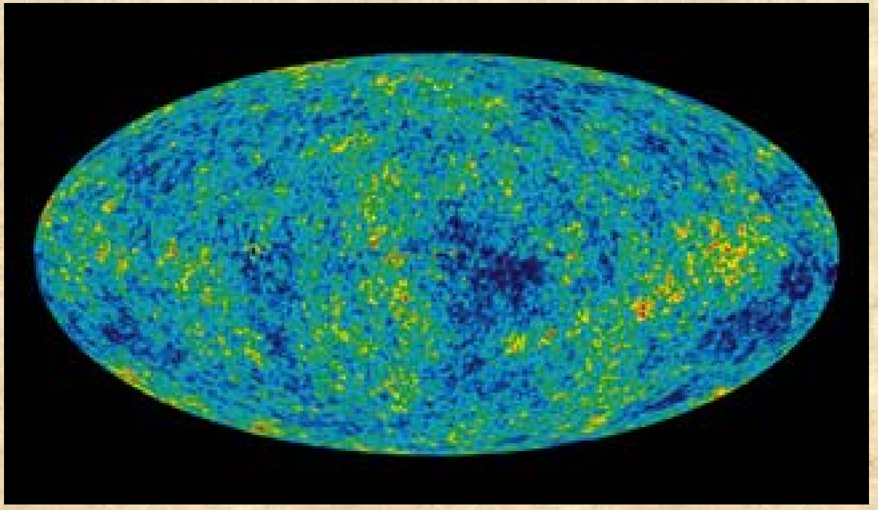
Evolution of inhomogeneities is expected to explain cosmological *structure formation* and lead to *observable results*.



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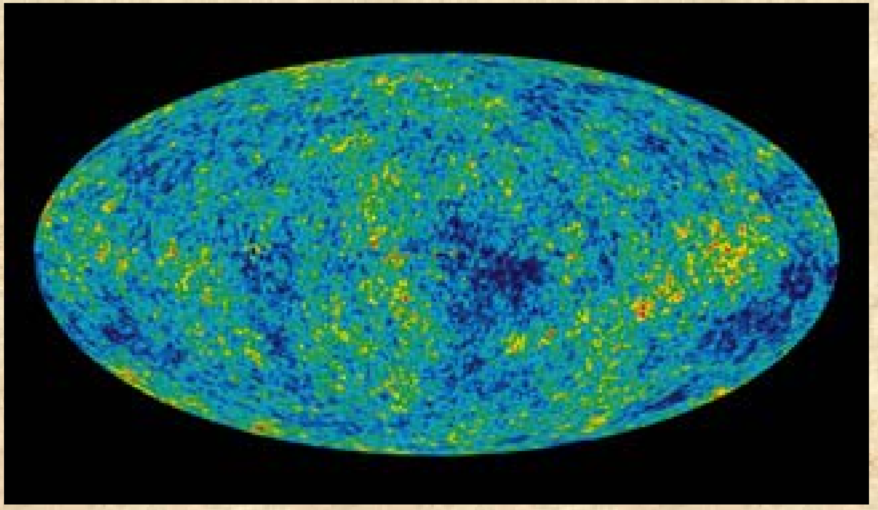

Effective approximation allows to extract predictions of the underlying quantum theory without going into consideration of quantum states.

Effective Approximation. Strategy.

Classical Theory — **Classical Constraints & $\{\,,\,\}_{PB}$**

↓

Quantization — **Quantum Operators & $[\,,\,]$**

↓

Effective Theory — **Quantum variables:**

$$(q,p) \rightarrow G_q^{a,n} = < (\hat{q} - < q >)^{n-a} (\hat{p} - < p >)^a >$$

expectation values, spreads, deformations, etc.

Truncation — **Expectation Values**

classically well behaved expressions → **Classical Expressions**

classically diverging expressions → $\left(\text{Classical Expressions}\right) \times \left(\text{Correction Functions}\right)$

Effective Approximation. Summary.

Classical Constraints & Poisson Algebra

quantization ↓

Constraint Operators & Commutation Relations

effective ↓ approximation

Effective Constraints & Effective Poisson Algebra

(differs from classical Poisson Algebra)

Effective Equations of Motion

(Bojowald, Hernandez, MK, Singh, Skirzewski
Phys. Rev. D, **74**, 123512, 2006;
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Anomaly Issue

Anomalies. Source of Corrections.

Basic Variables

- Densitized triad E_i^a
- Ashtekar connection $A_a^i = \Gamma_a^i + \gamma K_a^i$
- Scalar field φ
- Field momentum π

Diffeomorphism Constraint

intact

Hamiltonian constraint

$$H_g[N] = \frac{1}{16\pi G}\int_\Sigma \mathrm{d}^3x \frac{N}{\sqrt{|\det E|}}\left(\epsilon_{ijk}F^i_{ab}E^a_j E^b_k - 2(1+\gamma^2)K^i_a K^j_b E^{[a}_i E^{b]}_j\right)$$

$$H_m[N] = \int d^3x N\left[\frac{\pi^2}{2\sqrt{|\det E|}} + \frac{E^a_i E^b_i}{2\sqrt{|\det E|}}\partial_a\phi\partial_b\phi + \sqrt{|\det E|}U(\phi)\right]$$

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α(E) D(E) σ(E)

Anomalies. Restrictions on Correction Functions.

Corrected constraint algebra

$$H^Q = H^Q_g + H^Q_m, \qquad D^Q \equiv D = D_g + D_m$$

$$\begin{aligned}\{H^Q[N], D[N^a]\} = -H^Q[\tilde{N}] \ &+ \ \int d^3x N_c \partial N^j \left(E^c_i \delta^a_j - E^c_j \delta^a_i\right) \\ &\times \ \left[\frac{\partial\alpha}{\partial E^a_i}(\ldots) + \frac{\partial D}{\partial E^a_i}(\ldots) + \frac{\partial\sigma}{\partial E^a_i}(\ldots)\right]\end{aligned}$$

$$\begin{aligned}\{H^Q[N_1], H^Q[N_2]\} \ &= \ D_g[\alpha^2(N_1\partial^a N_2 - N_2\partial^a N_1)] \\ &+ \ D_m[D\sigma(N_1\partial^a N_2 - N_2\partial^a N_1)]\end{aligned}$$

Anomalies. Restrictions on Correction Functions.

Anomaly free conditions:

$$\alpha, D, \sigma = f\left(\mathcal{E}(E_i^a)\right)$$

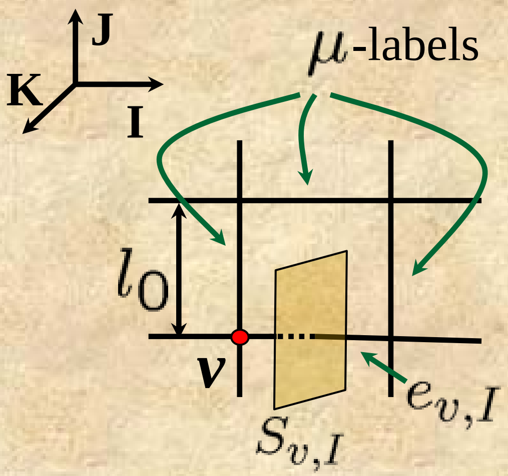
$$\left(E_i^c \delta_j^a - E_j^c \delta_i^a\right) \frac{\partial \mathcal{E}}{\partial E_i^a} = 0$$

$$\alpha^2 = D\sigma$$

f=arbitrary function

Effective Constraints. Lattice formulation.

(scalar mode/longitudinal gauge, Bojowald, Hernandez, MK, Skirzewski, 2007)



Fluxes

$$E_i^I = \tilde{p}^I(x)\delta_i^I$$

$$p_{v,I} \approx l_0^2 \tilde{p}^I(v)$$

(integrated over $S_{v,I}$)

Holonomies

$$K_I^i = \tilde{k}_I(x)\delta_I^i$$

$$\eta_{v,I} \approx \exp\left(i l_0 \tilde{k}_I(v)\right)$$

(integrated over $e_{v,I}$)

Basic operators:

$$\hat{p}_{v,I}|\ldots,\mu_{v',J},\ldots\rangle = 4\pi\gamma\ell_{\mathrm{P}}^2\mu_{v,I}|\ldots,\mu_{v',J}+1,\ldots\rangle\,.$$

$$\hat{\eta}_{v,I}|\ldots,\mu_{v',J},\ldots\rangle = |\ldots,\mu_{v,I}+1,\ldots\rangle\,.$$

Effective Constraints. Hamiltonian.

Curvature

$$F(A) = dA + AA = (\gamma dK + \gamma^2 KK) + (d\Gamma + \Gamma\Gamma) + \gamma(\Gamma K + K\Gamma)$$

$$\frac{FEE - KKEE}{\sqrt{|\det E|}} = [\underbrace{(\gamma dK - KK)}_{H_K = \sum_v H_{K,v}} + \underbrace{(d\Gamma + \Gamma\Gamma)}_{H_\Gamma}] \frac{EE}{\sqrt{|\det E|}}$$

($\gamma dK \to 0$)

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$\gamma dK \to 0$

Hamiltonian

$$\hat{H}_{\mathcal{K},\mathbf{v}} = \frac{-N(\mathbf{v})}{64\pi\gamma^2 G}\sum_{IJK}\sum_{\sigma_I\in\{\pm 1\}} \{[\; s^{-}_{\mathbf{v},\sigma_I I,\sigma_J J} s_{\mathbf{v},\sigma_J J} c_{\mathbf{v}+\sigma_I \mathbf{I},\sigma_J J}$$

$$+ s^{+}_{\mathbf{v},\sigma_I I,\sigma_J J} c_{\mathbf{v},\sigma_J J} s_{\mathbf{v}+\sigma_I \mathbf{I},\sigma_J J}]\hat{B}_{\mathbf{v},\sigma_K K}\}$$

$$c_{\mathbf{v},I} = \cos(\frac{\gamma}{2}k_I(v)), \qquad s_{\mathbf{v},I} = \sin(\frac{\gamma}{2}k_I(\mathbf{v}))$$

$$s^{\pm}_{\mathbf{v},\sigma_I I,\sigma_J J} := \sin\left(\frac{\gamma}{2}(k_{\sigma_I I}(\mathbf{v}) \pm k_{\sigma_I I}(\mathbf{v} + \sigma_J \mathbf{J})\right)$$

Effective Constraints. Hamiltonian.

Curvature

$$F(A) = dA + AA = (\gamma dK + \gamma^2 KK) + (d\Gamma + \Gamma\Gamma) + \gamma(\Gamma K + K\Gamma)$$

0

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Inverse triad operator

$$4\sin(\gamma k_I/2)\cos(\gamma k_I/2)\sin(\gamma k_J/2)\cos(\gamma k_J/2)$$

+higher curvature corrections

$$c_{\mathbf{v},I} = \cos\left(\frac{\gamma}{2}k_I(v)\right), \quad s_{\mathbf{v},I} = \sin\left(\frac{\gamma}{2}k_I(\mathbf{v})\right)$$

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Effective Constraints. Inverse triad corrections.

$$e^i_I = \frac{1}{2}\epsilon^{ijk}\epsilon_{IJK}\frac{E^J_j E^K_k}{\sqrt{|\det E|}} \propto \{A^i_I, \int \sqrt{|\det E|}d^3x\} \longrightarrow \mathrm{tr}\left(\tau^i h_{v,I}[h^{-1}_{v,I},\hat{V}]\right)$$

$$e^i_I \equiv e_I\delta^i_I \qquad \hat{e_I} := \hat{B}_{v,I} = \frac{1}{2\pi i\gamma\ell_{\mathrm{P}}^2}\mathrm{tr}\left(\tau^i h_{v,I}[h^{-1}_{v,I},\hat{V}]\right)$$

$$\hat{B}_{\mathbf{v},I}|\ldots,\mu_{\mathbf{v},I},\ldots\rangle := \left(4\pi\gamma\ell_{\mathrm{P}}^2\right)^{1/2} 2\sqrt{|\mu_{\mathbf{v},J}||\mu_{\mathbf{v},K}|}\left(\sqrt{|\mu_{\mathbf{v},I}+1/2|}-\sqrt{|\mu_{\mathbf{v},I}-1/2|}\right)|\ldots,\mu_{\mathbf{v},I},\ldots\rangle$$

Effective Constraints. Inverse triad corrections.

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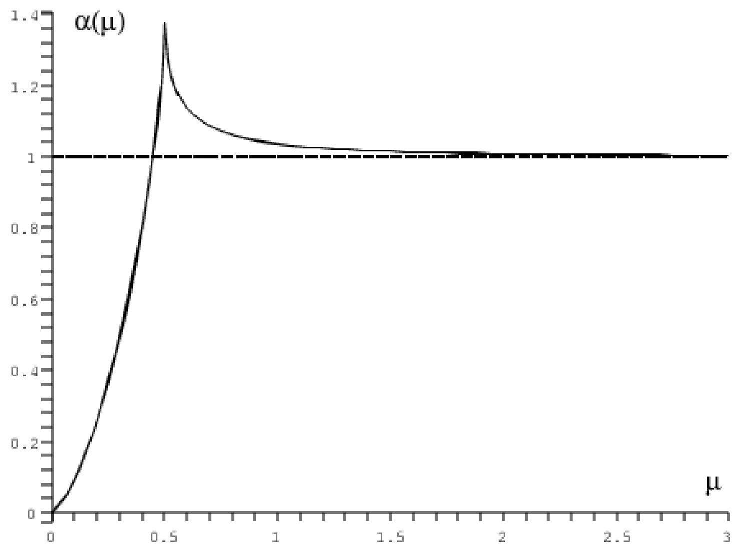
$$e_{I(class)} = \sqrt{\frac{p^J p^K}{p^I}} \propto \sqrt{\frac{\mu^J\mu^K}{\mu^I}},$$

$$e_{I(qc)} = \alpha(p) e_{I(class)}$$

$$\alpha = 2\sqrt{|\mu|}\left(\sqrt{|\mu+1/2|}-\sqrt{|\mu-1/2|}\right)$$



Effective Constraints. Inverse triad corrections.

Generalization

$$\frac{1}{\sqrt{|\det E|}} = \frac{(\det e)^k}{\sqrt{|\det E|^{(k+1)/2}}} \propto \left(\epsilon^{IJK}\epsilon_{ijk}\{A_I^i, V^r\}\{A_J^j, V^r\}\{A_K^k, V^r\}\right)$$

for $k \geq 1$ **and** $r_k = \frac{2k-1}{3k} \geq \frac{1}{3}$

$$\{A_a^i, V^r\} = 4\pi\gamma G\, r V^{r-1} e_a^i$$

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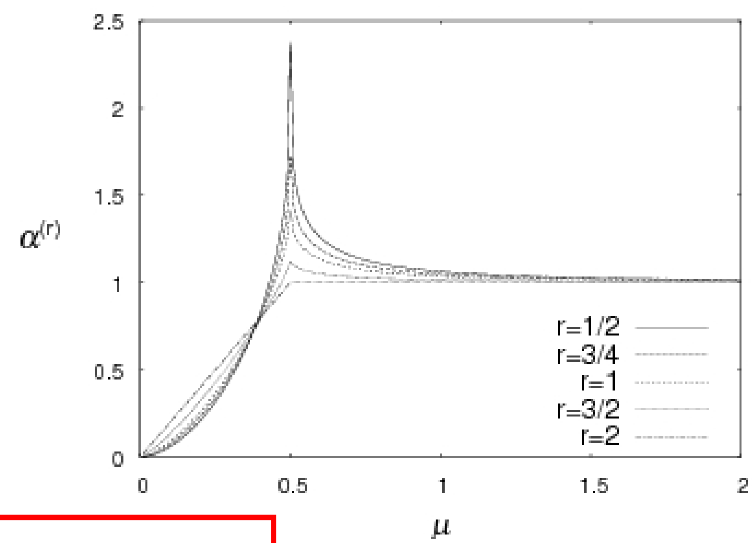

$$\alpha^{(r)} = \frac{2}{r}|\mu|^{1-\frac{r}{2}}\left(|\mu+1/2|^{\frac{r}{2}} - |\mu-1/2|^{\frac{r}{2}}\right)$$

Effective Constraints. Inverse triad corrections.

Higher *j*-representations

$$\alpha^{(r,j)} = \frac{6}{rj(j+1)(2j+1)}|\mu|^{1-\frac{r}{2}} \sum_{m=-j}^{j} m\,|\mu+m|^{r/2}$$

for large *j*

$$\alpha^{(r,j)} = \frac{6\tilde{\mu}^{1-\frac{r}{2}}}{r(r+2)(r+4)}\Big[(\tilde{\mu}+1)^{\frac{r}{2}+1}(r+2-2\tilde{\mu}) + \mathrm{sgn}(\tilde{\mu}-1)|\tilde{\mu}-1|^{\frac{r}{2}+1}(r+2+2\tilde{\mu})\Big]$$

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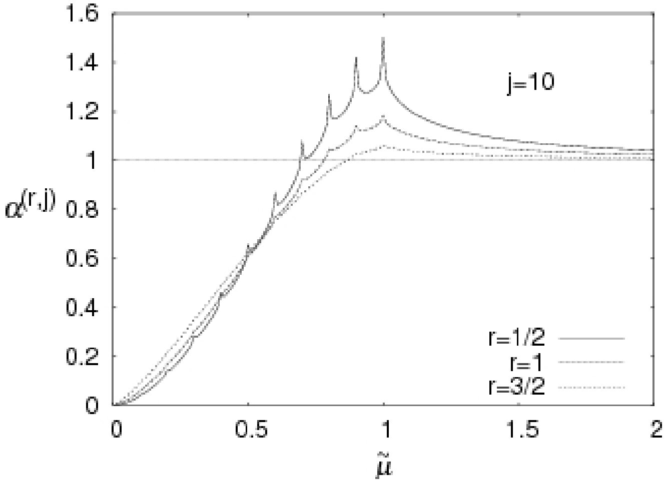

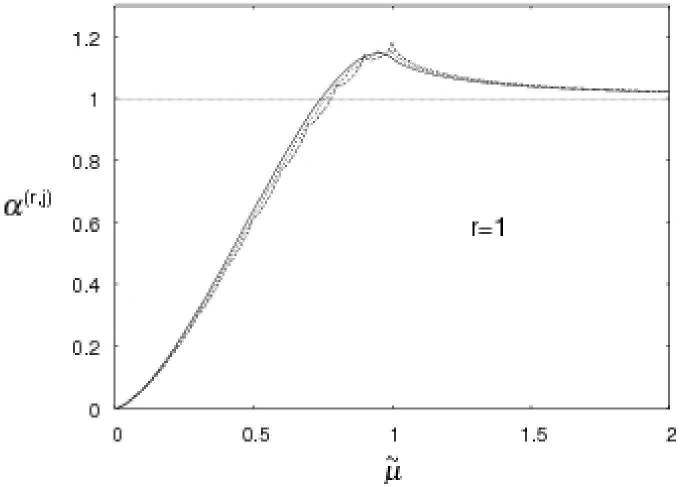

Asymptotics:

$$\alpha^{(r,j)} \approx 1 + \frac{1}{32\tilde{\mu}^2} \frac{(r-2)(r-4)}{3} \frac{4(3j^2+3j-1)}{5}, \quad \tilde{\mu} \to \infty$$

$$\alpha^{(r,j)} \approx (2\tilde{\mu})^{2-\frac{r}{2}}, \qquad \tilde{\mu} \to 0, \qquad \tilde{\mu} := \mu/j$$

Summary.

1. Can check anomalies order by order. Do not have to work with full setting.

2. Indications that ambiguities are restricted.

3. There is a consistent set of corrected constraints which are first class.

4. Cosmology: can formulate equations of motion in terms of gauge invariant variables.