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Research Article

Some Identities on the q-Genocchi Polynomials of Higher-Order and q-Stirling Numbers by the Fermionic p-Adic Integral on $\mathbb{Z}p$

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A systemic study of some families of q-Genocchi numbers and families of polynomials of Nörlund type is presented by using the multivariate fermionic p-adic integral on $\mathbb{Z}p$. The study of these higher-order q-Genocchi numbers and polynomials yields an interesting q-analog of identities for Stirling numbers.

1. Introduction

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p, \mathbb{Q}_p, \mathbb{C}$, and $\mathbb{C}_p$ denote the ring of p-adic rational integers, the field of p-adic rational numbers, the complex number field, and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = 1/p$.

When one talks of q-extension, q is variously considered as an indeterminate, a complex $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, then one normally assumes $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume $|q-1|_p < 1$. In this paper, we use the following notation:

$$[x]_q = \frac{1-q^x}{1-q}, \qquad [x]_{-q} = \frac{1-(-q)^x}{1+q}, \tag{1.1}$$

see [1–10]. Hence $\lim_{q\to 1}[x]_q = x$ for all $x \in \mathbb{Z}_p$.

The q-factorial is defined as $[n]_q! = [n]_q[n-1]_q \cdots [2]_q[1]_q$, and the Gaussian binomial coefficient is defined by the standard rule

$$\binom{n}{k}_q = \frac{[n]_q!}{[n-k]_q![k]_q!} = \frac{[n]_q[n-1]_q \cdots [n-k+1]_q!}{[k]_q!}, \tag{1.2}$$

(see [7, 9]). Note that $\lim_{q\to 1}\binom{n}{k}_q = \binom{n}{k} = n!/(n-k)!k! = n(n-1)\cdots(n-k+1)/k!$. It readily follows from (1.2) that

$$\binom{n+1}{k}_q = \binom{n}{k-1}_q + q^k\binom{n}{k}_q = q^{n-k+1}\binom{n}{k-1}_q + \binom{n}{k}_q, \tag{1.3}$$

(see [4, 7]).

The q-binomial formulas are known,

$$\begin{aligned}(b;q)_n &= (1-b)(1-bq)\cdots\left(1-bq^{n-1}\right) = \sum_{i=0}^{n}\binom{n}{i}_q q^{\binom{i}{2}}(-1)^i b^i,\\ \frac{1}{(b;q)_n} &= \frac{1}{(1-b)(1-bq)\cdots(1-bq^{n-1})} = \sum_{i=0}^{\infty}\binom{n+i-1}{i}_q b^i.\end{aligned} \tag{1.4}$$

We say that $f : \mathbb{Z}_p \to \mathbb{C}_p$ is uniformly differentiable function at a point $a \in \mathbb{Z}_p$, and we write $f \in UD(\mathbb{Z}_p)$, if the difference quotients $\Phi_f : \mathbb{Z}_p \times \mathbb{Z}_p \to \mathbb{C}_p$ such that $\Phi_f(x,y) = (f(x)-f(y))/(x-y)$ have a limit $f'(a)$ as $(x,y) \to (a,a)$. For $f \in UD(\mathbb{Z}_p)$, the q-deformed fermionic p-adic integral is defined as

$$I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)\,d\mu_{-q}(x) = \lim_{N\to\infty}\frac{1}{[p^N]_{-q}}\sum_{x=0}^{p^N-1} f(x)(-q)^x, \tag{1.5}$$

(see [7, 9]). Note that

$$I_{-1}(f) = \lim_{q\to 1} I_{-q}(f) = \int_{\mathbb{Z}_p} f(x)\,d\mu_{-1}(x). \tag{1.6}$$

For $n \in \mathbb{N}$, write $f_n(x) = f(x+n)$. Then, we have

$$I_{-1}(f_n) = (-1)^n I_{-1}(f) + 2\sum_{l=0}^{n-1}(-1)^{n-1-l} f(l). \tag{1.7}$$

Using (1.7), we can readily derive the Genocchi polynomials, $G_n(x)$, namely,

$$t\int_{\mathbb{Z}_p} e^{(x+y)t}d\mu_{-1}(y) = \frac{2t}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!}, \tag{1.8}$$

(see [1–27]). Note that $G_n(0) = G_n$ are referred to as the nth Genocchi numbers. Let us now introduce the Genocchi polynomials of Nörlund type as follows:

$$t^r\underbrace{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{r\text{ times}} e^{(x+x_1+\cdots+x_r)t}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r) = \left(\frac{2t}{e^t+1}\right)^r e^{xt} = \sum_{n=0}^{\infty} G_n^{(r)}(x)\frac{t^n}{n!}, \tag{1.9}$$

$$\left(\frac{e^t+1}{2t}\right)^r e^{xt} = \sum_{n=0}^{\infty} G_n^{(-r)}(x)\frac{t^n}{n!}, \tag{1.10}$$

(see [7, 9]). In the special case $x=0$, $G_n^{(-r)}(0) = G_n^{(-r)}$, and $G_n^{(r)}(0) = G_n^{(r)}$ are referred to as the Genocchi numbers of Nörlund type. Let $(Eh)(x) = h(x+1)$ be the shift operator. Then, the q-difference operator Δ_q is defined as

$$\Delta_q^n = \prod_{i=1}^{n}\left(E - q^{i-1}I\right), \quad \text{where } (Ih)(x) = h(x), \tag{1.11}$$

(see [4, 7, 9]). It follows from (1.11) that

$$f(x) = \sum_{n\geq 0}\binom{x}{n}_q \Delta_q^n f(0), \tag{1.12}$$

where $\Delta_q^n f(0) = \sum_{k=0}^{n}\binom{n}{k}_q q^{\binom{k}{2}} f(n-k)$ (see [5, 6, 10]). The q-Stirling number of the second kind (as defined by Carlitz) is given by

$$S_2(n,k;q) = \frac{q^{-\binom{k}{2}}}{[k]_q!}\sum_{j=0}^{k}(-1)^j q^{\binom{j}{2}}\binom{k}{j}_q [k-j]_q^n, \tag{1.13}$$

(see [7, 10]). By (1.12) and (1.13), we see that

$$S_2(n,k;q) = \frac{q^{-\binom{k}{2}}}{[k]_q!}\Delta_q^k 0^n, \tag{1.14}$$

(see [6, 10]).

In this paper, the q-extensions of (1.9) are considered in several ways. Using these q-extensions, we derive some interesting identities and relations for Genocchi polynomials and numbers of Nörlund type. The purpose of this paper is to present a systemic study of some families q-Genocchi numbers and polynomials of Nörlund type by using the multivariate fermionic p-adic integral on $\mathbb{Z}_p$.

2. q-Extensions of Genocchi Numbers and Polynomials of Nörlund Type

In this section, we assume that $q \in \mathbb{C}_p$ with $|1-q|_p < 1$. We first consider the q-extensions of (1.8) given by the rule

$$\sum_{n=0}^{\infty} G_{n,q}(x)\frac{t^n}{n!} = t\int_{\mathbb{Z}_p} e^{[x+y]_q t} d\mu_{-1}(y) = \sum_{n=0}^{\infty} \frac{2t}{(1-q)^n}\sum_{l=0}^{n}\left(\frac{\binom{n}{l}(-1)^l q^{lx}}{1+q^l}\right)\frac{t^n}{n!} = 2t\sum_{m=0}^{\infty}(-1)^m e^{[m+x]_q t}. \tag{2.1}$$

Thus, we obtain the following lemma.

Lemma 2.1. *If $n \geq 0$, then*

$$\frac{G_{n+1,q}(x)}{n+1} = 2\sum_{m=0}^{\infty}(-1)^m [m+x]_q^n = \frac{2}{(1-q)^n}\sum_{l=0}^{n}\frac{\binom{n}{l}(-1)^l q^{lx}}{1+q^l}. \tag{2.2}$$

By (1.14),

$$\begin{aligned}
[x]_q^n &= \sum_{k=0}^{n}\binom{x}{k}_q [k]_q! S_2(k, n-k; q) q^{\binom{k}{2}} \\
&= \sum_{k=0}^{n}[x]_q[x-1]_q\cdots[x-k+1]_q \frac{q^{\binom{k}{2}-\binom{n-k}{2}}}{[n-k]_q!}\Delta_q^{n-k}0^k \\
&= \sum_{k=0}^{n}\frac{q^{\binom{k}{2}-\binom{n-k}{2}}}{[n-k]_q!}\Delta_q^{n-k}0^k \frac{1}{(1-q)^k}\sum_{l=0}^{k}\binom{k}{l}_q q^{\binom{l}{2}}(-1)^l q^{l(x-k+1)}.
\end{aligned} \tag{2.3}$$

Thus, we have

$$\frac{G_{n+1,q}}{n+1} = \sum_{k=0}^{n}\frac{q^{\binom{k}{2}}S_2(k,n-k;q)}{(1-q)^k}\sum_{l=0}^{k}\binom{k}{l}_q q^{\binom{l}{2}}(-1)^l\sum_{m=0}^{l}\binom{l}{m}(q-1)^m\frac{G_{m+1,q}(1-k)}{m+1}, \tag{2.4}$$

and we obtain the following theorem.

Theorem 2.2. *If $n \geq 0$, then*

$$\frac{G_{n+1,q}}{n+1} = \sum_{k=0}^{n} \frac{q^{\binom{k}{2}} S_2(k, n-k; q)}{(1-q)^k} \sum_{l=0}^{k} \binom{k}{l}_q q^{\binom{l}{2}} (-1)^l \sum_{m=0}^{l} \binom{l}{m} (q-1)^m \frac{G_{m+1,q}(1-k)}{m+1}, \tag{2.5}$$

where $G_{n,q} = G_{n,q}(0)$ stand for the nth Genocchi numbers.

Consider a q-extensoin of (1.9) such that $G_{0,q}^{(r)}(x) = G_{1,q}^{(r)}(x) = \cdots = G_{r-1,q}^{(r)}(x) = 0$ and

$$\begin{aligned} \frac{G_{n+r,q}^{(r)}(x)}{r!\binom{n+r}{r}} &= \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + x_1 + \cdots + x_r]_q^n \, d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\ &= \frac{2^r}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{lx} \left(\frac{1}{1+q^l}\right)^r = 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m [m+x]_q^n. \end{aligned} \tag{2.6}$$

Let $F_q^{(r)}(t, x) = \sum_{n=0}^{\infty} G_{n,q}^{(r)}(x)(t^n/n!)$. Then,

$$F_q^{(r)}(t, x) = 2^r t^r \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m e^{[m+x]_q t}. \tag{2.7}$$

In the special case $x = 0$, the numbers $G_{n,q}^{(r)}(0) = G_{n,q}^{(r)}$ are referred to as q-extension of the Genocchi numbers of order r. In the sense of the q-extension in (1.10), consider the q-extension of Genocchi polynomials of Nörlund type given by

$$G_q^{(r)}(t, x) = F_q^{(-r)}(t, x) = \frac{1}{2^r t^r} \sum_{m=0}^{r} \binom{r}{m} e^{[m+x]_q t} = \sum_{n=0}^{\infty} G_{n,q}^{(-r)}(x) \frac{t^n}{n!}. \tag{2.8}$$

By (2.8), $G_{0,q}^{(-r)}(x) = G_{1,q}^{(-r)}(x) = \cdots = G_{r-1,q}^{(-r)}(x) = 0$ and $r!\binom{n}{r} G_{n-r,q}^{(r)}(x) = (1/2^r) \sum_{m=0}^{r} \binom{r}{m} [m+x]_q^n$. Therefore, we obtain the following theorem.

Theorem 2.3. *For $r \in \mathbb{N}$, and, $n \geq 0$, write*

$$2^r t^r \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m e^{[m+x]_q t} = \sum_{n=0}^{\infty} G_{n,q}^{(r)}(x) \frac{t^n}{n!}. \tag{2.9}$$

Then,

$$\begin{aligned} \frac{G_{n+r,q}^{(r)}(x)}{r!\binom{n+r}{r}} &= \frac{2^r}{(1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{lx} \left(\frac{1}{1+q^l}\right)^r = 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m} (-1)^m [m+x]_q^n, \\ r!\binom{n}{r} G_{n-r,q}^{(-r)}(x) &= \frac{1}{2^r (1-q)^n} \sum_{l=0}^{n} \binom{n}{l} (-1)^l q^{lx} \left(1+q^l\right)^r = \frac{1}{2^r} \sum_{m=0}^{r} \binom{r}{m} [m+x]_q^n. \end{aligned} \tag{2.10}$$

The numbers $G_{n,q}^{(-r)}(0) = G_{n,q}^{(-r)}$ are referred to as the q-extension of Genocchi numbers of Nörlund type. For $h \in \mathbb{Z}$ and $r \in \mathbb{N}$, introduce the extended higher-order q-Genocchi polynomials as follows:

$$\frac{G_{n+r,q}^{(h,r)}(x)}{r!\binom{n+r}{r}} = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{j=1}^{r}(h-j)x_j} [x + x_1 + \cdots + x_r]_q^n d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r). \tag{2.11}$$

Then,

$$\begin{aligned} \frac{G_{n+r,q}^{(h,r)}(x)}{r!\binom{n+r}{r}} &= \frac{2^r}{(1-q)^n} \sum_{l=0}^{n} \frac{\binom{n}{l}(-1)^l q^{lx}}{(-q^{h-1+l}; q^{-1})_r} = \frac{2^r}{(1-q)^n} \sum_{l=0}^{n} \frac{\binom{n}{l}(-1)^l q^{lx}}{(-q^{h-r+l}; q)_r} \\ &= 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m}_q (-1)^m q^{(h-r)m} [x+m]_q^n. \end{aligned} \tag{2.12}$$

Let $F_q^{(h,r)}(t,x) = \sum_{n=0}^{\infty} G_{n,q}^{(h,r)}(x)(t^n/n!)$. Then, we can readily see that

$$F_q^{(h,r)}(t,x) = 2^r t^r \sum_{m=0}^{\infty} \binom{m+r-1}{m}_q (-1)^m q^{(h-r)m} e^{[x+m]_q t}. \tag{2.13}$$

Therefore, we obtain the following theorem.

Theorem 2.4. *For $h \in \mathbb{Z}$ and $n \geq 0$, let*

$$2^r t^r \sum_{m=0}^{\infty} \binom{m+r-1}{m}_q (-1)^m q^{(h-r)m} e^{[x+m]_q t} = \sum_{n=0}^{\infty} G_{n,q}^{(h,r)}(x) \frac{t^n}{n!}. \tag{2.14}$$

Then,

$$\frac{G_{n+r,q}^{(h,r)}(x)}{r!\binom{n+r}{r}} = \frac{2^r}{(1-q)^n} \sum_{l=0}^{n} \frac{\binom{n}{l}(-1)^l q^{lx}}{(-q^{h-r+l}; q)_r} = 2^r \sum_{m=0}^{\infty} \binom{m+r-1}{m}_q (-1)^m q^{(h-r)m} [x+m]_q^n. \tag{2.15}$$

Let us now define the extended higher-order Nörlund type q-Genocchi polynomials as follows:

$$r!\binom{n}{r} G_{n-r,q}^{(h,-r)}(x) = \frac{1}{(1-q)^n} \sum_{l=0}^{n} \frac{\binom{n}{l}(-1)^l q^{lx}}{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{l(x_1+\cdots+x_r)} q^{\sum_{j=1}^{r}(h-j)x_j} d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r)}. \tag{2.16}$$

By (2.16),

$$r!\binom{n}{r}G_{n-r,q}^{(h,-r)}(x)=\frac{1}{2^r(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{lx}\left(-q^{h-r+l};q\right)_r$$
$$=\frac{1}{2^r}\sum_{m=0}^{r}\binom{r}{m}_q q^{\binom{m}{2}}q^{(h-r)m}[m+x]_q^n. \tag{2.17}$$

Let $F_q^{(h,-r)}(t,x)=\sum_{n=0}^{\infty}G_{n,q}^{(h,-r)}(x)(t^n/n!)$. Then, we have

$$F_q^{(h,-r)}(t,x)=\frac{1}{2^r t^r}\sum_{m=0}^{r}\binom{r}{m}_q q^{\binom{m}{2}}q^{(h-r)m}e^{[m+x]_q t}, \tag{2.18}$$

where, $G_{0,q}^{(h,-r)}(x)=G_{1,q}^{(h,-r)}(x)=\cdots=G_{r-1,q}^{(h,-r)}(x)=0$. Therefore, we obtain the following theorem.

Theorem 2.5. *For $h\in\mathbb{Z}$, $n\geq 0$, and $r\in\mathbb{N}$, write*

$$\frac{1}{2^r t^r}\sum_{m=0}^{r}\binom{r}{m}_q q^{\binom{m}{2}}q^{(h-r)m}e^{[m+x]_q t}=\sum_{n=0}^{\infty}G_{n,q}^{(h,-r)}(x)\frac{t^n}{n!}. \tag{2.19}$$

Then,

$$r!\binom{n}{r}G_{n-r,q}^{(h,-r)}(x)=\frac{1}{2^r(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{lx}\left(-q^{h-r+l};q\right)_r$$
$$=\frac{1}{2^r}\sum_{m=0}^{r}\binom{r}{m}_q q^{\binom{m}{2}}q^{(h-r)m}[m+x]_q^n, \tag{2.20}$$

where, $G_{0,q}^{(h,-r)}(x)=G_{1,q}^{(h,-r)}(x)=\cdots=G_{r-1,q}^{(h,-r)}(x)=0$.

For $h=r$,

$$\frac{G_{n+r,q}^{(r,r)}(x)}{r!\binom{n+r}{r}}=\frac{2^r}{(1-q)^n}\sum_{l=0}^{n}\frac{\binom{n}{l}(-1)^l q^{lx}}{(-q^l;q)_r}=2^r\sum_{m=0}^{\infty}\binom{m+r-1}{m}_q(-1)^m[x+m]_q^n, \tag{2.21}$$

$$r!\binom{n}{r}G_{n-r,q}^{(r,-r)}(x)=\frac{1}{2^r(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{lx}\left(-q^l;q\right)_r=\frac{1}{2^r}\sum_{m=0}^{r}\binom{r}{m}_q q^{\binom{m}{2}}[m+x]_q^n. \tag{2.22}$$

It can readily be seen that

$$\begin{aligned}\frac{q^{mx}2^r}{(-q^{m-r};q)_r} &= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{\sum_{j=1}^{r}(m-j)x_j+mx}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\\ &= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} \left([x+x_1+\cdots+x_r]_q(q-1)+1\right)^m q^{-\sum_{j=1}^{r}jx_j}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\\ &= \sum_{l=0}^{m}\binom{m}{l}(q-1)^l\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x+x_1+\cdots+x_r]_q^l q^{-\sum_{j=1}^{r}jx_j}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\\ &= \sum_{l=0}^{m}\binom{m}{l}(q-1)^l\frac{G_{l+r,q}^{(0,r)}(x)}{r!\binom{l+r}{r}}.\end{aligned} \tag{2.23}$$

By (2.23), $q^{mx}2^r/(-q^{m-r};q)_r = \sum_{l=0}^{m}\binom{m}{l}(q-1)^l(G_{l+r,q}^{(0,r)}(x)/r!\binom{l+r}{r})$. As is known,

$$I_{-1}(f_1)+I_{-1}(f)=2f(0), \quad \text{where} \quad f_1(x)=f(x+1). \tag{2.24}$$

It follows from (2.24) that

$$\begin{aligned}&q^{h-1}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x+1+x_1+\cdots+x_r]_q^n q^{\sum_{j=1}^{r}(h-j)x_j}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\\ &\quad= -\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x+x_1+\cdots+x_r]_q^n q^{\sum_{j=1}^{r}(h-j)x_j}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\\ &\qquad+2\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x+x_2+\cdots+x_r]_q^n q^{\sum_{j=1}^{r-1}(h-1-j)x_{j+1}}d\mu_{-1}(x_2)\cdots d\mu_{-1}(x_r).\end{aligned} \tag{2.25}$$

By (2.25),

$$q^{h-1}\frac{G_{n+r,q}^{(h,r)}(x+1)}{n+r}+\frac{G_{n+r,q}^{(h,r)}(x)}{n+r}=2G_{n+r-1,q}^{(h-1,r-1)}(x). \tag{2.26}$$

A simple manipulation shows that

$$\begin{aligned}&q^{x}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x+x_1+\cdots+x_r]_q^n q^{\sum_{j=1}^{r}(h-j+1)x_j}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\\ &\quad=(q-1)\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x+x_1+\cdots+x_r]_q^{n+1} q^{\sum_{j=1}^{r}(h-j)x_j}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)\\ &\qquad+\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x+x_1+\cdots+x_r]_q^n q^{\sum_{j=1}^{r}(h-j)x_j}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r).\end{aligned} \tag{2.27}$$

By (2.27), $q^x(G_{n+r,q}^{(h+1,r)}(x)/(n+1))=(q-1)(G_{n+r+1,q}^{(h,r)}(x)/(n+r+1))+(G_{n+r,q}^{(h,r)}(x)/(n+1))$.

Therefore, we obtain the following proposition.

Proposition 2.6. *For $h \in \mathbb{Z}$, $r \in \mathbb{N}$ and $n \geq 0$, the following equations*

$$q^{h-1}\frac{G_{n+r,q}^{(h,r)}(x+1)}{n+r}+\frac{G_{n+r,q}^{(h,r)}(x)}{n+r}=2G_{n+r-1,q}^{(h-1,r-1)}(x),$$
$$q^{x}\frac{G_{n+r,q}^{(h+1,r)}(x)}{n+1}=(q-1)\frac{G_{n+r+1,q}^{(h,r)}(x)}{n+r+1}+\frac{G_{n+r,q}^{(h,r)}(x)}{n+1} \tag{2.28}$$

hold. Moreover, $(q^{mx}2^r)/((-q^{m-r};q)_r)=\sum_{l=0}^{m}\binom{m}{l}(q-1)^l(G_{l+r,q}^{(0,r)}(x)/r!\binom{l+r}{r})$.

By (2.21),

$$\frac{G_{n+r,q^{-1}}^{(r,r)}(r-x)}{r!\binom{n+r}{r}}=\frac{2^r}{(1-q^{-1})^n}\sum_{l=0}^{n}\frac{\binom{n}{l}(-1)^l q^{-l(r-x)}}{(-q^{-l};q^{-1})_r}$$
$$=(-1)^n q^{n+\binom{r}{2}}\frac{2^r}{(1-q)^n}\sum_{l=0}^{n}\frac{\binom{n}{l}(-1)^l q^{x}}{(-q^{l};q)_r}=(-1)^n q^{n+\binom{r}{2}}\frac{G_{n+r,q}^{(r,r)}(x)}{r!\binom{n+r}{r}}. \tag{2.29}$$

Hence,

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[r-x+x_1+\cdots+x_r]_{q^{-1}}^{n}q^{-\sum_{j=1}^{r}(r-j)x_j}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)$$
$$=(-1)^n q^{n+\binom{r}{2}}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x+x_1+\cdots+x_r]_q^n q^{\sum_{j=1}^{r}(r-j)x_j}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r). \tag{2.30}$$

For $h=r$, $G_{n+r,q^{-1}}^{(r,r)}(0)=(-1)^n q^{n+\binom{r}{2}}G_{n+r,q}^{(r,r)}(r)$. It also follows from (2.26) that

$$q^{r-1}\frac{G_{n+r,q}^{(r,r)}(x+1)}{n+r}+\frac{G_{n+r,q}^{(r,r)}(x)}{n+r}=2G_{n+r-1,q}^{(r-1,r-1)}(x). \tag{2.31}$$

The Stirling numbers of the first kind are defined as

$$\prod_{k=1}^{n}\left(1+[k]_q z\right)=\sum_{k=0}^{n}S_1(n,k;q)z^k, \tag{2.32}$$

(see[6, 9]),

$$q^{\binom{m}{2}}\binom{r}{m}_q=\frac{q^{\binom{m}{2}}[r]_q\cdots[r-m+1]_q}{[m]_q!}=\frac{1}{[m]_q!}\prod_{k=0}^{m-1}\left([r]_q-[k]_q\right). \tag{2.33}$$

It can readily be seen that

$$\prod_{k=0}^{n-1}\left(z-[k]_q\right)=z^n\prod_{k=0}^{n-1}\left(1-\frac{[k]_q}{z}\right)=\sum_{k=0}^{n}S_1(n-1,k;q)(-1)^k z^{n-k}. \tag{2.34}$$

By (2.33) and (2.34),

$$\prod_{k=0}^{m-1}\left([r]_q-[k]_q\right)=\sum_{k=0}^{m}S_1(m-1,k;q)(-1)^k[r]_q^{m-k}. \tag{2.35}$$

Formulas (2.22) and (2.35) imply the following assertion.

Proposition 2.7. *For $r\in\mathbb{N}$ and $n\in\mathbb{Z}_+$,*

$$r!\binom{n}{r}G_{n-r,q}^{(r,-r)}(x)=\frac{1}{2^r[m]_q!}\sum_{m=0}^{r}\sum_{k=0}^{m}S_1(m-1,k;q)(-1)^k[r]_q^{m-k}[m+x]_q^n. \tag{2.36}$$

The generalized Genocchi numbers and polynomials of Nörlund type are defined by

$$\frac{2^r t^r}{(e^{w_1t}+1)(e^{w_2t}+1)\cdots(e^{w_rt}+1)}e^{xt}=\sum_{n=0}^{\infty}G_n^{(r)}(x\mid w_1,\ldots,w_r)\frac{t^n}{n!}, \tag{2.37}$$

and $G_n^{(r)}(w_1,\ldots,w_r)=G_n^{(r)}(0\mid w_1,\ldots,w_r)$. We can now also define a q-extension of (2.37) as follows. For $w_1,\ldots,w_r\in\mathbb{Z}_p$ and $\delta_1,\ldots,\delta_r\in\mathbb{Z}$, write

$$\frac{G_{n+r,q}^{(r)}(x\mid w_1,\ldots,w_r;\delta_1,\ldots,\delta_r)}{r!\binom{n+r}{r}}=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x_1w_1+\cdots+x_rw_r+x]_q^n\,d\mu_{-q^{\delta_1}}(x_1)\cdots d\mu_{-q^{\delta_r}}(x_r), \tag{2.38}$$

and $G_{n+r,q}^{(r)}(w_1,\ldots,w_r;\delta_1,\ldots,\delta_r)=G_{n+r,q}^{(r)}(0\mid w_1,\ldots,w_r;\delta_1,\ldots,\delta_r)$. Thus,

$$\frac{G_{n+r,q}^{(r)}(x\mid w_1,\ldots,w_r;\delta_1,\ldots,\delta_r)}{r!\binom{n+r}{r}}=\frac{[2]_{q^{\delta_1}}\cdots[2]_{q^{\delta_r}}}{(1-q)^n}\sum_{l=0}^{n}\frac{\binom{n}{l}(-1)^l q^{lx}}{\left(1+q^{\delta_1+lw_1}\right)\cdots\left(1+q^{\delta_r+lw_r}\right)}. \tag{2.39}$$

Another q-extension of Nörlund type generalized Genocchi numbers and polynomials is also of interest, namely,

$$\begin{aligned}&\frac{G_{n+r,q}^{*(r)}(x\mid w_1,\ldots,w_r;\delta_1,\ldots,\delta_r)}{r!\binom{n+r}{r}}\\&\quad=\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x_1w_1+\cdots+x_rw_r+x]_q^n q^{\delta_1x_1+\cdots+\delta_rx_r}\,d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r),\end{aligned} \tag{2.40}$$

and $G^{*(r)}_{n+r,q}(w_1,\ldots,w_r;\delta_1,\ldots,\delta_r) = G^{*(r)}_{n+r,q}(0 \mid w_1,\ldots,w_r;\delta_1,\ldots,\delta_r)$. By (2.40),

$$\frac{G^{*(r)}_{n+r,q}(x \mid w_1,\ldots,w_r;\delta_1,\ldots,\delta_r)}{r!\binom{n+r}{r}} = \frac{2^r}{(1-q)^n}\sum_{l=0}^{n}\frac{\binom{n}{l}(-1)^l q^{lx}}{(1+q^{\delta_1+lw_1})\cdots(1+q^{\delta_r+lw_r})}. \tag{2.41}$$

3. Further Remarks

For $h=0$, consider the following polynomials $G^{(0,r)}_{n+r,q}(x)/r!\binom{n+r}{r}$ and $r!\binom{n}{r}G^{(0,-r)}_{n+r,q}(x)$:

$$\begin{aligned}\frac{G^{(0,r)}_{n+r,q}(x)}{r!\binom{n+r}{r}} &= \int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}[x+x_1+\cdots+x_r]_q^n q^{-\sum_{j=1}^{r} jx_j} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r),\\ r!\binom{n}{r}G^{(0,-r)}_{n-r,q}(x) &= \frac{1}{(1-q)^n}\sum_{l=0}^{n}\frac{\binom{n}{l}(-1)^l q^{lx}}{\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p} q^{l(x_1+\cdots+x_r)}q^{-\sum_{j=1}^{r} jx_j} d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r)}.\end{aligned} \tag{3.1}$$

Then,

$$\begin{aligned}\frac{G^{(0,r)}_{n+r,q}(x)}{r!\binom{n+r}{r}} &= \frac{2^r}{(1-q)^n}\sum_{l=0}^{n}\frac{\binom{n}{l}(-1)^l q^{lx}}{(-q^{l-r};q)_r} = 2^r\sum_{m=0}^{\infty}\binom{m+r-1}{m}_q q^{-rm}(-1)^m[x+m]_q^n\\ r!\binom{n}{r}G^{(0,-r)}_{n-r,q}(x) &= \frac{1}{2^r(1-q)^n}\sum_{l=0}^{n}\binom{n}{l}(-1)^l q^{lx}\left(-q^{l-r};q\right)_r = \frac{1}{2^r}\sum_{m=0}^{r}\binom{r}{m}_q q^{\binom{m}{2}}q^{-rm}[m+x]_q^n.\end{aligned} \tag{3.2}$$

Let $F^{(0,r)}_q(t,x) = \sum_{n=0}^{\infty} G^{(0,r)}_{n,q}(x)(t^n/n!)$ and let $F^{(0,-r)}_q(t,x) = \sum_{n=0}^{\infty} G^{(0,-r)}_{n,q}(x)(t^n/n!)$. Then,

$$\begin{aligned}F^{(0,r)}_q(t,x) &= 2^r t^r\sum_{m=0}^{\infty}\binom{m+r-1}{m}_q q^{-rm}(-1)^m e^{[x+m]_q t},\\ F^{(0,-r)}_q(t,x) &= \frac{1}{2^r t^r}\sum_{m=0}^{r}\binom{r}{m}_q q^{\binom{m}{2}}q^{-rm}e^{[m+x]_q t}.\end{aligned} \tag{3.3}$$

Consider the following polynomials:

$$\frac{G^{(h,1)}_{n+1,q}(x)}{n+1} = \int_{\mathbb{Z}_p} q^{x_1(h-1)}[x+x_1]_q^n d\mu_{-1}(x_1) = \frac{2}{(1-q)^n}\sum_{l=0}^{n}\frac{\binom{n}{l}(-1)^l q^{lx}}{1+q^{l+h-1}}. \tag{3.4}$$

A simple calculation of the fermionic p-adic invariant integral on $\mathbb{Z}_p$ show that

$$\begin{aligned} q^x \int_{\mathbb{Z}_p} [x+x_1]_q^n q^{x_1(h-1)} d\mu_{-1}(x_1) \\ = (q-1)\int_{\mathbb{Z}_p} [x+x_1]_q^{n+1} q^{x_1(h-2)} d\mu_{-1}(x_1) + \int_{\mathbb{Z}_p} [x+x_1]_q^n q^{x_1(h-2)} d\mu_{-1}(x_1). \end{aligned} \tag{3.5}$$

By (3.5), $q^x G_{n+1,q}^{(h,1)}(x) = (q-1)(G_{n+2,q}^{(h-1,1)}(x)/2(n+2)) + G_{n+1,q}^{(h-1,1)}(x)$. It can readily be proved that

$$\int_{\mathbb{Z}_p} [x+x_1]_q^n q^{x_1(h-1)} d\mu_{-1}(x_1) = \sum_{j=0}^{n} \binom{n}{j} [x]_q^{n-j} q^{jx} \int_{\mathbb{Z}_p} [x_1]_q^j q^{x_1(h-1)} d\mu_{-1}(x_1). \tag{3.6}$$

By (3.6), $G_{n+1,q}^{(h,1)}(x)/(n+1) = \sum_{j=0}^{n} \binom{n}{j} [x]_q^{n-j} q^{jx} (G_{j+1,q}^{(h,1)}/(j+1))$. Using (2.24), we can also prove that

$$\int_{\mathbb{Z}_p} [x+x_1+1]_q^n q^{(x_1+1)(h-1)} d\mu_{-1}(x_1) + \int_{\mathbb{Z}_p} [x+x_1]_q^n q^{x_1(h-1)} d\mu_{-1}(x_1) = 2[x]_q^n. \tag{3.7}$$

Thus, $q^{h-1}(G_{n+1,q}^{(h,1)}(x)/(n+1)) + (G_{n+1,q}^{(h,1)}(x)/(n+1)) = 2[x]_q^n$. For $x=0$, we have $q^{h-1}(G_{n+1,q}^{(h,1)}(1)/(n+1)) + (G_{n+1,q}^{(h,1)}/(n+1)) = 2\delta_{n,0}$, where $\delta_{n,0}$ is the Kronecker delta.

It is easy to see that $G_{1,q}^{(h,1)} = \int_{\mathbb{Z}_p} q^{x_1(h-1)} d\mu_{-1}(x_1) = 2/(1+q^{h-1}) = 2/([2]_{q^{h-1}})$. By (3.4),

$$\begin{aligned} \frac{G_{n+1,q^{-1}}^{(h,1)}(1-x)}{n+1} &= \int_{\mathbb{Z}_p} [1-x+x_1]_{q^{-1}}^n q^{-x_1(h-1)} d\mu_{-1}(x_1) \\ &= (-1)^n q^{n+h-1} \frac{2}{(1-q)^n} \sum_{l=0}^{n} \frac{\binom{n}{l}(-1)^l q^{lx}}{1+q^{l+h-1}} \\ &= (-1)^n q^{n+h-1} \frac{G_{n+1,q}^{(h,1)}(x)}{n+1}. \end{aligned} \tag{3.8}$$

In particular, if $x = 1$, then $G_{n+1,q^{-1}}^{(h,1)}(0)/(n+1) = (-1)^n q^{n+h-1}(G_{n+1,q}^{(h,1)}(1)/(n+1)) = (-1)^{n-1} q^n (G_{n+1,q}^{(h,1)}/(n+1))$ for $n \geq 1$.

Recently, Kim has studied p-adic fermionic integral on $\mathbb{Z}_p$ connected with the problems of mathematical physics (see [6, 10, 11]), and our result are closely related to his results. In the future, we will try to study p-adic stochastic problems associated with our theorems. For example, p-adic q-Bernstein polynomials seem to be closely related to our results (see [6, 14, 20]).

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