Manuscript submitted to
AIMS' Journals
Volume **00**, Number **0**, Xxxx **XXXX**

Website: http://AIMsciences.org

pp. **000–000**

EXISTENCE OF WEAK SOLUTIONS TO THE CAUCHY PROBLEM OF A SEMILINEAR WAVE EQUATION WITH SUPERCRITICAL INTERIOR SOURCE AND DAMPING

Lorena Bociu

University of Nebraska-Lincoln
Lincoln, NE 68588-0130, USA

Petronela Radu

University of Nebraska-Lincoln
Lincoln, NE 68588-0130, USA

Abstract. In this paper we show existence of finite energy solutions for the Cauchy problem associated with a semilinear wave equation with interior damping and supercritical source terms. The main contribution consists in dealing with super-supercritical source terms (terms of the order of $|u|^p$ with $p \geq 5$ in $n = 3$ dimensions), an open and highly recognized problem in the literature on nonlinear wave equations.

1. **Introduction.**

Consider the Cauchy problem:

$$\begin{cases} u_{tt} - \Delta u + f(u) + g(u_t) = 0 \text{ a.e. } (x,t) \in \mathbb{R}^n \times [0, \infty); \\ (u, u_t)|_{t=0} = (u_0, u_1), \text{ a.e. } x \in \mathbb{R}^n. \end{cases} \tag{SW}$$

We are interested in the existence of weak solutions to (SW) on the finite energy space $H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$. We will work with the following notation : $|\cdot|_{p,\Omega}$ denotes the $L^p(\Omega)$ norm, while for the L^2 norm we simply use $|\cdot|_\Omega$; when there is no danger of confusion we simplify the notation $|\cdot|_{p,\Omega}$ to $|\cdot|_p$.

For the sake of exposition, we will focus on the most relevant case of dimension $n = 3$, but the analysis can be adapted to any other value of n. In order to classify the interior source and briefly describe previous results, assume that f is polynomially bounded at infinity, i.e. $|f(s)| \leq |s|^p$, for $|s| > 1$. In this case, we classify the interior source f based on the criticality of the Sobolev's embedding $H^1(\mathbb{R}^3) \to L^6(\mathbb{R}^3)$ as follows: (i) **subcritical**: $1 \leq p < 3$ and **critical**: $p = 3$. In these cases, f is locally Lipschitz from $H^1(\mathbb{R}^3)$ into $L^2(\mathbb{R}^3)$; (ii) **supercritical**: $3 < p < 5$. For this exponent, f is no longer locally Lipschitz, but the potential well energy associated with f is still well defined on the finite energy space; (iii) **super-supercritical**: $5 \leq p < 6$. The source is no longer within the framework of potential well theory, due to the fact that the potential energy may not be defined on the finite energy space.

2000 *Mathematics Subject Classification.* Primary: 35L15, 35L70; Secondary: 35L05.

Key words and phrases. wave equations, damping and source terms, weak solutions, energy identity.

1.1. **Assumptions.** Throughout the paper we will impose the following conditions on the source and damping terms:

(A_g) g is increasing and continuous with $g(0) = 0$. In addition, the following growth condition at infinity holds: there exist positive constants l_m, L_m such that for $|s| > 1$ we have $l_m|s|^{m+1} \leq g(s)s \leq L_m|s|^{m+1}$ with $m \geq 0$.

(A_f) $f \in C^1(\mathbb{R})$ and the following growth condition is imposed on f:

$$|f'(u)| \leq C|u|^{p-1} \text{ for } |s| > 1$$

where $p \in [1, 6)$ satisfies either

(a) $1 < p \leq 3$, $m \geq 0$

or

(b) $p > 3$ and $p + \dfrac{p}{m} \leq 6/(1 + 2\varepsilon)$ for some $\varepsilon > 0$, where $m > 0$ is the growth exponent from (A_g).

Remark 1. Note that the Assumption (A_f) allows for both types of supercriticality. Also, (A_f) guarantees that f is locally Lipschitz from $H^{1-\varepsilon}(\mathbb{R}^3) \to L^{\frac{m+1}{m}}(\mathbb{R}^3)$.

Definition 1.1. (Weak solution) Let $\Omega_T := \Omega \times (0, T)$, $T > 0$, where $\Omega \subset \mathbb{R}^3$ is an open connected set with smooth boundary $\partial\Omega$. Let f and g be two real valued functions which satisfy (A_f) and (A_g), and further suppose that $u_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega) \cap L^{m+1}(\Omega)$.

A *weak solution on* Ω_T of the boundary value problem

$$\begin{cases} u_{tt} - \Delta u + f(u) + g(u_t) = 0 \text{ in } \Omega_T; \\ (u, u_t)|_{t=0} = (u_0, u_1) \text{ in } \Omega; \\ u = 0 \text{ on } \partial\Omega \times (0, T). \end{cases} \tag{SWB}$$

is any function u satisfying

$$u \in C(0, T; H_0^1(\Omega)), \quad u_t \in L^2(\Omega_T) \cap L^{m+1}(\Omega_T),$$

and

$$\int_{\Omega_T} \Big(u(x, s)\phi_{tt}(x, s) + \nabla u(x, s) \cdot \nabla\phi(x, s) + f(u)\phi(x, s) \\ + g(u_t)\phi(x, s)\Big)\, dxds = \int_\Omega \Big(u_1(x)\phi(x, 0) - u_0(x)\phi_t(x, 0)\Big)dx$$

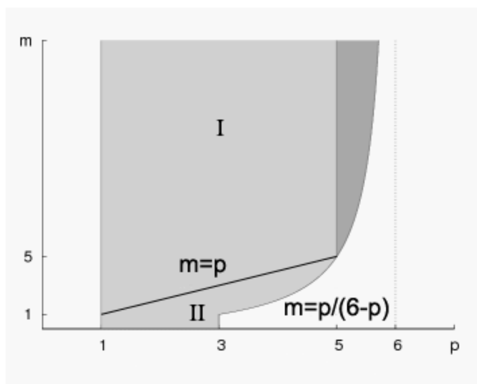
for every $\phi \in C_c^\infty(\Omega \times (-\infty, T))$.

Remark 2. A weak solution for the Cauchy problem (SW) is defined by taking in the above definition $\Omega = \mathbb{R}^n$ with no boundary conditions.

1.2. **Relationship to previous literature. Significance of results.** Semilinear wave equations with interior damping-source interaction have attracted a lot of attention in recent years. In the case of subcritical source f, local existence and uniqueness of solutions are standard and they follow from monotone operator theory [1]. In [8], the authors considered the case of polynomial damping and source, i.e. $g(u_t) = |u_t|^{m-1}u_t$ and $f(u) = -|u|^{p-1}u$ and showed that if the damping is strong enough ($m \geq p$), the solutions live forever, while in the complementary region $m < p$, the solutions blow-up in finite time. For supercritical interior sources, [2], [7] and [15] exhibited existence of weak solutions for a bounded domain Ω, under the restriction $p < 6m/(m + 1)$, while [17] obtained the same results for

$\Omega = \mathbb{R}^3$, and compactly supported initial data, with $p < 6m/(m+1)$. In this case, it was shown additionally by [14] that if the interior damping is absent or linear, the exponent p may be supercritical, i.e. $p < 5$; also, in [14] the initial data may not be compactly supported. The case of super-supercritical sources on a bounded domain was analyzed and resolved recently in [3], [4], [5]. The authors considered the wave equation with interior and boundary damping and source interactions, and proved existence and uniqueness of weak solutions. Moreover, they provided complete description of parameters corresponding to global existence and blow-up in finite time. We will provide more details on these results in the next section.

Our paper provides existence of solutions to wave equations on $\mathbb{R}^3$ for the case of super-supercritical sources. The method used will also provide an alternative proof in the case of supercritical (and below) interior sources. Thus our paper extends the known existence results to the super-supercritical case (we include the dark shaded regions $5 \leq p < 6$). We illustrate our results and improvements over previous literature in the following graph:



Note that for the range of exponents $m \geq p$ (region I above) one expects global existence of solutions, while for $m < p$ the solutions may blow up in finite time (according to the results of [8, 2, 3, 5] obtained on bounded domains).

2. **Preliminaries.**

Consider the system on an open bounded domain $\Omega \subset \mathbb{R}^3$:

$$\begin{cases} u_{tt} - \Delta u + f(u) + g(u_t) = 0 \text{ a.e. } (x,t) \in \Omega \times [0,\infty); \\ u(x,t) = 0, \text{ a.e. } (x,t) \in \partial\Omega \times [0,\infty); \\ (u, u_t)|_{t=0} = (u_0, u_1), \text{ a.e. } x \in \Omega. \end{cases} \tag{SWB}$$

We begin this section with the following theorems, which were proved in [14] and which will be used in the proof of our main result.

Theorem 2.1. (Existence and uniqueness of solutions for dissipative wave equations with Lipschitz source terms) *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary $\partial\Omega$, and let the functions f and g satisfy assumptions* (A_f), (A_g), *where f is globally Lipschitz. Let $u_0, u_1 \in H_0^1(\Omega) \times L^2(\Omega)$ and $T > 0$ arbitrary. Then the boundary value problem (SWB) admits a unique solution u on the time interval $[0,T]$ in the sense of the Definition 1.1, i.e.,*

$$u \in C(0,T; H_0^1(\Omega)) \cap L^{p+1}(\Omega_T), \quad u_t \in C(0,T; L^2(\Omega_T)) \cap L^{m+1}(\Omega_T).$$

The finite speed of propagation property is known to hold for wave equations with nonlinear damping and/or with source terms *of good sign*, i.e. their contribution to the energy of the system is decretive. The following theorem states that the property remains true for source terms of *arbitrary* sign, as long as they are Lipschitz (for a proof see [14]).

Theorem 2.2. (Finite speed of propagation) *Consider the problem (SWB) under the hypothesis of Theorem 2.1. Then*

(1) *if the initial data u_0, u_1 is compactly supported inside the ball $B(x_0, R) \subset \Omega$, then $u(x,t) = 0$ for all points $x \in \Omega$ outside $B(x_0,\ R+t)$;*

(2) *if $(u_0, u_1), (v_0, v_1)$ are two pairs of initial data with compact support, with the corresponding solutions $u(x,t)$, respectively $v(x,t)$, and $u_0(x) = v_0(x)$ for $x \in B(x_0,R) \subset \Omega$, then $u(x,t) = v(x,t)$ inside $B(x_0,\ R-t)$ for any $t < R$.*

We conclude this section by stating the following result which appears in [3, 4] and whose analog on $\mathbb{R}^3$ we will prove in the next section.

Theorem 2.3. (Local existence and uniqueness in the case of interior damping-source interactions) *Consider (SWB) under assumptions (A_f), (A_g) above and let $u_0 \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega)$. If $p > 3$, we additionally assume that $f \in C^2(\mathbb{R})$, and $|f''(s)| \leq C|s|^{p-2}$, for $|s| > 1$. Then there exists a* ***local in time unique*** *weak solution $u \in C[(0,T_M), H_0^1(\Omega)] \cap C^1[(0,T_M), L^2(\Omega)]$, where the maximal time of existence $T_M > 0$depends on initial data $|(u_0,u_1)|_{H_0^1(\Omega)\times L^2(\Omega)}$, and l_m given by (A_g).*

Remark 3. The condition $|f''(s)| \leq C|s|^{p-2}$ is needed for the uniqueness, but not for the existence of solutions.

3. **Local in time existence of solutions to the Cauchy problem.** Our main result states:

Theorem 3.1. (Existence of weak solutions) *Let $(u_0, u_1) \in H_0^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ and consider the Cauchy problem*

$$\begin{cases} u_{tt} - \Delta u + f(u) + g(u_t) = 0 \ \ a.e.\ \ in\ \mathbb{R}^n \times [0,\infty); \\ u|_{t=0} = u_0; \qquad u_t|_{t=0} = u_1. \end{cases} \tag{SW}$$

where f and g satisfy (A_f)-(A_g). Then, there exists a time $T > 0$ such that (SW) admits a weak solution on $[0,T]$ in the sense of Definition 1.1. The existence time T depends on the energy norm of the initial data and on the constant l_m given by (A_g).

Proof. We identify the following steps:

3.1. **Local existence on bounded domains.** Consider for now the problem (SWB) where Ω is an open, bounded domain with smooth boundary. First we will solve the existence problem on such a domain; in the second step we will cut the initial data in pieces defined on small domains; finally, we will show how to piece together the solutions defined on these small domains to obtain existence of solutions on the entire space $\mathbb{R}^3$.

Approximation of f: We consider the following approximation of equation (SWB), with $n \to \infty$ as the parameter of approximation:

$$\begin{cases} u^n_{tt} - \Delta u^n + f_n(u^n) + g(u^n_t) = 0 \text{ in } \Omega \times [0,\infty) \\ u^n = 0 \text{ on } \partial\Omega \times [0,\infty) \\ u^n(0) = u_0 \in H^1(\Omega) \text{ and } u^n_t(0) = u_1 \in L^2(\Omega). \end{cases} \tag{1}$$

We construct the approximating functions f_n as follows: let η be a cutoff smooth function such that: $(i) 0 \leq \eta \leq 1$, $(ii) \eta(u) = 1$, if $|u| \leq n$, $(iii) \eta(u) = 0$, if $|u| > 2n$, and $(iv) |\eta'(u)| \leq C/n$. Now construct $f_n : H^{1-\varepsilon}(\Omega) \to L^{\frac{m+1}{m}}(\Omega)$, $f_n(u) := f(u)\eta(u)$. This means that

$$f_n(u) = \begin{cases} f(u) & , \ |u| \leq n \\ f(u)\eta(u) & , \ n < |u| < 2n. \\ 0 & , \text{otherwise.} \end{cases}$$

Claim 1: f_n is locally Lipschitz from $H^{1-\varepsilon}(\Omega) \to L^{\frac{m+1}{m}}(\Omega)$ (uniformly in n). In the sequel we will use the notation $\tilde{m} = \frac{m+1}{m}$. In order to prove the claim, we consider the following three cases:

Case 1: $|u|$, $|v| \leq n$. Then $|f_n(u) - f_n(v)|_{\tilde{m}} = |f(u) - f(v)|_{\tilde{m}}$ and we already know that f is locally Lipschitz from $H^{1-\varepsilon}(\Omega) \to L^{\tilde{m}}(\Omega)$.

Case 2: $n \leq |u|$, $|v| \leq 2n$. Then we have the following computations:

$$\begin{aligned} |f_n(u) - f_n(v)|_{\tilde{m}} &= |f(u)\eta(u) - f(v)\eta(v)|_{\tilde{m}} \\ &\leq |f(u)\eta(u) - f(v)\eta(u) + f(v)\eta(u) - f(v)\eta(v)|_{\tilde{m}} \\ &\leq |f(u) - f(v)|_{\tilde{m}} + \Big(\int_\Omega [|f(v)|\eta(u) - \eta(v)|]^{\tilde{m}} \, dx \Big)^{m/(m+1)} \\ &\leq |f(u) - f(v)|_{\tilde{m}} + \Big(\int_\Omega [|v|^{p-1}|v| \max_\xi |\eta'(\xi)|u - v|]^{\tilde{m}} \, dx \Big)^{m/(m+1)} \end{aligned} \tag{2}$$

Now using the definition of the cutoff function η and the fact that $|v| \leq 2n$, we can see that $|v| \max_\xi |\eta'(\xi) \leq C$ and thus (2) becomes

$$|f_n(u) - f_n(v)|_{\tilde{m}} \leq |f(u) - f(v)|_{\tilde{m}} + \Big(\int_\Omega |v|^{(p-1)\tilde{m}} |u - v|^{\tilde{m}} \, dx \Big)^{m/(m+1)}. \tag{3}$$

For the second term on the right side of (3), we use Hölder's Inequality with p and $p/(p-1)$, the fact that $p(m+1)/m \leq 6/(1+2\varepsilon)$, and Sobolev's Imbedding $H^{1-\varepsilon}(\Omega) \to L^{\frac{6}{1+2\varepsilon}}(\Omega)$ to obtain

$$\begin{aligned} |f_n(u) - f_n(v)|_{\tilde{m}} &\leq |f(u) - f(v)|_{\tilde{m}} + C|v|^{p-1}_{\frac{6}{1+2\varepsilon}} |u - v|_{\frac{6}{1+2\varepsilon}} \\ &\leq |f(u) - f(v)|_{\tilde{m}} + C|v|^{p-1}_{H^{1-\varepsilon}(\Omega)} |u - v|_{H^{1-\varepsilon}(\Omega)} \end{aligned}$$

which proves that f_n is locally Lipschitz $H^{1-\varepsilon}(\Omega) \to L^{\frac{m+1}{m}}(\Omega)$.

Case 3: If $|u| \leq n$ and $n < v \leq 2n$, then we have

$$\begin{aligned} |f_n(u) - f_n(v)|_{\tilde{m}} &= |f(u) - f(v)\eta(v)|_{\tilde{m}} \\ &\leq |f(u) - f(v)|_{\tilde{m}} + \Big(\int_\Omega |f(v)|1 - \eta(v)| \, dx \Big)^{m/(m+1)}. \end{aligned} \tag{4}$$

In (4), we can replace $1 = \eta(u)$, since $|u| \leq n$ and then the calculations follow exactly as in case 2.

Claim 2: For each n, f_n is Lipschitz from $H^1(\Omega) \to L^2(\Omega)$. Again, we consider the three cases:

Case 1: $|u| \le n$ and $|v| \le n$. Then

$$|f_n(u) - f_n(v)|_\Omega = |f(u) - f(v)|_\Omega \le \Big(\int_\Omega C|u-v|^2[|u|^{p-1} + |v|^{p-1} + 1]^2 \, dx \Big)^{1/2}. \tag{5}$$

Using Hölder's Inequality with 3 and 3/2, the fact that $|u| \le n$ and $|v| \le n$ and Sobolev's Imbedding $H^1(\Omega) \to L^6(\Omega)$, (5) becomes

$$|f_n(u) - f_n(v)|_\Omega \le C_n |\nabla(u-v)|_\Omega$$

Case 2: $n < |u|$, $|v| \le 2n$. Then we use the calculations performed in case 2 of Claim 1 and obtain

$$|f_n(u) - f_n(v)|_\Omega \le |f(u) - f(v)|_\Omega + \Big(\int_\Omega C|v|^{2(p-1)}|u-v|^2 \, dx \Big)^{1/2}.$$

Now reiterating the strategy used in Case 1, we obtain the desired result. As before, the case when $|u| \le n$ and $n < |v| \le 2n$ reduces to case 2.

Claim 3: $|f_n(u) - f(u)|_{L^{\frac{m+1}{m}}(\Omega)} \to 0$ as $n \to \infty$ for all $u \in H^1(\Omega)$. This can be easily seen, since $|f_n(u) - f(u)| = |f(u)||\eta(u) - 1|$ shows that $f_n(u) \to f(u)$ a.e. (because f is continuous and $\eta \to 1$ as $n \to \infty$). Then we also have that $|f_n(u) - f(u)|^{\frac{m+1}{m}} \le 2^{\frac{m+1}{m}} |f(u)|^{\frac{m+1}{m}}$ and $f(u) \in L^{\frac{m+1}{m}}(\Omega)$, for $u \in H^1(\Omega)$. Thus by Lebesgue Dominated Convergence Theorem, $f_n \to f$ in $L^{\frac{m+1}{m}}(\Omega)$.

Since g and f_n satisfy the assumptions of Theorem 2.1, then its result holds true for each n with $T_M(|(u_0, u_1)|_{H^1(\Omega)\times L^2(\Omega)}, l_m)$ (with T_M uniform in n), i.e for each n, there exists a pair $(u^n(t), u^n_t(t)) \in C(0,T; H^1(\Omega) \times L^2(\Omega))$ which solves the approximated problem (1). Thus $u^n(t)$ satisfies the following variational equality: for any $\phi \in H^1(\Omega) \cap L^{m+1}(\Omega)$, we have:

$$\frac{d}{dt}(u^n_t(t), \phi)_\Omega + (u^n(t), \phi)_{H^1(\Omega)} + (f_n(u^n(t)), \phi)_\Omega + (g(u^n_t(t)), \phi)_\Omega) = 0. \tag{6}$$

We will prove that this sequence of solutions u^n has a subsequence that converges to an appropriate limit that will be the solution to the original problem (SWB).

By using the regularity properties of the solutions u^n, we apply the energy identity to the "n"-problem and obtain that for each $0 < T < T_{max}$, we have

$$\frac{1}{2}\Big(|u^n_t(T)|^2_\Omega + |\nabla u^n(T)|^2_\Omega\Big) + \int_{\Omega_T} f_n(u^n(t)) u^n_t(t) dx dt + \int_{\Omega_T} g(u^n_t(t)) u^n_t(t) dx dt = \frac{1}{2}\Big(|u^n_t(0)|^2_\Omega + |\nabla u^n(0)|^2_\Omega\Big) \tag{7}$$

A-priori bounds: We will show that for any positive $T_0 < T_{max}$, $u^n(t) \in L^\infty(0,T; H^1(\Omega))$ and $u^n_t(t) \in L^\infty(0,T; L^2(\Omega))$, for all $0 < T \le T_0$.

Remember the assumptions on g and f:

- $g(s)s \ge l_m |s|^{m+1}$ for $|s| \ge 1$
- f_n is locally Lipschitz: $H^1(\Omega) \to L^{\frac{m+1}{m}}(\Omega)$

Going back to (7), we estimate the terms involving the source f_n by using Hölder's Inequality with $\tilde{m} = \frac{m+1}{m}$ and $m+1$, followed by Young's Inequality with the corresponding components. For simplicity, in the following computations we use $u(t)$ instead of $u^n(t)$.

$$\begin{aligned}\int_{\Omega_T} f_n(u(t))u_t(t)dx &\le \int_0^T |f_n(u(t)|_{\tilde{m}} \cdot |u_t(t)|_{m+1}dt \\ &\le \varepsilon_1 \int_0^T |u_t(t)|_{m+1}^{m+1}dt + C_{\varepsilon_1}\int_0^T |f_n(u)|_{\tilde{m}}^{\tilde{m}}dt \\ &\le \varepsilon_1 \int_0^T |u_t(t)|_{m+1}^{m+1}dt + C_{\varepsilon_1}L_{f_n}^{\tilde{m}}(K)\int_0^T |\nabla u(t)|_{\Omega}^{\tilde{m}}dt + C_{\varepsilon_1}C_{f_n}T\end{aligned} \tag{8}$$

Combining (7) with (8) and using the growth conditions imposed on g, we obtain:

$$\begin{aligned}&\frac{1}{2}\Big(|u_t(T)|_\Omega^2 + |\nabla u(T)|_\Omega^2\Big) + l_m\int_0^T |u_t(t)|_{m+1}^{m+1}\,dt - C_{g,f}T \\ &\le \frac{1}{2}\Big(|u_t(0)|_\Omega^2 + |\nabla u(0)|_\Omega^2\Big) + \varepsilon_1\int_0^T |u_t(t)|_{m+1}^{m+1}\,dt + C_{\varepsilon_1}L_f^{\tilde{m}}(K)\int_0^T |\nabla u(t)|_\Omega^{\tilde{m}}\,dt.\end{aligned}$$

Choosing $\varepsilon_1 < \frac{l_m}{2}$ and since $m > 0$, we obtain that for all $T < T_{max}$, we have

$$|u_t^n(T)|_\Omega^2 + |\nabla u^n(T)|_\Omega^2 \le [|u_t^n(0)|_\Omega^2 + |\nabla u^n(0)|_\Omega^2 + \overline{C}T]\cdot e^{\tilde{C}T} \tag{9}$$

where $\overline{C} = C(g, f, \varepsilon_1, m)$ and $\tilde{C} = C(\varepsilon_1, m, K)$. This is obvious for $\tilde{m} = 2$ (i.e. $m = 1$). If $\tilde{m} > 2$ (i.e. $m < 1$), then we obtain the same result by writing $\int_0^T |\nabla u(t)|_\Omega^{\tilde{m}}\,dt = \int_0^T |\nabla u(t)|_\Omega^{\tilde{m}-2}\cdot|\nabla u(t)|_\Omega^2\,dt$ and using the fact that $|\nabla u(t)|_\Omega \le K$. Last case, if $\tilde{m} < 2$ (i.e. $m > 1$), then we can estimate $\int_0^T |\nabla u(t)|_\Omega^{\tilde{m}}\,dt \le \int_0^T |\nabla u(t)|_\Omega^2\,dt + T$ and then we obtain (9) again.

Also, we have

$$\int_0^T |u_t^n(t)|_{L^{m+1}(\Omega)}^{m+1} \le C_{|u_0|_{H^1(\Omega)},|u_1|_\Omega,T_{max}}. \tag{10}$$

From (10), combined with the growth assumptions imposed on the damping g, we obtain that

$$\begin{aligned}\int_{\Omega_T} |g(u_t^n(t))|^{\tilde{m}}dxdt &\le \int_{\Omega_T} L_m^{\tilde{m}}|u_t^n(t)|^{m+1}\,dxdt \\ &= \int_0^T |u_t(t)|_{L^{m+1}(\Omega)}^{m+1}\,dt \le C_{|\nabla u_0|_\Omega,|u_1|_\Omega,T_{max}}\end{aligned}$$

Therefore, for any $T < T_{max}$, on a subsequence we have

$$\begin{aligned}(u^n, u_t^n) &\to (u^n, u_t^n) \text{ weakly}^* \ L^\infty(0,T;H^1(\Omega)\times L^2(\Omega)) \\ u_t^n &\to u_t \text{ weakly in } \ L^{m+1}(0,T;\Omega) \\ g(u_t^n) &\to g^* \text{ weakly in } L^{\frac{m+1}{m}}(0,T;\Omega)\text{ , for some } g^* \in L^{\frac{m+1}{m}}(0,T;\Omega).\end{aligned}$$

We want to show that $g^* = g(u_t)$. In order to do that, consider u^k and u^n be the solutions to the approximated problem corresponding to the parameters k and n. For sake of notation, let $\tilde{u}(t) = u^n(t) - u^k(t)$ and $\tilde{u}_t(t) = u_t^n(t) - u_t^k(t)$. Then from the energy identity we obtain that for any $T < T_{max}$ we have

$$\frac{1}{2}|\tilde{u}_t(T)|_\Omega^2 + \frac{1}{2}|\tilde{u}(T)|_{H^1(\Omega)}^2 + \int_{\Omega_T}(f_n(u^n(t)) - f_k(u^k(t)))\tilde{u}_t(t)\,dxdt$$

$$+\int_{\Omega_T}(g(u_t^n(t))-g(u_t^k(t)))\tilde{u}_t(t)\,dxdt=0 \tag{11}$$

First we will show that $\int_{\Omega_T}(f_n(u^n(t))-f_k(u^k(t)))(\tilde{u}_t(t))\,dxdt \to 0$ as $k,n\to\infty$. Recall that $\tilde{m}=\frac{m+1}{m}$ and $|\cdot|_s=|\cdot|_{L^s(\Omega)}$. Using Hölder's Inequality with $\tilde{m}$ and $m+1$, we obtain:

$$\Big|\int_{\Omega_T}[f_n(u^n(t))-f_k(u^k(t))]\tilde{u}_t(t)dxdt\Big| \le \Big|\int_{\Omega_T}[f_n(u^n(t))-f_n(u(t))]\tilde{u}_t(t)dxdt\Big|$$
$$+\Big|\int_{\Omega_T}[f_n(u(t))-f(u(t))]\tilde{u}_t(t)dxdt\Big|+\Big|\int_{\Omega_T}[f(u(t))-f_k(u(t))]\tilde{u}_t(t)dxdt\Big|$$
$$+\Big|\int_{\Omega_T}[f_k(u(t))-f_k(u^k(t))]\tilde{u}_t(t)dxdt\Big| \le \int_0^T|f_n(u^n(t))-f_n(u(t))|_{\tilde{m}}|\tilde{u}_t(t)|_{m+1}dt$$
$$+\int_0^T|f_n(u(t))-f(u(t))|_{\tilde{m}}|\tilde{u}_t(t)|_{m+1}dt+\int_0^T|f(u(t))-f_k(u(t))|_{\tilde{m}}|\tilde{u}_t(t)|_{m+1}dt$$
$$+\int_0^T|f_k(u(t))-f_k(u^k(t))|_{\tilde{m}}|\tilde{u}_t(t)|_{m+1}dt$$

Now we use the fact that f is locally Lipschitz from $H^{1-\varepsilon}(\Omega)\to L^{\tilde{m}}(\Omega)$ and obtain:

$$\int_{\Omega_T}[f_n(u^n(t))-f_k(u^k(t))]\tilde{u}_t(t)dxdt \le \int_0^T L(K)|u^n(t)-u(t)|_{H^{1-\varepsilon}(\Omega)}|\tilde{u}_t(t)|_{m+1}dt$$
$$+\int_0^T|f_n(u(t))-f(u(t))|_{\tilde{m}}|\tilde{u}_t(t)|_{m+1}dt+\int_0^T|f(u(t))-f_k(u(t))|_{\tilde{m}}|\tilde{u}_t(t)|_{m+1}dt$$
$$+\int_0^T L(K)|u^k(t)-u(t)|_{H^{1-\varepsilon}(\Omega)}|\tilde{u}_t(t)|_{m+1}dt. \tag{12}$$

We know that $u^n(t)\to u(t)$ weakly* in $L^\infty(0,T;H^1(\Omega))$ and since the embedding $H^1(\Omega)\subset H^{1-\varepsilon}(\Omega)$ is compact and u_t^n is bounded in $L_{m+1}(0,T;\Omega)$, we get that $u^n(t)\to u(t)$ strongly in $L_{\tilde{m}}([0,T];H^{1-\varepsilon}(\Omega))$ (by Aubin's Compactness Theorem). We also know that $|f_n(u)-f(u)|_{\tilde{m}}\to 0$ as $n\to\infty$ (and same for k) and that $|u_t^n(t)-u_t^k(t)|_{m+1,\Omega}\le C$ for $t<T_{max}$. Thus from (12) we obtain the desired result

$$\int_{\Omega_T}[f_n(u^n(t))-f_k(u^k(t))]\tilde{u}_t(t)\,dxdt \to 0 \text{ as } k,n\to\infty.$$

Now we let $k,n\to\infty$ in (11) and remembering that g is monotone, we obtain:

$$\lim_{k,n\to\infty}\Big[|u^n(T)-u^k(T)|^2_{H^1(\Omega)}+|u_t^n(T)-u_t^k(T)|^2_\Omega\Big]=0 \tag{13}$$

and

$$\lim_{k,n\to\infty}\int_{\Omega_T}[g(u_t^n(t))-g(u_t^k(t))]\tilde{u}_t(t)\,dx=0. \tag{14}$$

Since now we know that $u_t^n\to u_t$ weakly in $L^{m+1}(0,T;\Omega)$ and $g(u_t^n)\to g^*$ weakly in $L^{\frac{m+1}{m}}(0,T;\Omega)$, and we also showed that $\limsup_{k,n\to\infty}(g(u_t^n)-g(u_t^k),u_t^n-u_t^k)\le 0$, then by Lemma 1.3 (p.42) in [1], we obtain that $g^*=g(u_t)$ and $(g(u_t^n),u_t^n)_\Omega\to(g(u_t),u_t)_\Omega$.

Since $|f_n(u) - f(u)|_{\tilde{m}} \to 0$ as $n \to \infty$, it follows that $f_n(u^n(t)) \to f(u(t))$ in $L^{\tilde{m}}(\Omega)$, as $u^n \to u$ weakly in $H^1(\Omega)$.

We are now in the position to pass to the limit in (6) and obtain the desired equality on bounded domains.

3.2. **Cutting the initial data.** Consider now a pair of initial data $(u_0, u_1) \in H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ and let K be an upper bound on the energy norm of the initial data, more precisely take K such that

$$|\nabla u_0|_{\mathbb{R}^3} + |u_1|_{\mathbb{R}^3} < K. \tag{15}$$

We find r such that

$$\begin{aligned} &|\nabla u_0|_{B(x_0,r)} < \frac{K}{4}, \quad |u_1|_{B(x_0,r)} < \frac{K}{4}, \\ &2(C^*\omega_3)^{\frac{1}{3}} \left(|\nabla u_0|_{B(x_0,r)} + |u_0|_{B(x_0,r)}\right) < \frac{K}{4}, \end{aligned} \tag{16}$$

where ω_3 is the volume of the unit ball in $\mathbb{R}^3$ and C^* is the constant from the Sobolev inequality (which does not depend on x_0 nor r). It can be easily shown that the above inequalities are satisfied by r chosen such that

$$\begin{aligned} &|u_0|_{B(x_0,r)} < \frac{K}{8(C^*\omega_3)^{\frac{1}{3}}}, \quad |u_1|_{B(x_0,r)} < \frac{K}{4}, \\ &|\nabla u_0|_{B(x_0,r)} < \min\left\{\frac{K}{4}, \frac{K}{8(C^*\omega_3)^{\frac{1}{3}}}\right\}. \end{aligned} \tag{17}$$

The fact that r can be chosen independently of x_0 is motivated by the equi-integrability of the functions $u_0, \nabla u_0, u_1$. For each of the functions $u_0, \nabla u_0, u_1$ we apply the following result of classical analysis:

If $f \in L^1(A)$, with A a measurable set, then for every given $\varepsilon > 0$, there exists a number $\delta > 0$ such that $\int_E |f(x)|dx < \varepsilon$, for every measurable set $E \subset A$ of measure less than δ (see [6]).

Note that δ in the above result does not depend on E, hence r does not vary with x_0.

From Theorem 2.3 it follows that the solution exists up to time $T(K)$ (which depends on K, but it does not depend on x_0) on all balls $B(x_0, r)$, $x_0 \in \mathbb{R}^3$, provided that $u_0 \in H_0^1(B(x_0, r))$. In order to obtain that u_0 has zero trace on $\partial B(x_0, r)$ we multiply it by a smooth cut-off function θ such that

$$\theta(x) = \begin{cases} 1, & |x - x_0| \leq r/2 \\ 0, & |x - x_0| \geq r \end{cases}$$

and

$$|\theta|_{\infty, B(x_0,r)} \leq 1, \quad |\nabla\theta|_{\infty, B(x_0,r)} \leq \frac{2}{r}. \tag{18}$$

Such θ can be obtained from a mollification which approximates the Lipschitz function

$$\theta_0(x) = \begin{cases} 1, & |x - x_0| \leq r/2 \\ 2 - 2|x - x_0|/r, & r/2 \leq |x - x_0| \leq r \\ 0, & |x - x_0| \geq r. \end{cases}$$

We denote by

$$u_0^{x_0} = \theta u_0, \quad u_1^{x_0} = u_1,$$

and by u^{x_0} the solution generated by $(u_0^{x_0}, u_1^{x_0})$. In order to show that

$$|\nabla u_0^{x_0}|_{B(x_0)} + |u_1^{x_0}|_{B(x_0)} < K \tag{19}$$

we start with the following estimate

$$|\nabla u_0^{x_0}|_{2,B(x_0,r)} \leq |\theta|_{\infty,B(x_0,r)} |\nabla u_0|_{2,B(x_0,r)} + |\nabla \theta|_{\infty,B(x_0,r)} |u_0|_{2,B(x_0,r)}.$$

By (18), (16), Hölder's inequality, followed by Sobolev's inequality we conclude that:

$$\begin{aligned} |\nabla u_0^{x_0}|_{2,B(x_0,r)} &< \frac{K}{4} + |B(x_0,r)|^{\frac{1}{3}} \frac{2}{r} |u_0|_{6,B(x_0,r)} \\ &\leq \frac{K}{4} + 2(C^* \omega_3)^{\frac{1}{3}} \left(|\nabla u_0|_{2,B(x_0,r)} + |u_0|_{2,B(x_0,r)} \right) \overset{(16)}{\leq} \frac{K}{4} + \frac{K}{4} = \frac{K}{2}. \end{aligned}$$

Thus we showed that the pair $(u_0^{x_0}, u_1^{x_0})$ satisfies (19).

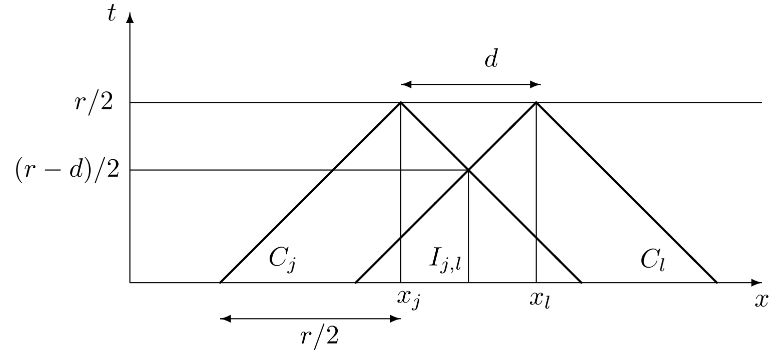
3.3. **Patching the small solutions.** The key argument that we use in order to construct the solution to the Cauchy problem from the "partial" solutions to the boundary value problems set on the balls $B(x_0, r)$ constructed in section 3.2 uses an idea due to Crandall and Tartar. They first used this type of argument to obtain global existence of solutions for a Broadwell model with *arbitrarily large* initial data starting from solutions with small data (see [18]). Subsequently, the second author has recast it in the framework of semilinear wave equations and showed local existence of solutions for (SW) on the entire space $\mathbb{R}^3$ (see [14]); the argument may also be employed on bounded domains as it was done in [15].

Step 1. Construction of partial solutions. Consider a lattice of points in $\mathbb{R}^3$ denoted by x_j situated at distance $d > 0$ from each other, such that in every ball of radius d we find at least one x_j. Next construct the balls $B_j := B(x_j, r/2)$, where r is given by (16) and inside each B_j take a snapshot of the initial data. More precisely, construct $(u_0^{x_j}, u_1^{x_j})$ by the procedure used in subsection 3.2. On each of the balls $B(x_j, r)$ we use Theorem 2.1 for the approximated problem given by the system (1) to obtain existence of solutions $u^{x_j,n}$ up to a time $T(K)$ independent of x_j and of n. These solutions will satisfy the estimate (7) on $B(x_j, r + T(K))$. Following the arguments from section 3.1 we pass to the limit in the sequence of approximations $u^{x_j,n}$ on each of the balls B_j and obtain a solution u^{x_j}.

Step 2. Patching the small solutions. For $j \in \mathbb{N}$ let

$$C_j := \{(y,s) \in \mathbb{R}^3 \times [0,\infty); |y - x_j| \leq r/2 - s\}$$

be the backward cones which have their vertices at $(x_j, r/2)$. For d small enough (i.e. for $0 < d < r/2$) any two neighboring cones C_j and C_l will intersect.



The intersection of the cones C_j and C_l

For every set of intersection

$$I_{j,l} := C_j \cap C_l$$

the maximum value for time contained in it is equal to $(r-d)/2$ (see figure above). For $t < r/2$ we define the piecewise function:

$$u(x,t) := u^{x_j}(x,t), \text{ if } (x,t) \in C_j. \tag{20}$$

This solution is defined only up to time $(r-d)/2$, since the cones do not cover the entire strip $\mathbb{R}^3 \times (0, r/2)$. By letting $d \to 0$ we can obtain a solution well defined up to time $r/2$. Thus, we have u defined up to time $r/2$, which is the height of all cones C_j. Every pair $(x,t) \in \mathbb{R}^3 \times (0, r/2)$ belongs to at least one C_j, so in order to show that this function from (20) is well defined, we need to check that it is single-valued on the intersection of two cones. Also, we need to show that the above function is the solution generated by the pair of initial data (u_0, u_1). Both proofs will be done in the next step.

Step 3. The solution given by (20) is valid. To show that u defined by (20) is a proper function, we use the same result of uniqueness given by the finite speed of propagation. First note that for $n \geq 3$ the intersection $I_{j,l}$ is not a cone, but it is contained by the cone $C_{j,l}$ with the vertex at $((x_j + x_l)/2, (r-d)/2)$ of height $(r-d)/2$. In this cone we use the uniqueness asserted by the finite speed of propagation as follows. Recall that the approximations f_n are Lipschitz inside the balls $\{B(x_j, r)\}_{j \in \mathbb{R}^3}$, hence the finite speed property holds for the solutions $u^{j,n}$. First note that the cones $C_{j,l}$ contain the sets $I_{j,l}$, but $C_{j,l} \subset C_j \cup C_l$. In C_j and C_l we have the two solutions $u^{j,n}$ and $u^{l,n}$ hence, in $C_{j,l}$ we now have defined two functions which can pose as solutions. Since $u^{j,n}$ and $u^{l,n}$ start with the same initial data ($(u_0^j, u_1^j) = (u_0, u_1) = (u_0^l, u_1^l)$ on $B_j \cap B_l$), hence they are equal in $C_{j,l}$; since $I_{j,l} \subset C_{j,l}$ we proved $u^{j,n} = u^{l,n}$ on $I_{j,l}$. By letting $n \to \infty$ we get $u^j = u^l$ on $I_{j,l}$. Therefore, u is a single-valued (proper) function.

Finally, the fact that this constructed function u is a solution to the Cauchy problem (SW) is immediate since it satisfies both, the wave equation and the initial conditions.

Remark 4. (**on global existence**) The above method of using cutoff functions and "patching" solutions based on the finite speed of propagation property will work the same way to prove global existence on bounded domains. Since we can choose the height of the the cones as large as we wish the solutions exist globally in time.

Remark 5. (**on uniqueness**) In [4] the authors showed under the same assumptions (A_f), (A_g) that the boundary value problem (SWB) admits a unique solution (in fact, the result was shown in the presence of damping and source terms in the interior *and* on the boundary). The methods employed here seem to preclude us from obtaining a corresponding uniqueness result for the Cauchy problem (SW) since they are obtained by passing to the limit in sequences of approximations.

□

Acknowledgements. We would like to thank the referee for his/her valuable comments and suggestions.

REFERENCES

[1] V. Barbu, "Nonlinear semigroups and differential equations in Banach spaces", Editura Academiei, Bucuresti Romania and Noordhoff International Publishing, Leyden Netherlands (1976).

[2] V. Barbu, I. Lasiecka, and M. A. Rammaha, *On nonlinear wave equations with degenerate damping and source terms*, Trans. Amer. Math. Soc., **357** (2005), 2571–2611.

[3] L. Bociu, *Local and Global Wellposedness of Weak Solutions for the Wave Equation with Nonlinear Boundary and Interior Sources of Supercritical Exponents and Damping*, Nonlinear Analysis, Theory, Methods and Applications, 2008, doi:10.1016/j.na.2008.11.062.

[4] L. Bociu and I. Lasiecka, *Uniqueness of Weak Solutions for the Semilinear Wave Equations with Supercritical Boundary/Interior Sources and Damping*, Discrete and Continuous Dynamical Systems A, **22** (2008), no. 4, 835–860.

[5] L.Bociu and I.Lasiecka, *Blow-up of Weak Solutions for the Semilinear Wave Equations with Nonlinear Boundary and Interior Sources and Damping*, *Applicationes Mathematicae*, **35** (2008), no. 3, 281–304.

[6] L. C. Evans and R. F. Gariepy, "Measure Theory and Fine Properties of Functions", Studies in Advanced Mathematics, (1992).

[7] E. Feireisl, *Global attractors for semilinear damped wave equations with supercritical exponent*, Journal of Differential Equations, **116** (1995), pp. 431 - 447.

[8] V. Georgiev and G. Todorova, *Existence of a solution of the wave equation with nonlinear damping and source terms*, J.Differential Equations, **109**, (1994), 295–308.

[9] (R. T. Glassey), *Blow-up theorems for nonlinear wave equations*, Math.Z., **132** (1973), 183–203.

[10] F. John, *Blow-up of solutions of nonlinear wave equations in three space dimensions*, Manuscripta Math., **28** (1979), 235–268.

[11] I. Lasiecka and D. Tataru, *Uniform boundary stabilization of semilinear wave equations with nonlinear boundary damping*, Differential and Integral Equations, vol.6, no.3, (1993), 507–533.

[12] I. Lasiecka and D. Toundykov, *Energy decay rates for the semilinear wave equation with nonlinear localized damping and source terms*, Nonlinear Analysis, **64** (2006), no. 8, 1757–1797.

[13] J.-L. Lions and W. A. Strauss, *On some nonlinear evolution equations*, Bull. Soc.Math. France, **93** (1965), 43–96.

[14] P. Radu, *Weak Solutions to the Cauchy Problem of a Semilinear Wave Equation with Damping and Source Terms*, Advanced Differential Equations, **10**, no.11 (2005), 1261–1300.

[15] P. Radu, *Weak Solutions to the Initial Boundary Value Problem of a Semilinear Wave Equation with Damping and Source Terms*, Applicationes Mathematicae, **35**, no.3 (2008), 355–378.

[16] J. Schaeffer, *The equation $u_{tt} - \Delta u = |u|^p$ for the critical value of p*, Proc. Roy. Soc. Edinburgh Sect. A, **101A** (1985), 31–44.

[17] J. Serrin, G. Todorova, and E. Vitillaro, *Existence for a nonlinear wave equation with damping and source terms*, Differential Integral Equations, **16** (2003), no. 1, 13–50.

[18] L. Tartar, *Existence globale pour un système hyperbolique semi linéaire de la théorie cinétique des gaz*, Séminaire Goulaouic-Schwartz (1975/1976), Equations aux dérivées partielles et analyse fonctionnelle, Exp. No. 1, 11 pp. Centre Math., École Polytech., Palaiseau, 1976.

[19] L. Tartar, " Topics in nonlinear analysis", Publications Mathématiques d'Orsay, **78, 13**, Université de Paris-Sud, Département de Mathématiques, Orsay, 1978, ii+271 pp.

Received June 2008; revised April 2009.

E-mail address: `lbociu2@math.unl.edu`
E-mail address: `pradu@math.unl.edu`