Trajectory piece-wise quasi-linear approximation of large non-linear dynamic systems

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Abstract: In this work we extend the trajectory piece-wise linear (TPWL) approximation of large, non-linear and input-affine dynamical systems to non-linear systems with non-linear input operators. The new technique is called the trajectory piece-wise quasi-linear (TPWQ) approximation. We explain the motivation for this technique in light of previously reported under-performance of the TPWL method and show that its a more general alternative, developing it is formulation and demonstrating its effectiveness in the process.

Keywords: large dynamical systems; model order reduction; MOR; non-linear systems; simulation; modelling; trajectory piece-wise linear; TPWL; quasi-linear; trajectory piece-wise quasi-linear; TPWQ.

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1 Introduction

Realistic and accurate mathematical-description of complex natural and artificial processes gives rise to dynamic system models of large dimensions. The simulation of such models requires expenditure of considerable amounts of computational resources and time. Additionally, the simulation may have to be carried out repeatedly, e.g., for determination of optimal design or operating parameters. Approximation of the large systems thus being crucial, model order reduction (MOR) techniques are a means to realise this objective. They result in a dimensionally reduced system with input-output response characteristics similar to the large dynamic system.

It has been observed that many large linear time invariant (LTI) systems are poorly controllable and observable and their

dynamics are restricted to a smaller subspace. MOR methods for such systems are different ways to find the dominant subspace and then *project* the large system dynamics onto them (Antoulas, 2005). Important techniques for MOR of LTI systems include; Krylov subspace projection and singular value decomposition (SVD)-based methods. Krylov-based methods provide a numerically robust and cheap alternative for constructing the dominant subspace. They have evolved out of the application of numerical linear algebra algorithms for iterative eigenvalue computation, to implicitly match the *moments* of the original and reduced systems at particular frequencies. The projection matrix, whose columns span the dominant subspace, is made up iteratively. However the resulting reduced order systems have no guaranteed error bound, the procedure is not automatic and features like stability and passivity are not retained. SVD-based methods require the solution of Lyapunov equations for the computation of controllability and observability grammians W_c and W_o. The reduced order system is obtained by a projection of the original system onto the dominant Eigen-space of (W_cW_o). The exact solution of the Lyapunov equations requires dense computations and hence these methods are limited to systems of dimension in thousands (Antoulas, 2005). SVD-based methods have a number of desirable properties, a global error bound exists and stability is preserved. MOR techniques for LTI systems are well researched and have been extensively covered in Antoulas (2005).

Most processes, whether natural or man-made, being inherently non-linear, simulation and control that includes model non-linearities is important. Model reduction of non-linear systems though an area in infancy, popular non-linear model reduction tools include: approximating the system non-linearity, system-identification-based methods, proper orthogonal decomposition (POD) and trajectory piece-wise linear (TPWL).

Approximating the system non-linearity (Chen, 1999; Mbarek et al., 2012) generates reduced-order models valid only around the operating point of the system and is applicable only to weakly non-linear systems. System-identification-based methods (Pierri et al., 2011) are generally limited to cases wherein the governing equations of the non-linear system are unknown or partially known.

POD was first introduced in Lumley (1967) to identify coherent structures in dynamical systems. In it, one or more 'training' simulations of the non-linear system is performed and state-vectors are saved in the 'snap-shot' matrix at a number of time steps from these simulations. The basis functions are then generated through an SVD of the snapshot matrix, which is analogous to an eigen-decomposition of the covariance matrix formed from the snapshots. However, effective dimension reduction is limited to problems with linear or bi-linear terms as the evaluation cost of the projected non-linear function depends on the dimension of the original system (Chaturantabut and Sorensen, 2010).

A relatively recent development is TPWL (Rewienski and White, 2001), wherein the non-linear system is approximated at multiple points by sub-models as it evolves along a training trajectory in the state space. These sub-models are then used to construct the reduced order model. This offsets the drawback of local linearisations, creating a model with global fidelity, as long as the non-linear system stays near training trajectory(s). The approximations may be obtained by linearisation or polynomial expansion followed by MOR, and the final model is expressed as their weighted sum. Since, instead of the original non-linear system, linear systems are projected onto the dominant subspace, the computational cost of projection and evaluation of the reduced-order model is made comparable to that of projecting an LTI system.

TPWL has been generally applied to input-affine non-linear systems. In this work, we develop a modified technique for non-linear systems with non-linear input operators, called the Trajectory piece-wise quasi-linear (TPWQ) approximation. In TPWQ, instead of locally approximating the non-linear system by linearisations or polynomial expansions, we propose partially linearising the non-linear term. This partial-linearisation is with respect to states and is called quasi-linearisation, as the non-linearity with respect to the input is retained. TPWQ is applied for MOR of a circuit consisting of a chain of inverters, for which TPWL has been reported to be ineffective, and it shows a substantially improved reproduction of non-linear system dynamics.

The organisation of this paper is as follows. In the next section we review TPWL, highlighting it is salient features, modifications and shortcomings. In Section 3, the motivation for quasi-linearisation, and the formulation of TPWQ is presented. In Section 4, description of the circuit model which is used to demonstrate TPWQ is given. This is followed by simulation results and conclusions.

2 Review of the TPWL method

The complete process of MOR using TPWL consists of the following sequence of steps:

1 Finding the non-linear system trajectory in response to a training input. An approximate trajectory suffices for most applications.

2 Looking for meaningful states along the trajectory at which local approximations are to be created, these points are called the linearisation points (LPs).

3 Approximating the non-linear system at selected LPs. In Rewienski and White (2001), order-1 Taylor series approximants are used.

4 Identification and evaluation of the subspace in which the system dynamics lie predominantly.

5 MOR of the local sub-models by projection onto the dominant subspace.

6 Expression of the non-linear system as a weighted sum of the reduced linear sub-models.

2.1 Mathematical formulation

Consider the following input-affine non-linear dynamical system

$$\left.\begin{aligned}\dot{x} &= f(x) + Bu \\ y &= Cx\end{aligned}\right\} \tag{1}$$

where $x \in R^n$ is a vector of system states, $f : R^n \to R^n$ is the non-linear vector field, $B \in R^{n\times p}$ is the input matrix, $C \in R^{n\times q}$ is the output matrix and $y \in R^q$ is the output. Assuming m LPs $[x_0, ..x_i, ..x_{m-1}]$ are selected, the Taylor series first-order approximation of $f(x)$ about x_i is given by

$$\tilde{f}(x) = f(x_i) + A_i(x - x_i) \tag{2}$$

where A_i is the Jacobian of $f(x)$ evaluated at x_i. The linear system at x_i is hence given by:

$$\left.\begin{aligned}\dot{x} &= f(x_i) + A_i(x - x_i) + Bu \\ y &= Cx\end{aligned}\right\} \tag{3}$$

The dynamics are restricted to the dominant subspace by the state transformation $x \simeq Vz$. V is the projection matrix with its columns spanning the dominant subspace of dimension r, where $r \ll n$ and $V^TV = I$. With this, the reduced sub-model at x_i is:

$$\left.\begin{aligned}\dot{z} &= A_{ir}z + V^T(f(x_i) - A_ix_i) + B_ru \\ y &= C_rz\end{aligned}\right\} \tag{4}$$

where $A_{ir} = V^TA_iV$, $B_r = V^TB$, and $C_r = CV$. Next, weights $w_i(z)$ are assigned to the reduced order models based on the information about the distances $||z - z_i||$ of the projected LPs z_i from the current state z. The motivation is that the dominant model $[A_{ir}, B_r, C_r]$ at a state z is the one corresponding to the LP z_i which is closest to it. Hence, the final TPWL model is expressed as the weighted sum:

$$\left.\begin{aligned}\dot{z} &= \textstyle\sum_{i=0}^{m-1} w_i(z)(A_{ir}z + V^T(f(x_i) - A_ix_i)) \\ &\quad + B_ru \\ y &= C_rz\end{aligned}\right\} \tag{5}$$

2.2 Salient features

2.2.1 Selection of LPs

In Rewienski and White (2001), LPs are selected on the approximate non-linear system trajectory. This selection is based on state evolution, i.e., a point on the approximate system trajectory is selected as an LP if its 2-norm distance from previously selected LPs is greater than a threshold. The procedure is given in Algorithm 1. Another important algorithm (Voss et al., 2007) selects LPs on the exact non-linear system trajectory. The selection is done by comparing the responses of the non-linear system and the previous sub-model. When the difference is beyond a threshold, a new LP is selected. The pseudo-code is given in Algorithm 2.

Algorithm 1 LP selection on the approximate trajectory

1: $i \leftarrow 0$, $j \leftarrow 1$. x_0 is the initial state, T the number of simulation time-steps and δ is an appropriately selected constant.
2: For $j < T$ simulate (4), i.e., compute z for time-steps $t = j$ till the state Vz_j is close to any of the LPs, i.e., While $min_{(0\le k\le i)} \frac{||Vz_j - x_k||}{||x_k||} \le \delta$
3: $i \leftarrow i + 1$. $x_i = Vz_j$ is taken as the next LP. Step (3) is repeated.

Algorithm 2 LP selection on the exact trajectory

1: $i \leftarrow 0$, $j \leftarrow 1$. x_0 is the initial state, T the number of simulation time-steps and δ is an appropriately selected constant.
2: **while** $j < T$ **do** ▷ *Compute x and z for $t = j$ till x_j and Vz_j are close.*
3: **repeat**
4: Simulate (1) and (4)
5: **until** $\frac{||Vz_j - x_j||}{||x_j||} \le \delta$
6: **end while**
7: $i \leftarrow i + 1$. x_j is taken as the next LP. Step (3) is repeated.

2.2.2 Identifying the dominant subspace

The most popular method to form the projection matrix V is to construct the Krylov basis using Arnoldi algorithm for the linear system obtained at x_0 (Rewienski and White, 2001). The pseudo-code is given in Algorithm 3.

Algorithm 3 Finding the projection basis V

1: $\dot{x} = f(x_0) + A_0(x - x_0) + Bu$, $y = Cx$ ▷ *Linear System at x_0.*
2: If K_p is the p_{th} order Krylov subspace for this system, we have:
3: $span(V_1) = K_p[A_0^{-1},\ A_0^{-1}B]$
4: $span(V_2) = K_p[A_0^{-1},\ A_0^{-1}(f(x_0) - A_0x_0)]$
5: $V = [V_1\ V_2\ x_0]$
6: Orthogonalise the columns of V using SVD, and retain columns corresponding to singular values larger than some ϵ.

Finding the dominant subspace based on initial linearisation only is suitable if the non-linear system dynamics evolve dominantly in the subspace spanned by V. Projection accuracy can be increased by merging projection bases V_i constructed at LPs x_i and then reducing the set using SVD as shown in Rewienski and White (2002). Other methods for finding the projection matrix have also been used. TBR-based approaches have been tried in different forms depending on the non-linear problem (Vasilyev et al., 2006). POD has been found to be useful for use in conjunction with TPWL (Gratton and Willcox, 2004), especially if the non-linear system is simulated for finding LPs, during which snapshots of the system states can be stored and re-used for finding the projection basis.

2.2.3 Weighing procedure

The weighing procedure is based on the distances $||z - z_i||$ in the reduced space. The functions used are hard switching functions, i.e., they are significant for only very close neighbours.

Algorithm 4 Weighing procedure

1: For $i = 1, .., m$.
2: $d_i = ||z - z_i||$, $q = min(d_i)$
3: For $q \neq 0$, $\tilde{w}_i = e^{-\beta d_i/q}$
4: If $q = 0$ (because some $d_j = 0$) then $w_j = 1$ and $w_i = 0$ for $i \neq j$.
5: Normalise $\tilde{w}_i$, $w_i = \tilde{w}_i / \sum_{i=1}^{m} \tilde{w}_i$

These initially introduced, sharply varying, state dependant weights tend to be used in almost all applications. However, there are instances when smoother weight functions of similar nature give good results (Dong and Roychowdhury, 2005; Tiwary and Rutenbar, 2005).

2.2.4 Applications

There are diverse examples of successful application of trajectory-based methods to various fields of non-linear system simulation and reduction. Examples are, circuits (Dong and Roychowdhury, 2005; Tiwary and Rutenbar, 2005; Voss et al., 2007); CFD (Cardoso, 2009; Gratton and Willcox, 2004); power electronics (Qu and Chapman, 2006); moving non-linear electromagnetic devices (Albunni et al., 2008); and MEMS (Rewienski and White, 2003; Vasilyev et al., 2006).

2.3 Modifications and shortcomings

Most of the applications involving TPWL have used linear systems as local approximations at various trajectory points. However, it is not necessary that superposed linear systems would give good results in all cases. For example, in Dong and Roychowdhury (2003) the order of Taylor series expansion at each point is increased to 2 or 3. This method called piecewise polynomial (PWP), leads to a better reproduction of small signal and inter-modulation distortion, while retaining accuracy in transient analysis. This is because for small signal disturbances the TPWL model does not capture non-linear effects like distortion as the region boundaries between two linear systems are not crossed. Since PWP offers a better local approximation, it shows improved results in comparison to TPWL.

For application to industrial circuits TPWL model is made robust to a wide range of inputs by training across a varied input spectrum (Tiwary and Rutenbar, 2005). The huge number of linear sub-models created, are collected into clusters and interpolation is done between close neighbours only.

Other important extensions of TPWL are parameterised model order reduction (PMOR) using TPWL (Bond and Daniel, 2007) and preserving or enforcing stability of the TPWL model (Bond and Daniel, 2009).

It has been pointed out (Rewienski and White, 2006) that TPWL has been generally used for very damped systems, and may sometimes fail to give satisfactory results. For example, for a non-linear fluid dynamics problem, conventional TPWL has not been found very effective when the non-linearity is significant and leads to a very large shock motion (Gratton and Willcox, 2004). Further, in Striebel and Rommes (2011) it is shown that the conventional TPWL method fails to generate a satisfactory reduced order model of a circuit consisting of a series of inverters.

In the following sections we develop and demonstrate a modification of TPWL to make it an acceptable approximation tool for reduction of more general non-linear systems.

3 The TPWQ approximation

Let us consider a more general class of non-linear dynamical systems given by:

$$\left.\begin{aligned} \dot{x} &= F(x,u) \\ y &= Cx \end{aligned}\right\} \tag{6}$$

To get the TPWL formulation corresponding to (6) we linearise $F(x,u)$ at an LP (x_i, u_i)

$$\begin{aligned} F(x,u) = F(x_i,u_i) &+ A(x_i,u_i)(x - x_i) \\ &+ B(x_i,u_i)(u - u_i) \end{aligned} \tag{7}$$

where $A(x_i,u_i)$ is the Jacobian of $F(x,u)$ with respect to x evaluated at (x_i,u_i) and $B(x_i,u_i)$ is the Jacobian of $F(x,u)$ with respect to u evaluated at (x_i,u_i). The linear system at (x_i,u_i) is hence,

$$\left.\begin{aligned} \dot{x} &= A(x_i,u_i)x + B(x_i,u_i)u + D(x_i,u_i) \\ y &= Cx \end{aligned}\right\} \tag{8}$$

where $D(x_i,u_i) = F(x_i,u_i) - A(x_i,u_i)x_i - B(x_i,u_i)u_i$. Again, suppose the dominant dynamics are restricted to a subspace with the basis given by the columns of the orthogonal matrix V, substituting $x = Vz$ in (8) and pre-multiplying by V^T we obtain the reduced sub-model at x_i,

$$\left.\begin{aligned} \dot{z} &= V^T A(x_i,u_i)Vz + V^T B(x_i,u_i)u \\ &\quad + V^T D(x_i,u_i)) \\ y &= CVz \end{aligned}\right\} \tag{9}$$

The system matrices of (9) are functions of both u_i and x_i unlike (4). Using the distance $||z - z_i||$ for weight-assignment to find the TPWL model at the state z, when the sub-model does not depend on z_i alone may not be correct. This is especially true when the training and evaluations inputs are not the same. To circumvent this difficulty, it is proposed that the system (6) be linearised with respect to the states alone, this quasi-linearisation at x_i yields:

$$F(x,u) = F(x_i,u) + A(x_i,u)(x - x_i) \tag{10}$$

where $A(x_i, u)$ is the Jacobian of $F(x, u)$ with respect to x evaluated at x_i. Hence the quasi-linear system at x_i would be given by

$$\left.\begin{aligned} \dot{x} &= A(x_i, u)x + B(x_i, u) \\ y &= Cx \end{aligned}\right\} \tag{11}$$

where $B(x_i, u) = F(x_i, u) - A(x_i, u)x_i$. Restricting the dynamics in (11) to the subspace spanned by V and pre-multiplying by V^T gives the sub-model at each x_i:

$$\left.\begin{aligned} \dot{z} &= V^T A(x_i, u)Vz + V^T B(x_i, u) \\ y &= CVz \end{aligned}\right\} \tag{12}$$

In simplfied notation

$$\left.\begin{aligned} \dot{z} &= A_{ir}(u)z + B_{ir}(u) \\ y &= C_r z \end{aligned}\right\} \tag{13}$$

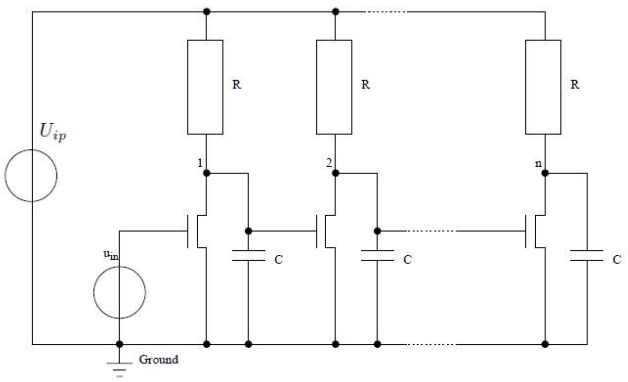
where $A_{ir}(u) = V^T A(x_i, u)V$, $B_{ir}(u) = V^T B(x_i, u)$, $C_r = CV$. The TPWQ model would hence be given as a weighted sum of the submodels (13), with the weights as usual being computed such that the dominant model $[A_{ir}(u), B_{ir}(u), C_r]$ is the one corresponding to the LP z_i which is closest to the current state of the system z. Note that since the dependence of the system matrices on u_i has been done away with, the procedure of weight assignment becomes valid. The TPWQ model would hence be given by:

$$\left.\begin{aligned} \dot{z} &= \textstyle\sum_{i=0}^{m-1} w_i(z)(A_{ir}(u)z + B_{ir}(u)) \\ y &= C_r z \end{aligned}\right\} \tag{14}$$

4 Non-linear test model

The non-linear dynamical system chosen for testing the new method is the ladder circuit shown in Figure 1. The circuit consists of a chain of inverters, with each inverter made of a MOSFET, a resistor and a capacitor. It exhibits significantly non-linear characteristics and is marked by *latency*, i.e., for some fixed time step only a small part of the circuits is active, whereas the major part remains passive.

Figure 1 Chain of inverters



The basic building block of the circuit is the MOSFET. The MOSFET is modelled as a voltage controlled current source, the descriptive relation being:

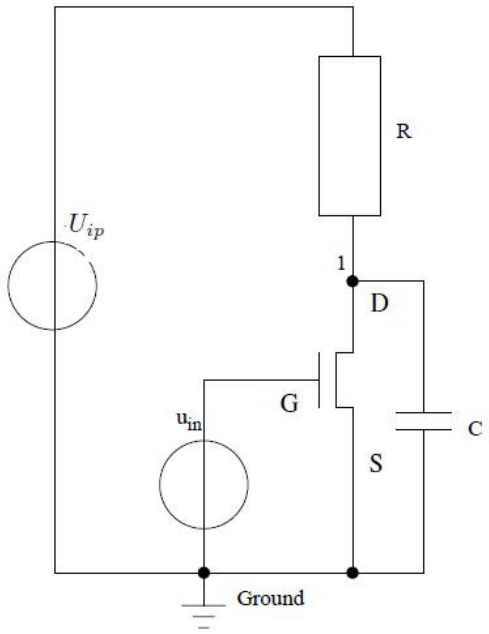
$$i_{ds} = k\ f(u_g, u_d, u_s) \tag{15}$$

where

$$f(u_g, u_d, u_s) = g_1 - g_2 \tag{16}$$

and $g_1 = max(u_g - u_s - U_{th}, 0)$, $g_2 = max(u_g - u_d - U_{th}, 0)$, i_{ds} is the drain to source current, u_g, u_d, u_s are the gate, drain and source voltages respectively and U_{th} is the threshold voltage of the MOSFET. A single inverter shown in Figure 2 consists of a voltage source, a resistance, a capacitance and a MOSFET.

Figure 2 Single inverter circuit



The values of parameters selected are $R = 5$ kΩ, $C = 0.2$ pF, $k = 2 \times 10^{-4}$ A/V^2, $U_{th} = 1$ V. For the circuit in Figure 2, assuming all leakage currents of the MOSFET to be zero, i.e.,

$$i_{ds} = i_{gb} = i_{gd} = 0 \tag{17}$$

the equation at gate can be written as

$$(C\dot{x}_1 + i_{ds})R + x_1 = U_{ip} \tag{18}$$

where x_1 is the voltage at node-1, which is the drain. Using (15) in (18) and substituting the parameter values we obtain:

$$\dot{x}_1 = 10^9(U_{ip} - x_1 - f(u_{in}, x_1, 0)) \tag{19}$$

Taking a scaling in time by 10^9

$$\dot{x}_1 = U_{ip} - x_1 - f(u_{in}, x_1, 0) \tag{20}$$

Hence (20) describes the dynamics of the voltage $\dot{x}_1$ with the time variable in nanoseconds. For a more detailed description of the circuit see Gunther and Rentrop (1993). A series of such inverters is connected to get the ladder circuit shown in

Figure 1. The governing equation for such an n-dimensional ladder circuit for nodes other than node-1 would be:

$$\dot{x}_k = U_{ip} - x_k - f(x_{k-1}, x_k, 0) \tag{21}$$

where $k = 2, 3..n$. Rewriting equations (20) and (21) together, we obtain the model of the circuit:

$$\dot{x} = GU_{ip} - x - F(x, u_{in}) \tag{22}$$

where $G = [1\ 1\ ...1]^T$ is an $n \times 1$ vector and $x = [x_1\ x_2\ ....x_n]^T$ represents the state vector comprising the drain voltages of all the MOSFETs and

$$F(x, u_{in}) = \begin{pmatrix} f(u_{in}, x_1, 0) \\ f(x_1, x_2, 0) \\ f(x_2, x_3, 0) \\ ..... \\ ..... \\ f(x_{n-1}, x_n, 0) \end{pmatrix} \tag{23}$$

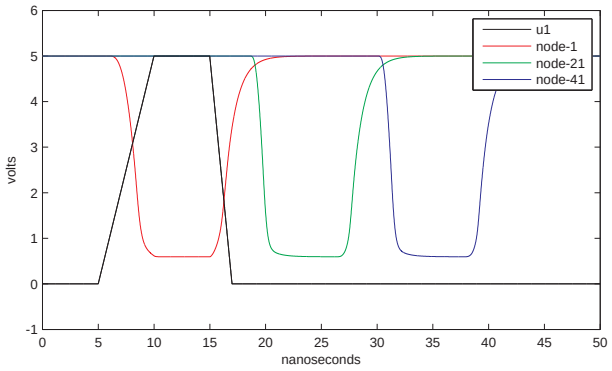
and f is defined as in equation (16).

Next, the inverter circuit is simulated. Figure 3 shows the output-response of the circuit of size $n = 100$ at different nodes. The input given to the circuit is the pulse given by:

$$u_{in} = u_1 = \begin{cases} t - 5 \text{ for } 5 \le t \le 10 \\ 5 \text{ for } 10 \le t \le 15 \\ 2.5(17 - t) \text{ for } 15 \le t \le 17 \\ 0 \text{ otherwise} \end{cases} \tag{24}$$

The time in (24) is measured in nanoseconds, and $U_{ip} = 5\ V$.

Figure 3 The input and output-response at different nodes for a 100 node circuit (see online version for colours)



4.1 TPWL formulation of the circuit

The dynamical system is in the form given by (6). To obtain linear submodels, a linearisation of the non-linear function F in (22) with respect to both the input u_{in} and the state x has to be done. Hence, linearising F given by (23) at $x = x_0$, $u_{in} = u_{in(0)}$ gives:

$$F(x, u_{in}) \simeq F(x_0, u_{in(0)}) + M_0(x - x_0) + B_0(u_{in} - u_{in(0)}) \tag{25}$$

where M_0 is the Jacobian of F with respect to the states x and B_0 is the Jacobian with respect to the input u_{in}, both being evaluated at the LP $(x_0, u_{in(0)})$. Using the approximation (25) the linear system at $(x_0, u_{in(0)})$ can be written as:

$$\dot{x} = A_0 x - B_0 u_{in} + D_0 \tag{26}$$

where

$$A_0 = -(I + M_0),$$
$$D_0 = -F(x_0, u_{in(0)}) + M_0 x_0 + B_0 u_{in(0)} + GU_{ip}.$$

The next step is to select m such linear systems (26) along a training trajectory. Once done, the resultant TPWL approximation is thus:

$$\left.\begin{array}{l} \dot{x} = \sum_{i=0}^{m-1} w_i(x)(A_i x - B_i u_{in} + D_i) \\ y = Cx \end{array}\right\} \tag{27}$$

Although in Voss et al. (2007) it is claimed that the TPWL model of the circuit is a satisfactory approximation, later studies have shown results to the contrary. Striebel and Rommes (2011) show that the trajectories produced by TPWL wiggle notably. Additionally, the model performs poorly when the training and evaluation input differ, e.g., when a chain of impulses is applied, the TPWL model misses every other impulse running through the system. The important point is that this under-performance is evident even if the linear systems used to form the TPWL model are unreduced, i.e., they have the same dimension as the original non-linear system.

In the coming sections we show that the TPWQ model proves to be a very good approximation of the non-linear system.

4.2 TPWQ formulation of the circuit

If (23) is linearised with respect to the states alone, the first-order Taylor series expansion at $x = x_0$ would be:

$$F(x, u_{in}) \simeq F(x_0, u_{in}) + M_0(u_{in})(x - x_0) \tag{28}$$

where $M_0(u_{in})$ is the jacobian of F with respect to the states x evaluated at x_0. It retains an explicit dependance on the input u_{in}. Hence the quasi-linear system at x_0 can be written as

$$\dot{x} = A_0(u_{in})x + B_0(u_{in}) \tag{29}$$

where

$$A_0(u_{in}) = -(I + M_0(u_{in})),$$
$$B_0(u_{in}) = GU_{ip} - F(x_0, u_{in}) + M_0(u_{in})x_0.$$

The system given by (29) is linear with respect to the states but non-linear with respect to the input. Proceeding as we did for the model in (27), after m such models (29) have been created, the TPWQ model can be expressed as:

$$\left.\begin{array}{l} \dot{x} = \sum_{i=0}^{m-1} w_i(x)(A_i(u_{in})z + B_i(u_{in})) \\ y = Cx \end{array}\right\} \tag{30}$$

5 Numerical simulations

In this section we show a comparison of the performance of TPWL and TPWQ. For both the techniques LPs selection is done using Algorithm 2. This is followed by their superposition along the evaluation trajectory using the weight assignment scheme given in Algorithm 3. Since the primary difference between TPWL and TPWQ is that of linearisation and quasi-linearisation, and that results in a difference in their performance, the non-linear system is expressed as a weighted sum of unreduced linear systems. This also stems from the observation that the under-performance of the TPWL model for this circuit is obvious after the linearisation stage only, and is not connected to reduction of the linear systems.

5.1 Test conditions

The training input is u_1 given by (24) and shown in Figure 3. The initial conditions are:

$$x_i = \begin{cases} 0.594 \text{ if } i \bmod 2 = 0 \\ 5 \text{ if } i \bmod 2 = 1 \end{cases} \tag{31}$$

Evaluation of the TPWL and TPWQ models is done using the same pulse u_1, or a sequence of pulses. Two different pulse-sequences are selected. In the first one given by (32) a thick pulse follows the training-input.

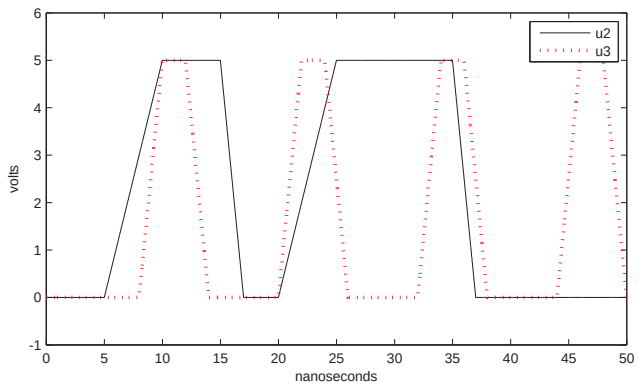
$$u_2 = \begin{cases} t-5 \text{ for } 5 \le t \le 10 \\ 5 \text{ for } 10 \le t \le 15 \\ 2.5(17-t) \text{ for } 15 \le t \le 17 \\ t-20 \text{ for } 20 \le t \le 25 \\ 5 \text{ for } 25 \le t \le 35 \\ 2.5(37-t) \text{ for } 35 \le t \le 37 \\ 0 \text{ otherwise} \end{cases} \tag{32}$$

The third evaluation input u_3 consists of a sequence of similar thin pulses, with the first pulse given by:

$$u_3 = \begin{cases} 2.5(t-8) \text{ for } 8 \le t \le 10 \\ 5 \text{ for } 10 \le t \le 12 \\ 2.5(14-t) \text{ for } 12 \le t \le 14 \\ 0 \text{ otherwise} \end{cases} \tag{33}$$

Figure 4 shows the two evaluation inputs.

Figure 4 Evaluation inputs (see online version for colours)



As primary purpose is to show the improvement gained at the linearisation step, the dimension selected for the non-linear system is not very large. A circuit with ten successive inverters is chosen, so the non-linear system has dimension 10.

5.2 Results

The following abbreviated words are used in the tables:

1 % error $y = 100 \times ||y_{NL} - y_{APPX}||/||y_{NL}||$ where y_{NL} is the output of the non-linear system and y_{APPX} is the output of the approximate model (TPWL or TPWQ) collected over the complete simulation time

2 % error $x = 100 \times ||x_{NL} - z_{APPX}||/||x_{NL}||$ where x_{NL} is the state-vector of the non-linear system and z_{APPX} is the state-vector of the approximate model (TPWL or TPWQ) collected over the complete simulation time.

Figure 5 shows a comparison in the performance of TPWL and TPWQ when the output voltage is taken at node-1 and training and evaluation is done using u_1. The figure clearly shows unsatisfactory performance of the TPWL model, with jumps in the output. In contrast, the TPWQ response is perfectly following the non-linear system response. This comparison is quantitatively evident from the errors in the two approximations given in Table 1.

Figure 5 Output taken at node-1 – training and evaluation using u_1 (see online version for colours)

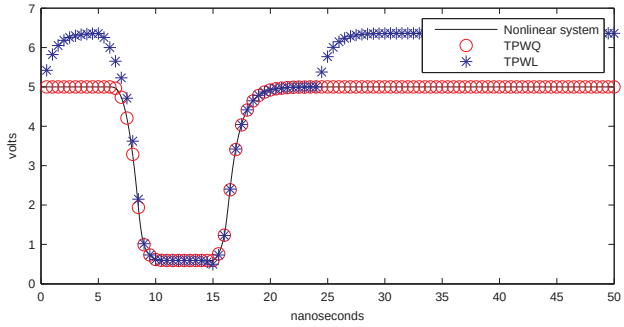


Table 1 Errors of the approximate models for training and evaluation using u_1 and output taken at node-1

Strategy	*% error y*	*% error x*
TPWL	23.02	10.32
TPWQ	0.09	0.11

Figure 6 shows a similar plot, but with the evaluation input u_2, training input remaining u_1 and output at node-1. The TPWL output has jumps and wiggles, and is unable to reproduce the falling edge of the nonlinear system output correctly. When the output is taken at node-5, as shown in Figure 7, the performance of the TPWL model is again unsatisfactory, although the jumps in the output are absent. In both the cases TPWQ performs substantially better as evident from Table 2 as well.

Figure 6 Output at node-1 – training input u_1, evaluation input u_2 (see online version for colours)

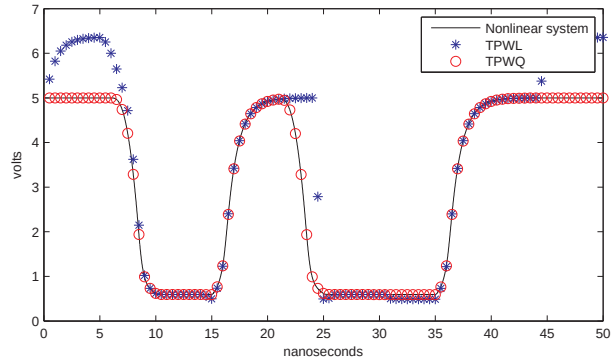


Figure 7 Output at node-5 – training input u_1, evaluation input u_2 (see online version for colours)

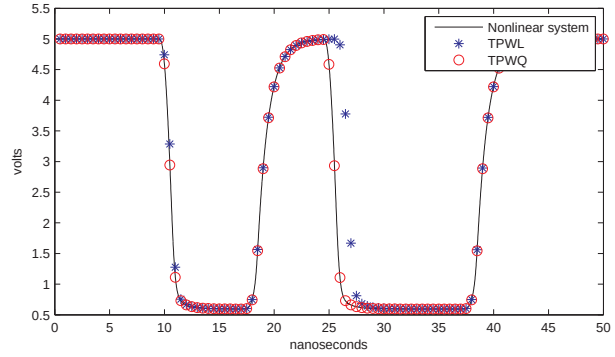


Table 2 Errors of the approximate models for training input u_1 and evaluation input u_2 and output taken at nodes 1 and 5

Strategy	*% error y (node-1)*	*% error y (node-5)*	*% error x*
TPWL	22.75	14.78	10.94
TPWQ	0.16	0.15	0.17

Figures 8 and 9 show the results when the two approximations are tested for the evaluation input u_3, while keeping the training input unchanged from u_1. Outputs are again taken at two nodes, 1 and 5. The TPWL model response is poor, it even misses the third pulse completely. TPWQ performs much better in comparison. Table 3 shows the errors.

Figure 8 Output at node-1 – training input u_1, evaluation input u_3 (see online version for colours)

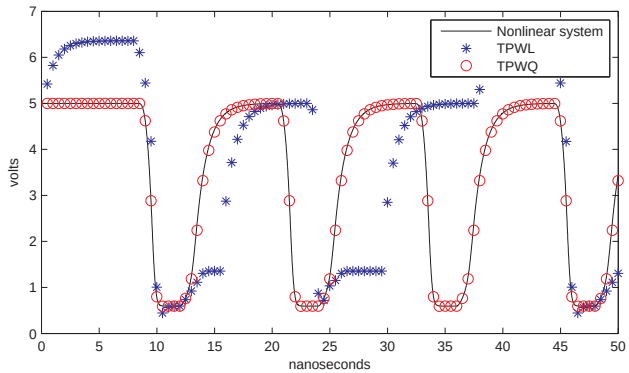


Figure 9 Output at node-5 – training input u_1, evaluation input u_3 (see online version for colours)

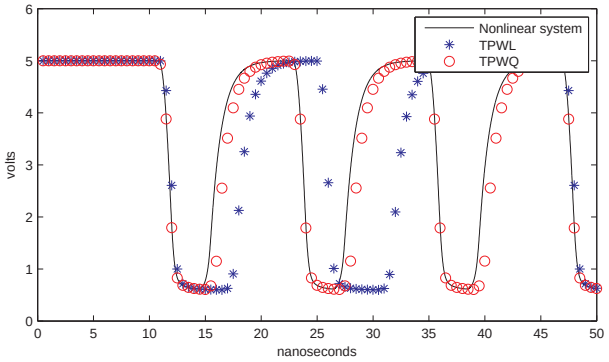


Table 3 Errors of the approximate models for training input u_1 and evaluation input u_3 and output taken at nodes 1 and 5

Strategy	*% error y (node-1)*	*% error y (node-5)*	*% error x*
TPWL	51.46	48.58	50.24
TPWQ	1.15	10.73	9.93

It is very evident that the performance improvement obtained using the TPWQ model is significant.

6 Conclusions

In this work we have successfully demonstrated that the TPWQ method can serve as an important tool for MOR of more general non-linear systems. We have shown its effectiveness on a problem for which conventional TPWL was almost dismissed as inadequate. However, to develop TPWQ as a complete MOR strategy, the quasi-linear systems need to be reduced before they are superposed and the associated computational costs need to be evaluated and compared. We feel that considerable scope is present in the further-development and study of the proposed method.

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