Combinatorial proofs of a kind of binomial and $q$-binomial coefficient identities

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Abstract. We give combinatorial proofs of some binomial and $q$-binomial identities in the literature, such as

\[ \sum_{k=-\infty}^{\infty} (-1)^k q^{(9k^2+3k)/2} \binom{2n}{n+3k} = (1 + q^n) \prod_{k=1}^{n-1} (1 + q^k + q^{2k}) \quad (n \geq 1), \]

and

\[ \sum_{k=0}^{\infty} \binom{3n}{2k} (-3)^k = (-8)^n. \]

Two related conjectures are proposed at the end of this paper.

1 Introduction

There are many different $q$-analogues of the following binomial coefficient identity

\[ \sum_{k=-\infty}^{\infty} (-1)^k \binom{2n}{n+2k} = 2^n, \quad (1.1) \]

\textsuperscript{*}This work was supported in part by Shanghai “Chenguang” Project (#2007CG29) and the National Science Foundation of China (#10801054).
in the literature. Here is a list of such identities:

\[
\sum_{k=-\infty}^{\infty} (-1)^k q^{2k^2} \left[ \frac{2n}{n + 2k} \right] = (-q; q^2)_n, \tag{1.2}
\]
\[
\sum_{k=-\infty}^{\infty} (-1)^k q^{2k^2 + k} \left[ \frac{2n}{n + 2k} \right] = (1 + q^n)(-q; q^2)_{n-1}, \tag{1.3}
\]
\[
\sum_{k=-\infty}^{\infty} (-1)^k q^{2k^2 + 2k} \left[ \frac{2n}{n + 2k} \right] = (1 + q)(-q; q^2)_{n-1}q^{n-1}, \tag{1.4}
\]
\[
\sum_{k=-\infty}^{\infty} (-1)^k q^{3k^2 + k} \left[ \frac{2n}{n + 2k} \right] = (-q; q)_n, \tag{1.5}
\]
\[
\sum_{k=-\infty}^{\infty} (-1)^k q^{(5k^2 + k)/2} \left[ \frac{2n}{n + 2k} \right] = \sum_{k=0}^{\infty} q^{k^2} \left[ \frac{n}{k} \right], \tag{1.6}
\]
\[
\sum_{k=-\infty}^{\infty} (-1)^k q^{(3k^2 + k)/2} \left[ \frac{2n}{n + 2k} \right] = \sum_{k=0}^{\infty} q^{nk} \left[ \frac{n}{k} \right], \tag{1.7}
\]

where the \(q\)-shifted factorials are defined by \((a; q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1})\) and the \(q\)-binomial coefficients are defined as

\[
\left[ \frac{n}{k} \right] = \begin{cases} 
\frac{(q; q)_n}{(q; q)_k(q; q)_{n-k}}, & \text{if } 0 \leq k \leq n, \\
0, & \text{otherwise}. 
\end{cases}
\]

Identities (1.2)–(1.4) can be proved by using the \(q\)-binomial theorem and \(i^2 = -1\) or other methods. For (1.2), see Ismail, Kim and Stanton [5, Proposition 2(2)], Berkovich and Warnaar [2, §7], and Sills [6, (3.3)]. For (1.3), see [5, Proposition 2(3)]. The identity (1.5) corresponds to Slater’s Bailey pair \(C(1)\). Identities (1.6) and (1.7) were discovered by Bressoud [3, (1.1) and (1.5)], and the former is usually known as a finite form of the first Rogers-Ramanujan identity.

For each of the identities (1.2)–(1.7), one can change \(q\) to \(q^{-1}\) to find a new identity of the same type. The identities (1.2)–(1.4) are “self-dual”, (1.6) and (1.7) are dual, and the dual of (1.5) is as follows:

\[
\sum_{k=-\infty}^{\infty} (-1)^k q^{k^2+k} \left[ \frac{2n}{n + 2k} \right] = q\left( \frac{n}{2} \right)(-q; q)_n.
\]

This identity is known as the Bailey pair \(C(5)\) in Slater’s list.

An identity similar to (1.1) is

\[
\sum_{k=-\infty}^{\infty} (-1)^k \left( \frac{2n}{n + 3k} \right) = \begin{cases} 
1, & \text{if } n = 0, \\
2 \cdot 3^{n-1}, & \text{if } n \geq 1, 
\end{cases} \tag{1.8}
\]
which also has two different $q$-analogues as follows:

$$\sum_{k=-\infty}^{\infty} (-1)^k q^{(9k^2+3k)/2} \binom{2n}{n+3k} = \begin{cases} 1, & \text{if } n = 0, \\ (1 + q^n) \frac{(q^3; q^3)_{n-1}}{(q; q)_{n-1}}, & \text{if } n \geq 1, \end{cases}$$

$$\sum_{k=-\infty}^{\infty} (-1)^k q^{(9k^2+9k)/2} \binom{2n}{n+3k} = \begin{cases} 1, & \text{if } n = 0, \\ 1 + q, & \text{if } n = 1, \\ (1 + q + q^2)(1 + q^n) \frac{(q^3; q^3)_{n-2}}{(q; q)_{n-2}} q^{n-2}, & \text{if } n \geq 2. \end{cases}$$

Like (1.2)–(1.4), Identities (1.9) and (1.10) can be proved by the $q$-binomial theorem. Identity (1.9) is equivalent to the Bailey pair $J(2)$ in [8], and can also be found in [5, Proposition 2(5)]. This identity was utilized by Berkovich and Warnaar [2] to prove a ‘perfect’ Rogers-Ramanujan identity.

There exists another not-so-famous binomial coefficient identity similar to (1.1) and (1.8) as follows:

$$\sum_{k=0}^{\infty} \binom{n}{2k} (-3)^k = \begin{cases} (-2)^n, & \text{if } n \equiv 0 \pmod{3}, \\ (-2)^{n-1}, & \text{if } n \equiv 1 \pmod{3}, \\ (-2)^{n-1}, & \text{if } n \equiv 2 \pmod{3}. \end{cases}$$

The main purpose of this paper is to give combinatorial proofs of the identities (1.1)–(1.4), (1.8)–(1.11), and some of their companions which appeared in the literature, such as

$$\sum_{k=-\infty}^{\infty} (-1)^k \binom{2n+1}{n+3k} = 3^n.$$  

However, we are unable to give combinatorial proofs of (1.5)–(1.7).

## 2 Proofs of (1.1)–(1.4)

**Proof of (1.1).** Let $S = \{a_1, \ldots, a_{2n}\}$ be a set of $2n$ elements, and let

$$\mathcal{F} = \{A \subseteq S: \#A \equiv n \pmod{2}\},$$

$$\mathcal{G} = \{A \subseteq S: \#(A \cap \{a_{2i-1}, a_{2i}\}) = 1 \text{ for all } i = 1, \ldots, n\}.$$

It is easy to see that $\mathcal{G} \subseteq \mathcal{F}$ and $\#\mathcal{G} = 2^n$. For any $A \in \mathcal{F}$, we associate $A$ with a sign $\text{sgn}(A) = (-1)^{(|A| - n)/2}$. It is clear that

$$\sum_{k=-\lfloor n/2 \rfloor}^{\lfloor n/2 \rfloor} (-1)^k \binom{2n}{n+2k} = \sum_{A \in \mathcal{F}} \text{sgn}(A) = \sum_{A \in \mathcal{F} \setminus \mathcal{G}} \text{sgn}(A) + \sum_{A \in \mathcal{G}} \text{sgn}(A).$$
Clearly, $\text{sgn}(A) = 1$ for $A \in \mathcal{G}$. What remains is to construct a sign-reversing involution on the set $\mathcal{F} \setminus \mathcal{G}$.

For any $A \in \mathcal{F} \setminus \mathcal{G}$, choose the first number $i$ such that $\#(A \cap \{a_{2i-1}, a_{2i}\}) \neq 1$, i.e., $A$ contains both $a_{2i-1}$ and $a_{2i}$ or none of them. Let $A'$ be a subset of $S$ obtained from $A$ as follows:

$$A' = \begin{cases} A \cup \{a_{2i-1}, a_{2i}\}, & \text{if } \{a_{2i-1}, a_{2i}\} \cap A = \emptyset, \\ A \setminus \{a_{2i-1}, a_{2i}\}, & \text{if } \{a_{2i-1}, a_{2i}\} \subseteq A. \end{cases}$$

(2.1)

It is obvious that $A' \in \mathcal{F} \setminus \mathcal{G}$, and $A \mapsto A'$ is the desired involution. $\square$

For $A \in S$, we associate it with a weight $||A|| = \sum_{a \in A} a$. By the $q$-binomial theorem (cf. Andrews [1, Theorem 3.3])

$$(z; q)_N = \sum_{j=0}^{N} \binom{N}{j} (-1)^j z^j q^{j(j-1)/2},$$

we have

$$\sum_{A \subseteq [n] \setminus \mathcal{A}} q^{||A||} = \sum_{\#A = k} \binom{n}{k} q^{(k+1)/2}.$$  

(2.2)

Here and in what follows $[n] := \{1, \ldots, n\}$. Now we can give proofs of (1.2)–(1.4).

**Proof of (1.2).** Let $\{a_{2i-1}, a_{2i}\} = \{- (2i - 1)/2, (2i - 1)/2\}$ for $i = 1, \ldots, n$. Since $a_{2i-1} + a_{2i} = 0$, the involution in the proof of (1.1) is indeed weight-preserving and sign-reversing. It follows that

$$\sum_{A \subseteq \mathcal{F}} \text{sgn}(A) q^{||A||} = \sum_{A \subseteq \mathcal{F} \setminus \mathcal{G}} \text{sgn}(A) q^{||A||} + \sum_{A \subseteq \mathcal{G}} \text{sgn}(A) q^{||A||}.$$  

(2.3)

It is easy to see that $S$ is obtained from $[2n]$ by a shift $-(n+1)/2$. By (2.2), the left-hand side of (2.3) equals

$$\sum_{k=-[n/2]}^{[n/2]} \sum_{A \subseteq S \setminus \mathcal{G}} \text{sgn}(A) q^{||A||} = \sum_{k=-[n/2]}^{[n/2]} (-1)^k \binom{2n}{n + 2k} q^{(n+2k+1)/2} q^{-(n+2k)(2n+1)/2}.$$  

On the other hand, the right-hand side of (2.3) is given by

$$\prod_{i=1}^{n} (q^{-(2i-1)/2} + q^{(2i-1)/2}) = (-q; q^2)_n q^{-n^2/2}.$$  

After simplification, we obtain (1.2). $\square$
Proof of (1.3). Note that the index $i$ in (2.1) is always less than $n$. Otherwise, $\#(A \cap \{a_{2i-1}, a_{2i}\}) = 1$ for $i = 1, \ldots, n - 1$ and $\#(A \cap \{a_{2n-1}, a_{2n}\}) \neq 1$, which is contradictory to the condition $\#A \equiv n \pmod{2}$. Thus, if we take $\{a_{2i-1}, a_{2i}\} = \{-i, i\}$ for $i = 1, \ldots, n - 1$ and $\{a_{2n-1}, a_{2n}\} = \{0, n\}$, then the involution in the proof of (1.1) is also weight-preserving and sign-reversing, and (2.3) still holds. Similarly as before, we obtain

$$\sum_{k=-[n/2]}^{[n/2]} (-1)^k \left[ \begin{array}{c} 2n \\ n + 2k \end{array} \right] q^{(n+2k+1)/2} q^{-n(n+2k)} = (q^0 + q^n) \prod_{i=1}^{n-1} (q^{-i} + q^i),$$

which is equivalent to (1.3). \qed

Proof of (1.4). Let $\{a_{2i-1}, a_{2i}\} = \{-2i - 1/2, 2i - 1/2\}$ for $i = 1, \ldots, n - 1$ and $\{a_{2n-1}, a_{2n}\} = \{(2n-1/2, (2n + 1/2)\}$. Then $S = \{i - (2n - 1)/2: i \in [2n]\}$ and the previous involution yields

$$\sum_{k=-[n/2]}^{[n/2]} (-1)^k \left[ \begin{array}{c} 2n \\ n + 2k \end{array} \right] q^{(n+2k+1)/2} q^{-n(n+2k)} = (q^{(2n-1)/2} + q^{(2n+1)/2}) \prod_{i=1}^{n-1} (q^{-(2i-1)/2} + q^{(2i-1)/2}),$$

which, after simplification, leads to (1.4). \qed

Similarly, if we set $S = \{a_1, \ldots, a_{2n+1}\}$ be a set of $2n + 1$ elements, and again let

$$\mathcal{F} = \{A \subseteq S: \#A \equiv n \pmod{2}\},$$

$$\mathcal{G} = \{A \subseteq S: \#(A \cap \{a_{2i-1}, a_{2i}\}) = 1 \text{ for all } i = 1, \ldots, n\},$$

then the same argument implies that

$$\sum_{k=-\infty}^{\infty} (-1)^k \left[ \begin{array}{c} 2n + 1 \\ n + 2k \end{array} \right] = 2^n.$$

Furthermore, letting $\{a_{2i-1}, a_{2i}\} = \{-2i - 1/2, 2i - 1/2\}, i = 1, \ldots, n$, and $a_{2n+1} = (2n + 1)/2$, we obtain

$$\sum_{k=-\infty}^{\infty} (-1)^k q^{2k^2} \left[ \begin{array}{c} 2n + 1 \\ n + 2k \end{array} \right] = (-q; q^2)_n$$

(see [5, Propositon 2(2)]); while letting $\{a_{2i-1}, a_{2i}\} = \{-i, i\}, i = 1, \ldots, n$, and $a_{2n+1} = 0$, we obtain

$$\sum_{k=-\infty}^{\infty} (-1)^k q^{2k^2-k} \left[ \begin{array}{c} 2n + 1 \\ n + 2k \end{array} \right] = (-q^2; q^2)_n.$$
Moreover, replacing $q$ by $q^{-1}$ in (2.4) and using the relation
\[
\begin{bmatrix} n \\ k \end{bmatrix}_{q^{-1}} = q^{k(n-k)} \begin{bmatrix} n \\ k \end{bmatrix}
\]
yields
\[
\sum_{k=\infty}^{n} (-1)^k q^{2k^2-2k} \begin{bmatrix} 2n+1 \\ n+2k \end{bmatrix} = (-q; q^n)_{n}. \]

3 Proofs (1.8)–(1.10)

Recall that the symmetric difference of two sets $A$ and $B$, denoted by $A \Delta B$, is the set of elements belonging to one but not both of $A$ and $B$ (cf. [4, p. 3]). In other words,
\[
A \Delta B := A \cup B \setminus (A \cap B) = (A \setminus B) \cup (B \setminus A).
\]
It is easy to see that $(A \Delta B) \Delta B = A$. Here we shall use the notation $A \Delta B$ to polish our description of certain involution.

Proof of (1.8). Let $S = \{a_1, \ldots, a_{2n}\} \ (n \geq 1)$, and let
\[
\mathcal{P} := \{A \subseteq S: \#A \equiv n \pmod{3}\}. \tag{3.1}
\]
For any $A \in \mathcal{P}$, we associate $A$ with a sign $\text{sgn}(A) = (-1)^{\#A-n}/3$. Then
\[
\sum_{k=-\lfloor n/3 \rfloor}^{\lfloor n/3 \rfloor} (-1)^k \binom{2n}{n+3k} = \sum_{A \in \mathcal{P}} \text{sgn}(A).
\]

We define a subset of $\mathcal{Q} \subseteq \mathcal{P}$ as follows:
\[
\mathcal{Q} := \{A \in \mathcal{P}: \#(A \cap \{a_1, \ldots, a_{2i+1}\}) \notin \{i-1, i+2\} \text{ for } i = 1, \ldots, n-1\}. \tag{3.2}
\]
We will show that the elements of $\mathcal{P} \setminus \mathcal{Q}$ cancel pairwise, i.e.,
\[
\sum_{A \in \mathcal{P} \setminus \mathcal{Q}} \text{sgn}(A) = 0. \tag{3.3}
\]
For any $A \in \mathcal{P} \setminus \mathcal{Q}$, there exist some numbers $i \leq n-1$ such that $\#(A \cap \{a_1, \ldots, a_{2i+1}\}) \in \{i-1, i+2\}$. Choose the smallest such $i$ and let
\[
A' = A \Delta \{a_1, \ldots, a_{2i+1}\}. \tag{3.4}
\]
Then $\#A' = \#A \pm 3$ and $A' \in \mathcal{P} \setminus \mathcal{Q}$. It is easy to see that $A \mapsto A'$ is a sign-reversing involution, and therefore (3.3) holds. It remains to evaluate the following summation
\[
\sum_{A \in \mathcal{Q}} \text{sgn}(A).
\]
For any $A \in \mathcal{A}$, we claim that
\[
\#(A \cap \{a_1, \ldots, a_{2i+1}\}) \in \{i, i+1\}, \text{ for all } i = 1, \ldots, n-1. \tag{3.5}
\]
Indeed, by definition, the statement (3.5) is obviously true for $i = 1$. Suppose it holds for $i-1$, i.e.,
\[
\#(A \cap \{a_1, \ldots, a_{2i-1}\}) \in \{i-1, i\},
\]
Then
\[
\#(A \cap \{a_1, \ldots, a_{2i+1}\}) \in \{i-1, i, i+1, i+2\}.
\]
By (3.2), we confirm our claim. In particular,
\[
\#(A \cap \{a_1, \ldots, a_{2n-1}\}) \in \{n-1, n\}. \tag{3.6}
\]
Thus by (3.1), we must have $\#A = n$ and so $\text{sgn}(A) = 1$. Note that we have 2 possible choices for $A \cap \{a_1\}$. By (3.5), we have 3 possible choices for each $A \cap \{a_{2i}, a_{2i+1}\}$, $i = 1, \ldots, n-1$. Finally, we only have one choice for $A \cap \{a_{2n}\}$ according to (3.6) and $\#A = n$. This proves that $\#\mathcal{A} = 2 \cdot 3^n$ and therefore completes the proof of (1.8). \hfill \Box

For $A \in S$, recall that its weight is defined by $||A|| = \sum_{a \in A} a$. In order to prove (1.9) and (1.10), we need to consider the following weighted sum
\[
\sum_{A \in \mathcal{P}} \text{sgn}(A)q^{|A|}
\]
on a particular $\mathcal{P}$. As one might have seen, the involution $A \mapsto A'$ in (3.4) is in general not weight-preserving. Nevertheless, a little modification will fix this problem. For any $A \in \mathcal{P} \setminus \mathcal{D}$, choose the same $i$ as in (3.4), and let $A''$ be constructed as follows:
- $a_1 \in A''$ if and only if $a_1 \notin A$;
- $a_{2j}, a_{2j+1} \in A''$ if $a_{2j}, a_{2j+1} \notin A$ ($j = 1, \ldots, i-1$);
- $a_{2j}, a_{2j+1} \notin A''$ if $a_{2j}, a_{2j+1} \in A$ ($j = 1, \ldots, i-1$);
- $a_{2j} \in A''$ and $a_{2j+1} \notin A''$ if $a_{2j} \in A$ and $a_{2j+1} \notin A$ ($j = 1, \ldots, i-1$);
- $a_{2j} \notin A''$ and $a_{2j+1} \in A''$ if $a_{2j} \notin A$ and $a_{2j+1} \in A$ ($j = 1, \ldots, i-1$);
- $a_k \in A''$ if and only if $a_k \in A$ ($2i+2 \leq k \leq 2n$).

It is clear that $\#A'' = \#A' = \#A \pm 3$. Furthermore, if we putting $a_1 = a_{2j} + a_{2j+1} = 0$ then $A \mapsto A''$ is a weight-preserving and sign-reversing involution. Now we can give proofs of (1.9) and (1.10) by selecting the set $\{a_1, \ldots, a_{2n}\}$ properly.

**Proof of (1.9).** Let $a_1 = 0, a_{2n} = n$ and $\{a_{2i}, a_{2i+1}\} = \{-i, i\}$ for $i = 1, \ldots, n-1$. Then the above involution $A \mapsto A''$ gives
\[
\sum_{A \in \mathcal{P} \setminus \mathcal{D}} \text{sgn}(A)q^{|A|} = 0.
\]
or
\[
\sum_{A \in \mathcal{P}} \text{sgn}(A)q^{||A||} = \sum_{A \in \mathcal{Q}} \text{sgn}(A)q^{||A||}.
\] (3.7)

By (2.2), the left-hand of (3.7) may be written as
\[
\sum_{k=-\lfloor n/3 \rfloor}^{\lfloor n/3 \rfloor} \sum_{\substack{A \subseteq S \atop \#A = n+3k}} \text{sgn}(A)q^{||A||} = \sum_{k=-\lfloor n/3 \rfloor}^{\lfloor n/3 \rfloor} (-1)^k \left[ \sum_{i=1}^{2n} \left( \frac{2n}{n+3k+1} \right)^{q} q^{-n-3k} \right] n.
\]

Let
\[\mathcal{Q}^*: = \{ A \subseteq \{a_1, \ldots, a_{2n-1}\} : \#(A \cap \{a_1, \ldots, a_{2i+1}\}) \neq \{i-1, i+2\}, i = 1, \ldots, n-1 \} .\]

Then (3.5) also holds for \( A \in \mathcal{Q}^* \). Moreover, for \( i = 1, \ldots, n-1 \), we have three choices for each \( A \cap \{a_{2i}, a_{2i+1}\} \), namely, \( \{a_{2i}\}, \{a_{2i+1}\}, \{a_{2i}, a_{2i+1}\} \) if \( \#(A \cap \{a_1, \ldots, a_{2i-1}\}) = i-1, \) and \( \emptyset, \{a_{2i}\}, \{a_{2i+1}\} \) if \( \#(A \cap \{a_1, \ldots, a_{2i-1}\}) = i \). Noticing that \( a_{2i} + a_{2i+1} = 0 \), we have
\[
\sum_{A \in \mathcal{Q}^*} q^{||A||} = \sum_{A \in \mathcal{Q}^*} q^{||A||} + \sum_{A \in \mathcal{Q}^*} q^{||A||} = 2 \prod_{i=1}^{n-1} (q^i + q^{-i} + q^0).
\]

It is not hard to image that there should exist a bijection from \( \{ A \in \mathcal{Q}^* : \#A = n-1 \} \) to \( \{ A \in \mathcal{Q}^* : \#A = n \} \) which preserves the weight. Indeed, our definition of the involution \( A \mapsto A'' \) on \( \mathcal{P} \setminus \mathcal{Q} \) can be simultaneously applied to \( \mathcal{Q}^* \), which yields the desired bijection. It follows that
\[
\sum_{A \in \mathcal{Q}^*} q^{||A||} = \prod_{i=1}^{n-1} (q^i + q^{-i} + q^0).
\]

Since
\[\mathcal{Q} = \{ A \in \mathcal{Q}^* : \#A = n \} \bigcup \{ A \cup \{a_{2n}\} : A \in \mathcal{Q}^*, \#A = n-1 \}\]
\[a_{2n} = n \text{ in this proof},\]
the right-hand of (3.7) equals
\[
\sum_{A \in \mathcal{Q}} q^{||A||} = \sum_{A \in \mathcal{Q}^*} q^{||A||} + q^n \sum_{A \in \mathcal{Q}^*} q^{||A||} = (1 + q^n) \prod_{i=1}^{n-1} (q^i + q^{-i} + q^0).
\]

The proof then follows after simplification. \( \square \)

**Proof of (1.10).** Suppose \( n \geq 3 \). Let \( a_1 = 0, a_{2n-2} = n-1, a_{2n-1} = n, a_{2n} = n+1 \) and \( \{a_{2i}, a_{2i+1}\} = \{-i, i\} \) for \( i = 1, \ldots, n-2 \). For any \( A \in \mathcal{P} \setminus \mathcal{Q} \), we claim that
\[
\#(A \cap \{a_1, \ldots, a_{2n-1}\}) \neq \{n-2, n+1\}.
\]
Otherwise, we have
\[ \# A \in \{ n - 2, n - 1, n + 1, n + 2 \}, \]
which is contrary to the definition (3.1). Therefore, the index \( i \) we choose for (3.4) is indeed less than \( n - 1 \). Since \( a_{2i} + a_{2i+1} = 0 \) (\( 1 \leq i \leq n - 2 \)) here, the previous involution \( A \mapsto A'' \) is still weight-preserving and sign-reversing, and thus (3.7) holds again. In this case, the left-hand of (3.7) equals
\[ \sum_{k=\lfloor n/3 \rfloor}^{\lfloor n/3 \rfloor} (-1)^k \binom{2n}{n+3k} q^{(n+3k+1)/2} q^{-(n+3k)(n-1)}. \]

To evaluate the right-hand side of (3.7), we introduce
\[ Q^\ast := \{ A \subseteq \{ a_1, \ldots, a_{2n-3} \} : \# (A \cap \{ a_1, \ldots, a_{2i+1} \}) \notin \{ i-1, i+2 \}, i = 1, \ldots, n-2 \}. \]
Then the same argument as \( Q^\ast \) implies that
\[ \sum_{A \in Q^\ast} q^{|A|} = \sum_{\# A = n-1} q^{|A|} \sum_{\# A = n-2} q^{|A|} = 2 \prod_{i=1}^{n-2} (q^i + q^{-i} + q^0). \quad (3.8) \]
Moreover, our definition for the involution \( A \mapsto A'' \) on \( \mathcal{P} \setminus \mathcal{Q} \) can also be applied to \( \mathcal{Q}^\ast \), and we have
\[ \sum_{A \in \mathcal{Q}^\ast} q^{|A|} = \sum_{\# A = n-1} q^{|A|} \sum_{\# A = n-2} q^{|A|} = \prod_{i=1}^{n-2} (q^i + q^{-i} + q^0). \quad (3.9) \]
It is easy to see that the right-hand of (3.7) equals
\[ \sum_{A \in \mathcal{Q}^\ast} q^{|A|} = \sum_{\# A = n-1} q^{|A|} (q^a_{2n-2} + q^a_{2n-1} + q^a_{2n}) \]
\[ + \sum_{\# A = n-2} q^{|A|} (q^{a_{2n-2}+a_{2n-1}} + q^{a_{2n-2}+a_{2n}} + q^{a_{2n-1}+a_{2n}}). \]
Substituting (3.8) and \( \{ a_{2n-2}, a_{2n-1}, a_{2n} \} = \{ n-1, n, n+1 \} \) into the above equation, we complete the proof of (1.10).

No doubt that we may define the involution \( A \mapsto A'' \) on the set \( \{ a_1, \ldots, a_{2n+1} \} \). Let \( \{ a_1, \ldots, a_{2n} \} \) be as in the proof of (1.9). Then putting \( a_{2n+1} = -n \) we obtain
\[ \sum_{k=-\infty}^{\infty} (-1)^k q^{(9k^2-3k)/2} \binom{2n+1}{n+3k} = \frac{(q^3; q^3)_n}{(q; q)_n}, \quad (3.10) \]
while putting \(a_{2n+1} = n+1\) we get

\[
\sum_{k=-\infty}^{\infty} (-1)^k q^{(9k^2+3k)/2} \left[ \frac{2n+1}{n+3k} \right] = \frac{(q^3; q^3)_{n-1}}{(q; q)_{n-1}} (1 + q^n + q^{n+1}) \quad (n \geq 1).
\]

Both (3.10) and (3.11) are \(q\)-analogues of (1.12). Finally, we point out that the following two identities:

\[
\begin{align*}
\sum_{k=-\infty}^{\infty} (-1)^k q^{(9k^2+9k)/2} & \left[ \frac{2n}{n+3k+1} \right] = \frac{(q^3; q^3)_{n-1}}{(q; q)_{n-1}} q^{n-1} (n > 0), \\
\sum_{k=-\infty}^{\infty} (-1)^k q^{(9k^2+3k)/2} & \left[ \frac{2n+1}{n+3k+1} \right] = \frac{(q^3; q^3)_{n}}{(q; q)_{n}} 
\end{align*}
\]

appearing in [5] can also be proved in the same way.

### 4 Proofs of (1.11)

**First Proof.** By the binomial theorem, we have

\[
\left( \sqrt{3} + i \right)^n = \sum_{k=0}^{\infty} \binom{n}{k} 3^{k/2} i^{n-k} \\
= i^n \sum_{k=0}^{\infty} \binom{n}{2k} (-3)^k + i^{n-1} \sqrt{3} \sum_{k=0}^{\infty} \binom{n}{2k+1} (-3)^k. \quad (4.1)
\]

On the other hand, there holds

\[
\left( \sqrt{3} + i \right)^n = 2^n \left( \cos \frac{\pi}{6} + i \sin \frac{\pi}{6} \right)^n = 2^n \left( \cos \frac{\pi n}{6} + i \sin \frac{n\pi}{6} \right). \quad (4.2)
\]

Comparing (4.1) and (4.2), we immediately get (1.11) and its companion

\[
\sum_{k=0}^{\infty} \binom{n}{2k+1} (-3)^k = \begin{cases} 
0, & \text{if } n \equiv 0 \pmod{3}, \\
(-2)^{n-1}, & \text{if } n \equiv 1 \pmod{3}, \\
(-1)^{n-2}, & \text{if } n \equiv 2 \pmod{3}.
\end{cases} \quad (4.3)
\]

**Second Proof.** Let \(\Gamma = \{a, b, c, d, e\}\) denote an alphabet. For a word \(w = w_1 \cdots w_n \in \Gamma^*\), its length \(n\) is denoted by \(|w|\). For any \(x \in \Gamma\), let \(|w|_x\) be the number of \(x\)'s appearing in the word \(w\). Let \(W_n\) denote the set of words \(w = w_1 \cdots w_n \in \Gamma^*\) satisfying the following conditions:

(i) \(|w|_a + |w|_b + |w|_c = |w|_d.\)
(ii) If we remove all $e$’s from $w$, then each $d$ is in the even position.

It is easy to see that there are \( \binom{n}{2k} 3^k \) words $w \in W_n$ such that $|w|_d = k$, and so

\[
\sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{2k} (-3)^k = \sum_{w \in W_n} (-1)^{|w|_d}.
\]

We call $(-1)^{|w|_d}$ the sign of the word $w$. In what follows, we shall construct an involution on $W_n$ which is sign-reversing for all non-fixed points.

For any word $w = w_1 \cdots w_n \in W_n$, let $u_i = w_{3i-2}w_{3i-1}w_{3i}$, $i = 1, \ldots, \lfloor n/3 \rfloor$. According to the conditions (i) and (ii), the subwords $u_i$ have at most 43 cases. Let us classify them into three types as follows:

$X$: ade, bde, cde, aed, bed, ced, ead, ebd;

$Y$: eee, aee, bee, cee, eae, ebe, ece, eea, eeb, eec, dee, ede, eed, ecd, ada, adb, adc, bda, bdb, bdc, cda, cdb, cdc, dad, dbd, dcd;

$Z$: eda, edb, edc, dea, deb, dec.

We claim that all the words in $W_n$ with a $u_i$ of type $Y$ cancel pairwise. Indeed, for such a word $w$, choose the smallest number $i$ such that $u_i$ is of type $Y$. Then we obtain a word $w'$ by replacing $u_i$ by $u_i'$, where $u_i \leftrightarrow u_i'$ is determined by the following table:

\[
\begin{array}{cccc}
\text{eee} & \leftrightarrow & \text{ecd} & \\
\text{ebe} & \leftrightarrow & \text{adb} & \\
\text{ede} & \leftrightarrow & \text{dbc} & \\
\text{ede} & \leftrightarrow & \text{cdc} & \\
\end{array}
\]

It is clear that $w' \in W_n$, $|w'|_d = |w|_d \pm 1$, and hence $w \mapsto w'$ is a sign-reversing involution.

On the other hand, for any word $w \in W_n$, we claim that if no $u_i$ in $w$ is of type $Y$, then no $u_i$ in $w$ is of type $Z$. In fact, by the definition of $w$, $u_1$ must be of type $X$ or $Y$. By the condition (ii), none of $dd$, $ded$, $deed$ can appear in $w$ and therefore no $u_i$ of type $X$ in $w$ can be followed by a $u_j$ of type $Z$. This proves the claim. It follows that the remained words in $W_n$ are just those all $u_i$ are of type $X$, and vice versa. Namely,

\[
\sum_{w \in W_n} (-1)^{|w|_d} = \sum_{w \in W_n \text{ all } u_i \text{ is of type } X} (-1)^{|w|_d}. \quad (4.4)
\]

Consider the right-hand side of (4.4) ($\text{RHS}(4.4)$ for short). Note that each $u_i$ has 8 possible choices. We have the following three cases:

- If $n \equiv 0 \pmod{3}$, then $|w|_d = n/3$ and $\text{RHS}(4.4) = (-8)^{n/3}$.
- If $n \equiv 1 \pmod{3}$, then $w$ must be ended by a letter $e$, $|w|_d = (n - 1)/3$, and $\text{RHS}(4.4) = (-8)^{(n-1)/3}$. 

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• If \( n \equiv 2 \pmod{3} \), then \( w \) may be ended by \( ee \), \( ad \), \( bd \), or \( cd \), and

\[
RHS(4.4) = (-8)^{(n-2)/3} + 3(-1)^{(n+1)/3}g^{(n-2)/3} = (-2)^{n-1}.
\]

This completes the proof.

The combinatorial proof of (4.3) is exactly analogous. We need only to replace the condition (i) by \( |w|_a + |w|_b + |w|_c = |w|_d - 1 \), and change “even” to “odd” in the condition (ii).

It is difficult to find \( q \)-analogues of (1.11) and (4.3). However, the mathematics software MAPLE hints us to propose the following two interesting conjectures.

**Conjecture 4.1** Let \( l, m, n \geq 0 \) and \( \epsilon \in \{0, 1\} \). Then

\[
\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k q^{2k^2 + 2mk} \left( \frac{n}{2k + \epsilon} \right)(1 + q + q^2)^k
\]

is divisible by \((1 + q)^{(n+2)/4}(1 + q^2)^{(n+4)/8}\).

**Conjecture 4.2** Let \( m, n \geq 0 \) and \( \epsilon \in \{0, 1\} \). Then

\[
\sum_{k=0}^{\lfloor n/2 \rfloor} (-1)^k q^{2k^2 + 2mk} \left( \frac{n}{2k + \epsilon} \right)(1 + q + q^2)^k
\]

is divisible by \((1 + q)^{\lfloor n/2 \rfloor}(1 + q^2)^{\lfloor n/4 \rfloor}\).

**References**


