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**Topological conformal field theory with a rational  $W$  potential  
and the dispersionless KP hierarchy**

S. Aoyama<sup>1</sup>

Instituut voor Theoretische Fysica  
Katholieke Universiteit Leuven  
Celestijnenlaan 200D  
B-3001 Leuven, Belgium

Y. Kodama<sup>2</sup>

Department of Mathematics  
Ohio State University  
Columbus, Ohio 43210, USA

**Abstract**

We present a new class of topological conformal field theories (TCFT) characterized by a rational  $W$  potential, which includes the minimal models of A and D types as its subclasses. An explicit form of the  $W$  potential is found by solving the underlying dispersionless KP hierarchy in a particular small phase space. We discuss also the dispersionless KP hierarchy in large phase spaces by reformulating the hierarchy, and show that the  $W$  potential takes a universal form, which does not depend on a specific form of the solution in a large space.

<sup>1</sup> e-mail : shogo%tf%fys@cc3.kuleuven.ac.be

<sup>2</sup> e-mail : kodama@math.ohio-state.edu

1. Since Witten's foundation topological field theories have undergone intensive investigations. Among them TCFT's <sup>[1]</sup> are most characteristic. Namely it was shown in ref. 2 that genus-zero 3-point functions  $c_{ijk} = \langle \phi_i \phi_j \phi_k \rangle$  of BRST invariant primaries  $\phi_i$  are such that

$$c_{0ij} = \text{const.}, \quad (\text{flatness of metric}), \quad (1a)$$

$$\partial_m c_{ijk} = \partial_i c_{mjk}, \quad (\text{integrability}), \quad (1b)$$

$$c_{ij}{}^m c_{mkl} = c_{jk}{}^m c_{mli}, \quad (\text{associativity}). \quad (1c)$$

Through the Landau-Ginzburg formulation they solved these equations by the A-D-E- minimal models and elucidated the intriguing connections of TCFT's with matrix models, dispersionless KP hierarchy, singularity theory, etc.. This pioneering work stimulated a number of people to study eqs. (1a)~(1c) and to find other types of TCFT<sup>[3,4]</sup>. As had already been shown in ref 5, TCFT's thus obtained may be coupled with  $2-d$  gravity. The studies in refs. 6, 7 and 8 showed that the Landau-Ginzburg formulation is also a suitable framework for this generalization.

In the Landau-Ginzburg picture TCFT's are completely determined by the  $W$  potential. All the solutions to eqs. (1a)~(1c) so far found in the literature<sup>[3]</sup> give the  $W$  potential in a polynomial form of several variables. However the potential of the D-model is exceptional, as expressed in a non-polynomial form by eliminating one of the variables <sup>[2]</sup>. There barely appears a non-polynomial piece as deforming the  $W$  potential of the A model<sup>[6,7,8]</sup>. A rational  $W$  potential of a more general form has been proposed in the recent work 9 for a characterization of a certain class of multi-matrix models. However, they did not discuss a connection to TCFT. A non-polynomial  $W$  potential was also studied for a Landau-Ginzburg description of the  $c = 1$  string in ref. 10. But it is not rational either and does not describe a TCFT.

In this note we show that the  $W$  potential in ref. 9 gives new types of TCFT satisfying eqs. (1a)~(1c). The novelty of these TCFT's is the presence of a finite number of positive and negative primaries. They contain the A-D- models as subclasses of this model. An explicit form of the  $W$  potential is found by properly solving the underlying dispersionless KP equation in a particular phase space. We here give a practical formula to evaluate genus-zero 3-point functions  $c_{ijk}$  in terms

of the  $W$  potential, and show that the model satisfies eqs. (1a)~(1c). Both positive and negative flows also appear in the dispersionless Toda hierarchy<sup>[11]</sup>, and Takasaki's extension<sup>[12]</sup> of the dispersionless BKP hierarchy, where they discussed the general structure of these hierarchies. Our model is however considered as a reduction of the dispersionless KP hierarchy extended by adding negative flows, rather than as the dispersionless BKP hierarchy<sup>[12]</sup>. A further extension of the dispersionless KP hierarchy including several types of flows has been discussed in ref. 4. A tau-function (free energy) is found, and is used to construct correlation functions. But it remained to be rather a formal expression.

To discuss a topological Landau-Ginzburg theory coupled with  $2-d$  gravity, it is indispensable to extend the analysis in the small phase space to that in larger ones. So far little has been understood in this case even for the A model. Following ref<sup>[13]</sup> we reformulate the dispersionless KP theory in the framework of quasi-linear system of partial differential equations. The reformulation turns out to be optimal to discuss on the issue. We are able to show a universal form of the  $W$  potential in the entire phase space.

2. We consider the dispersionless KP hierarchy

$$\frac{\partial W}{\partial t_i} = \{Q_i, W\} = \frac{\partial Q_i}{\partial p} \frac{\partial W}{\partial t_0} - \frac{\partial Q_i}{\partial t_0} \frac{\partial W}{\partial p}, \quad \text{for } -\infty < i < \infty, \quad (2)$$

with the  $W$  potential in a rational form[9]

$$W = \frac{1}{n+1} p^{n+1} + v_{n-1} p^{n-1} + \dots + v_0 + \frac{w_1}{p-s} + \dots + \frac{w_{m-1}}{(m-1)(p-s)^{m-1}} + \frac{w_m}{m(p-s)^m}. \quad (3)$$

where  $v_i$  and  $w_i$  are the functions of  $t_j$ ,  $-\infty < j < \infty$ . The Hamiltonian functions  $Q_i$  for  $t_i$  flows are defined by

$$\begin{aligned} Q_i &= \left[ \frac{1}{i+1} \lambda^{i+1} \right]_+, & \text{for } i \geq 0, \\ Q_{-1} &= \log(p-s), \\ Q_{-i} &= \left[ -\frac{1}{i-1} \mu^{i-1} \right]_-, & \text{for } i \geq 2, \end{aligned} \quad (4)$$

where  $\lambda$  and  $\mu$  are the semi-classical limits of the Lax operators given by

$$\begin{aligned}\lambda &= [(n+1)W]^{\frac{1}{n+1}} = p + O\left(\frac{1}{p}\right), \quad \text{for large } p, \\ \mu &= [mW]^{\frac{1}{m}} = \frac{\sqrt[m]{w_m}}{p-s} + O(1). \quad \text{for small } p-s.\end{aligned}\tag{5}$$

Here  $[..]_+$  and  $[..]_-$  indicate the parts of non-negative and negative powers in  $p$ , respectively. Eqs. (5) imply  $W = \frac{\lambda^{m+1}}{m+1} = \frac{\mu^m}{m}$  which may be considered as a Riemann-Hilbert problem on a genus zero surface (sphere) defined by the potential (3) [12]. The compatibility among the flows in (2), i.e.  $\frac{\partial}{\partial t_i} \frac{\partial}{\partial t_j} W = \frac{\partial}{\partial t_j} \frac{\partial}{\partial t_i} W$ , can be shown by the zero curvature conditions,

$$\frac{\partial Q_i}{\partial t_j} - \frac{\partial Q_j}{\partial t_i} + \{Q_i, Q_j\} = 0, \quad \text{for } -\infty < i, j < \infty,$$

which can be directly derived from (4). In ref. 9 the dispersionless KP hierarchy of this kind was discussed as characterizing a certain class of matrix models. But their hierarchy did not contain the negative  $t_{-i}$  flows with  $i \geq 2$ . Eq. (2) can not then give a  $W$  potential of TCFT except for the case  $m = 1$ , as it will be clear in this note.

We now define the fields  $\phi_i$  as Laurent polynomials of  $p$ ,

$$\phi_i = \frac{dQ_i}{dp}, \quad \text{for } -\infty < i < \infty.\tag{6}$$

For an arbitrary integer  $a$ , a set of  $(n+m+1)$  fields

$$\phi_a, \phi_{a+1}, \dots, \phi_{a+n+m-1}, \phi_{a+n+m},$$

form a basis of a Laurent polynomial ring with the ideal  $\frac{dW}{dp} (= W') = 0$ . When  $a = -(m+1)$  they are primaries, satisfying the flatness condition as shown below. The fusion algebra is found by calculating as

$$\phi_i \phi_j = c_{ij}^l \phi_l + W' Q_{ij}, \quad -(m+1) \leq i, j \leq n-1.\tag{7}$$

We propose to write 3-point functions as

$$\langle \phi_i \phi_j \phi_k \rangle = - \oint_C \left[ \frac{\phi_i \phi_j \phi_k}{W'} \right], \quad -\infty < i, j, k < \infty,\tag{8}$$

in which  $C$  is a contour surrounding roots of  $W' = 0$ . Note here that if  $W$  is a polynomial in  $p$ , i.e. the  $A$  model, the contour integral can be evaluated as the residue at  $p = \infty$ . Note also that  $W'$  is nilpotent in the numerator of the integrand, and the 3-point functions are faithful to the fusion algebra. Consequently they satisfy the associativity given by eq. (1c). Let  $\eta_{ij}$  be the metric defined as

$$\langle \phi_i \phi_j \phi_0 \rangle = \eta_{ij}, \quad -(m+1) \leq i, j \leq n-1. \quad (9)$$

Then from eqs. (7) and (8) it follows that

$$\langle \phi_i \phi_j \phi_k \rangle = c_{ij}^l \eta_{lk}, \quad -(m+1) \leq i, j, k \leq n-1. \quad (10)$$

We denote all other  $\phi_i$ 's with  $i \leq -(m+2)$  or  $n \leq i$  by

$$\begin{aligned} \sigma_N(\phi_i) &= c_{N,i} \phi_{N(n+1)+i}, \quad \text{for } 0 \leq i \leq n-1, \\ \sigma_N(\phi_{-i}) &= d_{N,i} \phi_{-(Nm+i)}, \quad \text{for } 2 \leq i \leq m+1, \end{aligned} \quad (11)$$

where

$$\begin{aligned} c_{N,i} &= [(i+1)(i+1+n+1)\cdots(i+1+(N-1)(n+1))]^{-1}, \\ d_{N,i} &= [(i+1)(i+1+m)\cdots(i+1+(N-1)m)]^{-1}, \end{aligned}$$

and  $N \geq 1$ . They are identified as the descendants for the primaries<sup>[6,7]</sup>. But note that the primary  $\phi_{-1}(= \sigma_0(\phi_{-1}))$  has no descendant. This is typical in the Toda hierarchy. Note also that  $\sigma_N(\phi_n) = \sigma_N(\phi_{-(m+1)})$  for all  $N$ 's by  $W' = 0$ .

3. We now study the dispersionless KP hierarchy (2) in a small phase space by restricting the flow parameters  $t_i$  only to those with  $-(m+1) \leq i \leq n-1$ . Let us assume that the potential  $W$  satisfies

$$\frac{\partial W}{\partial t_0} = 1. \quad (12)$$

which guarantees the flatness of the metric as shown below. Then eq. (2) becomes

$$\frac{\partial W}{\partial t_i} = \phi_i, \quad (13)$$

for  $-(m+1) \leq i \leq n-1$ , from which one can easily see separation of the variables,

$$s = t_{-(m+1)}, \quad (14a)$$

$$\frac{\partial}{\partial t_{-i}} v_j = 0, \quad \text{for } 1 \leq i \leq m+1, \quad 0 \leq j \leq n-1, \quad (14b)$$

$$\frac{\partial}{\partial t_i} w_j = 0, \quad \text{for } 0 \leq i \leq n-1, \quad 1 \leq j \leq m, \quad (14c)$$

Note that eq. (14a) is the unique solution for  $s$ . Furthermore from eq. (13) together with (6) and (7) it follows that

$$\frac{\partial \phi_j}{\partial t_i} = Q'_{ij}, \quad -(m+1) \leq i, j \leq n-1. \quad (15)$$

For  $ij \leq 0$ , this equation is evident since both sides vanish due to eqs. (14a)~(14c). If  $i \geq 0$  and  $j \geq 0$  it can be shown by noting that

$$Q_{ij} = \left[ \frac{\phi_i \phi_j}{W'} \right]_+ = [\phi_i \lambda^{j-n}]_+.$$

If  $i < 0$  and  $j < 0$  the proof goes in the same way.

In refs. 8 and 14 eq. (15) is referred as the flatness condition of the metric, and leads to the flatness of the metric defined by the period integral. We shall show that it is also a sufficient condition for the flatness of the metric defined by (9) with the residue formula (8). There is a gap between these two metrics. The flatness of our model may be shown by studying the relation between the residue formula and the period integral. Following ref. 8 we assume 1-point functions of the fields  $\phi_i$  by the period integral: for the primaries

$$\begin{aligned} \langle \phi_i \rangle &= \frac{1}{(i+n+2)(i+1)} \oint_{p=\infty} dp \lambda^{i+n+2}, & 0 \leq i \leq n-1, \\ \langle \phi_{-i} \rangle &= -\frac{1}{(i+m-1)(i-1)} \oint_{p=s} dp \mu^{i+m-1}, & 2 \leq i \leq m+1, \end{aligned} \quad (16)$$

and for all other fields

$$\begin{aligned} \sigma_N(\phi_i) &= c_{N+2,i} \oint_{p=\infty} dp W^{N+1+\frac{i+1}{n+1}}, & 0 \leq i \leq n-1, \\ \sigma_N(\phi_{-i}) &= d_{N+2,i} \oint_{p=s} dp W^{N+1+\frac{i-1}{m}}, & 2 \leq i \leq m+1, \end{aligned} \quad (17)$$

with  $c_{N+1,i}$  and  $d_{N+1,i}$  given in eq. (11). The period integral fails to define a 1-point function of the primary  $\phi_{-1}$ , since it diverges by a naive extension of the formula (16) to this case. An explicit form of  $\langle \phi_{-1} \rangle$  will be later given.

Let  $\langle \psi \rangle$  be one of these 1-point functions. Then we can show the following relation with the 3-point function defined by eq. (8):

$$\langle \phi_i \phi_j \psi \rangle = \frac{\partial}{\partial t_i} \frac{\partial}{\partial t_j} \langle \psi \rangle. \quad (18)$$

A similar formula appeared for the  $A$  model. But having the different contours on both sides eq. (18) is not obvious for the generalized model. The easiest way<sup>[8,14]</sup> to show this may be to differentiate  $\psi$  twice by the flow parameters and find the Gauss-Manin system

$$\frac{\partial}{\partial t_i} \frac{\partial}{\partial t_j} \langle \psi \rangle = c_{ij}^l \frac{\partial}{\partial t_l} \frac{\partial}{\partial t_0} \langle \psi \rangle, \quad \text{for } -(m+1) \leq i, j, \leq n-1, \quad (19)$$

by eq. (15). The same equation can be derived by using eq. (7) in the l.h.s. of eq. (18). This proves eq. (18) indirectly. We here give a direct proof. Consider the case when  $\psi = \phi_k$ ,  $0 \leq k < \infty$ . By eqs. (13) and (16) the r.h.s. of eq. (18) may be calculated as

$$\begin{aligned} \frac{\partial}{\partial t_i} \frac{\partial}{\partial t_j} \langle \phi_k \rangle &= \oint_{p=\infty} dp \frac{\phi_i \phi_j \phi_k}{W'} \\ &+ \oint_{p=\infty} dp \frac{\phi_i \phi_j \frac{d}{dp} [\frac{1}{k+1} \lambda^{k+1}]_-}{W'} + \oint_{p=\infty} dp [\frac{1}{k+1} \lambda^{k+1}] Q'_{ij}. \end{aligned} \quad (20)$$

If  $-(m+1) \leq i, j \leq -1$ , the second piece in the r.h.s is vanishing, while the remaining pieces add up to give

$$\frac{\partial}{\partial t_i} \frac{\partial}{\partial t_j} \langle \phi_k \rangle = c_{ij}^l \oint_{p=\infty} dp \frac{\phi_l \phi_k}{W'},$$

by eq. (7). Since this integral has no residue at  $p = s$ , we can analytically deform the contour around  $p = \infty$  to the one surrounding the roots of  $W' = 0$ . This leads to (18),

$$\frac{\partial}{\partial t_i} \frac{\partial}{\partial t_j} \langle \phi_k \rangle = -c_{ij}^l \oint_C dp \frac{\phi_l \phi_k}{W'} = - \oint_C dp \frac{\phi_i \phi_j \phi_k}{W'}.$$

If  $i$  or  $j$  takes other values, the second and third integrals in eq. (20) are either cancelled with each other by eq. (7) or trivially vanishing. The first integral has

no residue at  $p = s$ , so that it becomes  $\langle \phi_i \phi_j \phi_k \rangle$  by the analytical deformation of the contour. For the case when  $\psi = \phi_{-k}$ ,  $0 < k < \infty$ , the same proof goes through.

The meaning of eq. (18) is twofold. On one hand it provides us with a practical way to evaluate the residue of eq. (8). Without this, eq. (8) would be a formal definition. On the other hand eq. (15) guarantees the integrability

$$\langle \phi_i \phi_j \rangle = \frac{\partial}{\partial t_i} \langle \phi_j \rangle = \frac{\partial}{\partial t_j} \langle \phi_i \rangle, \quad (21)$$

for  $-(m+1) \leq i, j \leq -2$ ,  $0 \leq i, j \leq n-1$ . It is not evident a priori, for instance, in the case when  $ij < 0$ . Thus we obtain the unique higher point functions by differentiating the lowest ones given by eqs. (16) and (17). We were not able to define a 1-point function of  $\phi_{-1}$  by the period integral. But it can be found by solving eq. (18) with  $\psi = \phi_{-1}$ , or equivalently eq. (21) with  $i$  or  $j = -1$ :

$$\langle \phi_{-1} \rangle = \int_0^s dp [W]_+ - \frac{1}{m} \oint_{p=s} dp W \log[(p-s)^m W] + f(t_{-1}). \quad (22)$$

Here  $(p-s)^m$  works as a regulator without which the second integral diverges. The function  $f(t_{-1})$  is to be fixed by

$$\langle \phi_{-1} \phi_{-1} \phi_{-1} \rangle = \partial_{-1}^2 \langle \phi_{-1} \rangle.$$

Equipped with eq. (22) the formula (18) is now valid for  $\psi = \phi_{-1}$  as well. As the result we find the property of 3-point functions  $c_{ijk}$  given by eq. (1b). The flatness of the metric (9) can be also shown by means of this formula. For  $0 \leq j \leq n-1$  we calculate as

$$\begin{aligned} \frac{\partial \eta_{ij}}{\partial t_k} &= \frac{\partial}{\partial t_k} \frac{\partial}{\partial t_i} \frac{\partial}{\partial t_0} \langle \phi_j \rangle \\ &= (j-n) c_{ki}^l \oint_{p=\infty} dp \lambda^{j-2n-1} \phi_l = 0. \end{aligned}$$

For  $-(m+1) \leq j \leq 0$  the same thing can be shown.

By eqs. (8), (17) and (18) we can show the recursion relation for the descendants  $\sigma_N(\phi_i)$  with  $i \neq -1$

$$\langle \sigma_N(\phi_i) \phi_j \phi_k \rangle = \langle \sigma_{N-1}(\phi_i) \phi_l \rangle \langle \phi^l \phi_j \phi_k \rangle,$$



which is characteristic for TCFT's coupled with  $2-d$  gravity<sup>[5,7]</sup>. For a derivation it suffices to note the formulae

$$\begin{aligned} \langle \sigma_N(\phi_i)\phi_0 \rangle &= \langle \sigma_{N-1}(\phi_i) \rangle, \\ \sigma_N(\phi_i) &= \langle \sigma_N(\phi_i)\phi_0\phi^l \rangle \phi_l + W'Q, \end{aligned}$$

with a suitable Laurent polynomial  $Q$ .

4. Let us now solve eq. (13) to find an explicit form of the  $W$  potential. By eqs. (4) and (6) it turns into the equations for  $w_i$  and  $v_i$

$$\begin{aligned} \frac{\partial}{\partial t_{-k}} \frac{\partial}{\partial t_{-l}} w_i &= (i-1) \frac{\partial}{\partial t_{-(k+l-m-1)}} w_{i-1}, & 1 \leq i \leq m, \\ \frac{\partial}{\partial t_k} \frac{\partial}{\partial t_l} v_i &= (i+1) \frac{\partial}{\partial t_{k+l-n}} v_{i+1}, & 0 \leq i \leq n-1, \end{aligned}$$

respectively. By solving them recursively we obtain

$$\begin{aligned} w_i &= \sum_{l_1+\dots+l_i=(i-1)m+i} t_{-l_1} t_{-l_2} \cdots t_{-l_i}, \\ v_i &= t_i + \sum_{j=2}^{n-i-1} \frac{(i+j-1)!}{j(i!)} \sum_{\substack{l_1+\dots+l_j= \\ (j-1)n+i+j-1}} t_{l_1} \cdots t_{l_j}, & (23) \\ s &= t_{-(m+1)}. \end{aligned}$$

The  $W$  potential with these solutions gives a TCFT, for which 3-point functions are calculated by the residue formula (8) or eq. (18). As has been remarked, the primary  $\phi_{-1}$  has no descendant in this TCFT. Its 1-point function was not given by the period integral, but eq. (22) which contains logarithmic pieces  $\propto \log t_{-m}$ . These odd phenomena regarding  $\phi_{-1}$  are the consequences of the presence of the  $t_{-1}$  flow which is characteristic for the Toda hierarchy and the requirement of the fusion algebra (10), or equivalently the associativity (1c). If we look at subrings, the  $W$  potential with eqs. (23) gives a subclass of TCFT's, which does not have  $\phi_{-1}$ . The simplest one is the A model which has only positive primaries  $\phi_i$ ,  $0 \leq i \leq n-1$  [2]. It is given by the  $W$  potential (3) in which all the negative

flow parameters are switched off,  $w_i = 0$  for all  $i$ . Or we may consider the case where  $n$  and  $m$  are multiples of an integer  $M(\geq 2)$ , i.e.,  $n = Ma$  and  $m = Mb$ . The primaries

$$\phi_{-Mb}, \phi_{-M(b-1)}, \dots, \phi_{M(a-2)}, \phi_{M(a-1)}, \quad (24)$$

form a subring. The  $W$  potential of the TCFT with these primaries is given by eq. (23), where  $t_i = 0$  with  $i$  non-multiple of  $M$  (e.g.  $s = 0$ ). The primary  $\phi_{-1}$  is moded out, and hence the odd phenomena due to  $\phi_{-1}$  disappear. As a result, all the primaries (24) have the descendants defined according to eq. (11). For both fields the 1-point functions are given by the period integral of the  $W$  potential. They are evaluated to be polynomials of the flow parameters  $t_i$  corresponding to the primaries (24). The D model is the special case of this subclass of TCFT with  $M = 2$  and  $b = 1$  [2]. Thus the TCFT given by the  $W$  potential with eq. (23) is a natural generalization of the A-D- models.

5. So far eq. (2) was discussed in the small phase space. We may be naturally interested in solutions in larger spaces. The previous arguments can be generalized in a universal way. To do this we reformulate eq. (2) following the works in ref. 13. First of all we invert eqs. (5) in terms of the local coordinates  $\lambda$  and  $\mu$ :

$$p = \lambda - \frac{u_{n-1}}{\lambda} - \frac{u_{n-2}}{\lambda^2} - \dots - \frac{u_0}{\lambda^n} - O\left(\frac{1}{\lambda^{n+1}}\right),$$

$$p = s + \frac{u_{-m}}{\mu} + \frac{u_{-(m-1)}}{\mu^2} + \dots + \frac{u_0}{\mu^{m+1}} + O\left(\frac{1}{\mu^{m+2}}\right).$$

Note here that the coefficients in the higher orders are expressed in terms of  $u_i$ 's. For convenience, we write  $u_i$  and  $\phi_i$ ,  $-(m+1) \leq i \leq n-1$ , as column vectors

$$U = [u_{n-1}, u_{n-2}, \dots, u_{-m}, u_{-(m+1)}]^T,$$

$$\Phi = [\phi_{n-1}, \phi_{n-2}, \dots, \phi_{-m}, \phi_{-(m+1)}]^T,$$

with  $s = u_{-(m+1)}$ . There are recursion relations between these quantities which can be expressed as

$$\Phi^T(A - p \mathbb{I}) = [1, 0, \dots, 0, 0](\phi_{-(m+1)} - \phi_n),$$

where  $A$  is an  $(n + m + 1) \times (n + m + 1)$  matrix given by

$$\begin{pmatrix} 0 & 1 & \dots\dots & 0 & 0 & 0 & 0 & \dots\dots & 0 & 0 \\ -u_{n-1} & 0 & & & 0 & 0 & & & & 0 \\ \vdots & \vdots & & & \vdots & \vdots & & & & \vdots \\ \vdots & \vdots & & & \vdots & \vdots & & & & \vdots \\ -u_2 & & \dots\dots & 0 & 1 & & & & & \\ -u_1 & -u_2 & \dots\dots & -u_{n-1} & 0 & 1 & 0 & \dots\dots & 0 & 0 \\ 0 & 0 & \dots\dots & 0 & 0 & s & u_{-m} & \dots\dots & u_{-2} & u_{-1} \\ 0 & & & & 0 & 0 & s & \dots\dots & & u_{-2} \\ \vdots & & & & \vdots & \vdots & & & \vdots & \vdots \\ \vdots & & & & \vdots & \vdots & & & \vdots & \vdots \\ 0 & & & & 0 & 0 & & & s & u_{-m} \\ 1 & 0 & \dots\dots & 0 & 0 & 0 & 0 & \dots\dots & 0 & s \end{pmatrix}.$$

These recursion relations lead to a closed form for  $\phi_i$ ,  $-(m + 1) \leq i \leq n - 1$ :

$$\phi_{-i} = \frac{1}{(p - s)^i} \det \begin{vmatrix} u_{-m} & u_{-(m-1)} & \dots\dots & u_{-(m-i+3)} & u_{-(m-i+2)} \\ -(p - s) & u_{-m} & \dots\dots & u_{-(m-i+4)} & u_{-(m-i+3)} \\ \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots\dots & u_{-m} & u_{-(m-1)} \\ 0 & 0 & \dots\dots & -(p - s) & u_{-m} \end{vmatrix},$$

for  $2 \leq i \leq (m + 1)$ ,

$$\phi_i = \det \begin{vmatrix} p & -1 & \dots\dots & 0 & 0 \\ u_{n-1} & p & \dots\dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ \vdots & \vdots & & \vdots & \vdots \\ u_{n-i+2} & u_{n-i+3} & \dots\dots & p & -1 \\ u_{n-i+1} & u_{n-i+2} & \dots\dots & u_{n-1} & p \end{vmatrix},$$

for  $2 \leq i \leq n$ , and

$$\phi_{-1} = \frac{1}{p-s}, \quad \phi_0 = 1, \quad \phi_1 = p.$$

By noting

$$W' = \phi_n - \phi_{-(m+1)},$$

it can be shown that eq. (2) is equivalent to a set of the following equations

$$\frac{\partial U}{\partial t_i} = \phi_i(A) \frac{\partial U}{\partial t_0}, \quad -\infty < i < \infty, \quad (25)$$

$$\frac{\partial W}{\partial u_i} = \phi_i, \quad -(m+1) \leq i \leq n-1. \quad (26)$$

Here the quantities  $\phi_i(A)$  are  $(n+m+1) \times (n+m+1)$  matrices given by (6) with  $p$  substituted by  $A$ . The solution of (26) has been already found, i.e., eq. (23) in which  $t_i$  are replaced by  $u_i$ . With this we obtain a universal form of the  $W$  potential in terms of  $u_i$ . Thus the dispersionless KP hierarchy (2) is reduced to the quasi-linear system (25). Note that the variables  $u_i$ 's are the conserved densities for the system (25), and they can be expressed by the period integrals as well as the free energy as a consequence of the zero curvature conditions of  $Q_i$ 's, or equivalently the compatibility conditions of the flows in (2)<sup>[12,13]</sup>.

In order to construct some of the exact solutions of (25), we first note the following: Due to the Cayley-Hamilton theorem, any field  $\phi_M(A)$ ,  $M \leq -(m+2)$ ,  $n+1 \leq M$ , can be decomposed into the primaries,

$$\phi_M(A) = \sum_{-(m+1) \leq i \leq n-1} \Delta_i(U) \phi_i(A), \quad (27)$$

with appropriate coefficients  $\Delta_i(U)$ . This is equivalent to saying that  $\phi_i$ 's give a basis of the finite ring of Laurent polynomials by  $W' = 0$ . For instance, we have for  $M = n+1$

$$\phi_{n+1}(A) = \sum_{-(m+1) \leq i \leq n-1} u_i \phi_i(A). \quad (28)$$

Putting eqs. (25) and (27) together gives

$$\frac{\partial U}{\partial t_M} = \sum_{-(m+1) \leq i \leq n-1} \Delta_i(U) \frac{\partial U}{\partial t_i}. \quad (29)$$

We have infinitely many equations of this sort. They constrain solutions of the dispersionless KP hierarchy (2), and might be related to the Virasoro constraints.

In ref. 13 it was shown that from eq. (29) a solution of eq (25) in the small phase space with  $t_M = 1$  can be constructed as a hodograph form,

$$\Delta_i(U) = t_i, \tag{30}$$

The explicit forms of  $u_i$ 's are then obtained by inverting the algebraic equations (30). In particular, for  $M = n + 1$  we find the flat solution, i.e.  $u_i = t_i$  with  $t_{n+1} = 1$ . Of course we may be interested in more general solutions of eq. (25) with (29) in large spaces. An important point of the reformulation (25) and (26) is that one can bring any solution  $W$  of eq. (2) in the universal form by finding an appropriate solution  $u_i$ ,  $-(m + 1) \leq i \leq n - 1$  of eq. (25). Dependence on the flow parameters  $t_i$ ,  $-\infty < i < \infty$ , appears only implicitly through the solution  $u_i$ .

6. In this note we have studied the dispersionless KP hierarchy (2) with the rational  $W$  potential (3). In the small phase space it was solved by eqs. (23). We have shown that this solution gives a TCFT, for which the 3-point function  $c_{ijk}$  was given by the residue formula (8). The novelty of this TCFT is the presence of positive and negative primaries. We have given the proofs of the flatness of the metric  $c_{0ij}$  and the integrability  $\partial_m c_{ijk} = \partial_i c_{mjk}$  in some details, since they were not evident by simply generalizing the arguments for the A model. The key step in the proofs was to write the residue formula (8) in terms of the  $W$  potential, i.e., eq. (18). The TCFT thus obtained contains the A-D- models as subclasses of TCFT.

We have also discussed the dispersionless KP hierarchy (2) in the entire phase space. Through the reformulation the arguments in the small phase spaces were universally extended to larger ones. We have shown that any solution  $W$  of the dispersionless KP hierarchy (2) in the entire phase space can be brought into the universal form (23) with the flow parameters  $t_i$  replaced by an appropriate solution  $u_i$ ,  $-(m + 1) \leq i \leq n - 1$ , of eq. (25).

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