

# HUREWICZ THEOREM FOR EXTENSION DIMENSION

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ABSTRACT. We prove a new selection theorem for multivalued mappings of  $C$ -space. Using this theorem we prove extension dimensional version of Hurewicz theorem for a closed mapping  $f: X \rightarrow Y$  of  $k$ -space  $X$  onto paracompact  $C$ -space  $Y$ : if for finite  $CW$ -complex  $M$  we have  $e\text{-dim}Y \leq [M]$  and for every point  $y \in Y$  and every compactum  $Z$  with  $e\text{-dim}Z \leq [M]$  we have  $e\text{-dim}(f^{-1}(y) \times Z) \leq [L]$  for some  $CW$ -complex  $L$ , then  $e\text{-dim}X \leq [L]$ .

## 1. INTRODUCTION

The classical Hurewicz theorem states that for a mapping of finite-dimensional compacta  $f: X \rightarrow Y$  we have

$$\dim X \leq \dim Y + \dim f, \quad \text{where } \dim f = \max\{\dim(f^{-1}(y)) \mid y \in Y\}.$$

There are several approaches to extension dimensional generalization of Hurewicz theorem [6],[3],[1],[7],[8],[9].

Using the idea from [3] we improve Theorem 7.6 from [1]:

**Theorem 3.1.** *Let  $f: X \rightarrow Y$  be a closed mapping of a  $k$ -space  $X$  onto paracompact  $C$ -space  $Y$ . Suppose that  $e\text{-dim}Y \leq [M]$  for a finite  $CW$ -complex  $M$ . If for every point  $y \in Y$  and for every compactum  $Z$  with  $e\text{-dim}Z \leq [M]$  we have  $e\text{-dim}(f^{-1}(y) \times Z) \leq [L]$  for some  $CW$ -complex  $L$ , then  $e\text{-dim}X \leq [L]$ .*

The notion of extension dimension was introduced by Dranishnikov [4]: for a  $CW$ -complex  $L$  a space  $X$  is said to have *extension dimension*  $\leq [L]$  (notation:  $e\text{-dim}X \leq [L]$ ) if any mapping of its closed subspace  $A \subset X$  into  $L$  admits an extension to the whole space  $X$ .

To prove Theorem 3.1 we need an extension dimensional version of Uspenskij's selection theorem [11]. In section 2 we prove Theorem 2.8 on selections of multivalued mappings of  $C$ -space. Then Theorem 2.5 helps us to prove Theorem 2.9 — a needed version of Uspenskij's theorem.

Filtrations of multivalued maps are proved to be very useful for construction of continuous selections [10], [1]. And we state our selection theorems in terms

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of filtrations. Note that Valov [12] used filtrations to prove a selection theorem for mappings of finite  $C$ -spaces.

Let us recall some definitions and introduce our notations. A space  $X$  is called a  $k$ -space if  $U \subset X$  is open in  $X$  whenever  $U \cap C$  is relatively open in  $C$  for every compact subset  $C$  of  $X$ . The *graph* of a multivalued mapping  $F: X \rightarrow Y$  is the subset  $\Gamma_F = \{(x, y) \in X \times Y: y \in F(x)\}$  of the product  $X \times Y$ .

We denote by  $\text{cov}X$  the collection of all coverings of the space  $X$ . For a cover  $\omega$  of a space  $X$  and for a subset  $A \subseteq X$  let  $\text{St}(A, \omega)$  denote the star of the set  $A$  with respect to  $\omega$ . We say that a subset  $A \subset X$  *refines* a cover  $\omega \in \text{cov}X$  if  $A$  is contained in some element of  $\omega$ . A covering  $\omega' \in \text{cov}X$  *strongly star refines* a covering  $\omega \in \text{cov}X$  if for any element  $W \in \omega'$  the set  $\text{St}(W, \omega')$  refines  $\omega$ .

**Definition 1.1.** A topological space  $X$  is called  $C$ -space if for each sequence  $\{\omega_i\}_{i \geq 1}$  of open covers of  $X$ , there is an open cover  $\Sigma$  of  $X$  of the form  $\cup_{i=1}^{\infty} \sigma_i$  such that for each  $i \geq 1$ ,  $\sigma_i$  is a pairwise disjoint collection which refines  $\omega_i$ .

If the space  $X$  is paracompact, we can choose the cover  $\Sigma$  to be locally finite and every collection  $\sigma_i$  to be discrete.

**Definition 1.2.** A multivalued mapping  $F: X \rightarrow Y$  is said to be *strongly lower semicontinuous* (briefly, strongly l.s.c.) if for any point  $x \in X$  and any compact set  $K \subset F(x)$  there exists a neighborhood  $V$  of  $x$  such that  $K \subset F(z)$  for every  $z \in V$ .

**Definition 1.3.** Let  $L$  be a  $CW$ -complex. A pair of spaces  $V \subset U$  is said to be  $[L]$ -connected (resp.,  $[L]_c$ -connected) if for every paracompact space  $X$  (resp., compact metric space  $X$ ) of extension dimension  $\text{e-dim}X \leq [L]$  and for every closed subspace  $A \subset X$  any mapping of  $A$  into  $V$  can be extended to a mapping of  $X$  into  $U$ .

An increasing<sup>1</sup> sequence of subspaces  $Z_0 \subset Z_1 \subset \dots \subset Z$  is called a *filtration* of space  $Z$ . A sequence of multivalued mappings  $\{F_k: X \rightarrow Y\}$  is called a *filtration of multivalued mapping*  $F: X \rightarrow Y$  if  $\{F_k(x)\}$  is a filtration of  $F(x)$  for any  $x \in X$ .

**Definition 1.4.** A filtration of multivalued mappings  $\{G_i: X \rightarrow Y\}$  is said to be *fiberwise*  $[L]_c$ -connected if for any point  $x \in X$  and any  $i$  the pair  $G_i(x) \subset G_{i+1}(x)$  is  $[L]_c$ -connected.

## 2. SELECTION THEOREMS

The following notion of stably  $[L]$ -connected filtration of multivalued mappings provides a key property of the filtration for our construction of continuous selections.

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<sup>1</sup>We consider only increasing filtrations indexed by a segment of the integral series.

**Definition 2.1.** A pair  $F \subset H$  of multivalued mappings from  $X$  to  $Y$  is called *stably  $[L]$ -connected* if every point  $x \in X$  has a neighborhood  $O_x$  such that the pair  $F(O_x) \subset \bigcap_{z \in O_x} H(z)$  is  $[L]$ -connected.

We say that the pair  $F \subset H$  is called *stably  $[L]$ -connected with respect to a covering  $\omega \in \text{cov}X$* , if for any  $W \in \omega$  the pair  $F(W) \subset \bigcap_{x \in W} H(x)$  is  $[L]$ -connected.

A filtration  $\{F_i\}$  of multivalued mappings is called *stably  $[L]$ -connected* if every pair  $F_i \subset F_{i+1}$  is stably  $[L]$ -connected.

Clearly, any stably  $[L]$ -connected pair of multivalued maps of a space  $X$  is stably  $[L]$ -connected with respect to some covering of  $X$ .

We denote by  $Q$  the Hilbert cube. We identify a space  $Y$  with the subspace  $Y \times \{0\}$  of the product  $Y \times Q$  and denote by  $\text{pr}_Y$  the projection of  $Y \times Q$  onto  $Y$ .

**Definition 2.2.** For a subspace  $Z \subset Y \times Q$  we say that  $Y$  *projectively contains*  $Z$ . We say that a multivalued mapping  $F: X \rightarrow Y$  *projectively contains* a multivalued mapping  $G: X \rightarrow Y \times Q$  if for any point  $x \in X$  the set  $\text{pr}_Y \circ G(x)$  is contained in  $F(x)$ .

**Lemma 2.3.** *Let  $L$  be a finite CW-complex. If a topological space  $Y$  contains a compactum  $K$  of extension dimension  $\text{e-dim}K \leq [L]$  such that the pair  $K \subset Y$  is  $[L]_c$ -connected, then  $Y$  projectively contains a compactum  $K'$  of extension dimension  $\text{e-dim}K' \leq [L]$  such that  $K$  lies in  $K'$  and the pair  $K \subset K'$  is  $[L]$ -connected.*

*Proof.* There exists  $AE([L])$ -compactum  $K'$  of extension dimension  $\text{e-dim}K' \leq [L]$  containing the given compactum  $K$  [2]. Clearly, the pair  $K \subset K'$  is  $[L]$ -connected. Since  $\text{e-dim}K' \leq [L]$ , there exists a mapping  $p: K' \rightarrow Y$  extending the inclusion of  $K$  into  $Y$ .

It is easy to see that there exists a mapping  $q: K' \rightarrow Q$  such that  $q^{-1}(0) = K$  and  $q$  is an embedding on  $K' \setminus K$ . Now define an embedding  $j: K' \rightarrow Y \times Q$  as  $j = p \times q$ . Since  $q^{-1}(0) = K$ , the mapping  $j$  coincide with  $p$  on  $K$  which is inclusion on  $K$ .  $\square$

**Definition 2.4.** We say that a filtration  $F_0 \subset F_1 \subset \dots$  of multivalued mappings from  $X$  to  $Y$  *projectively contains* a filtration  $G_0 \subset G_1 \subset \dots$  of multivalued mappings from  $X$  to  $Y \times Q$  if for any point  $x \in X$  and any  $n$  the set  $\text{pr}_Y \circ G_n(x)$  is contained in  $F_n(x)$ .

**Theorem 2.5.** *For a finite CW-complex  $L$  any fiberwise  $[L]_c$ -connected filtration of strongly l.s.c. multivalued mappings of paracompact space  $X$  to a topological space  $Y$  projectively contains stably  $[L]$ -connected filtration of compact-valued mappings.*

*Proof.* For a given fiberwise  $[L]_c$ -connected filtration  $F_0 \subset F_1 \subset \dots$  of strongly l.s.c. multivalued mappings we construct stably  $[L]$ -connected filtration  $G_0 \subset G_1 \subset \dots$  of compact-valued mappings  $G_n: X \rightarrow Y \times Q^n$  as follows: successively for every  $n \geq 0$  we construct a covering  $\omega_n = \{W_\lambda^n\}_{\lambda \in \Lambda_n} \in \text{cov}X$  and a family of subcompacta  $\{K_\lambda^n\}_{\lambda \in \Lambda_n}$  of  $Y \times Q^n$ , and define the mapping  $G_n$  by the formula

$$G_n(x) = \cup\{K_\lambda^n \mid x \in W_\lambda^n\}.$$

First, we construct  $G_0$ , i.e. the covering  $\omega_0$  and the family  $\{K_\lambda^0\}_{\lambda \in \Lambda_0}$ . Since  $F_0$  is strongly l.s.c., there exists a locally finite open covering  $\omega_{-1} = \{W_\lambda^{-1}\}_{\lambda \in \Lambda_{-1}} \in \text{cov}X$  and a family  $\{M_\lambda^{-1}\}_{\lambda \in \Lambda_{-1}}$  of points in  $Y$  such that  $W_\lambda^{-1} \times M_\lambda^{-1} \subset \Gamma_{F_0}$  for any  $\lambda \in \Lambda_{-1}$ . Denote by  $H_0$  a multivalued mapping taking a point  $x \in X$  to the set  $H_0(x) = \cup\{M_\lambda^{-1} \mid x \in W_\lambda^{-1}\}$ . Note that  $H_0(x)$  is contained in  $F_0(x)$  and consists of finitely many points. By Lemma 2.3 for any  $x \in X$  there exists a compactum  $\widehat{H}_0(x) \subset F_1(x) \times Q$  of extension dimension  $e\text{-dim}\widehat{H}_0(x) \leq [L]$  such that the pair  $H_0(x) \subset \widehat{H}_0(x)$  is  $[L]$ -connected. Since  $F_1$  is strongly l.s.c., any point  $x \in X$  has a neighborhood  $\mathcal{O}_0(x)$  such that the product  $\mathcal{O}_0(x) \times \widehat{H}_0(x)$  is contained in  $\Gamma_{F_1} \times Q$ . Since  $X$  is paracompact, we can choose neighborhoods  $\mathcal{O}_0(x)$  in such a way that the covering  $\mathcal{O}_0 = \{\mathcal{O}_0(x)\}_{x \in X}$  strongly star refines  $\omega_{-1}$ . Let  $\omega_0 = \{W_\lambda^0\}_{\lambda \in \Lambda_0}$  be a locally finite open cover of  $X$  refining  $\mathcal{O}_0$ . For every  $\lambda \in \Lambda_0$  we fix a point  $x_\lambda$  such that  $W_\lambda^0 \subset \mathcal{O}_0(x_\lambda)$  and put  $M_\lambda^0 = \widehat{H}_0(x_\lambda)$ . For every  $\lambda \in \Lambda_0$  we fix  $\alpha(\lambda) \in \Lambda_{-1}$  such that  $\text{St}(W_\lambda^0, \mathcal{O}_0) \subset W_{\alpha(\lambda)}^{-1}$  and put  $K_\lambda^0 = M_{\alpha(\lambda)}^{-1}$ .

Inductive step of our construction is similar to the first step. Suppose that a covering  $\omega_{n-1} = \{W_\lambda^{n-1}\}_{\lambda \in \Lambda_{n-1}} \in \text{cov}X$  and a family  $\{M_\lambda^{n-1}\}_{\lambda \in \Lambda_{n-1}}$  of compacta in  $Y \times Q^{n-1}$  are already constructed such that  $e\text{-dim}M_\lambda^{n-1} \leq [L]$  and the product  $W_\lambda^{n-1} \times M_\lambda^{n-1}$  is contained in  $\Gamma_{F_n} \times Q^n$  for any  $\lambda \in \Lambda_{n-1}$ . Denote by  $H_n$  a multivalued mapping taking a point  $x \in X$  to the compactum  $H_n(x) = \cup\{M_\lambda^{n-1} \mid x \in W_\lambda^{n-1}\}$ . Note that  $H_n(x)$  is contained in  $F_n(x) \times Q^n$  and has extension dimension  $e\text{-dim}H_n(x) \leq [L]$ . By Lemma 2.3 for any  $x \in X$  there exists a compactum  $\widehat{H}_n(x) \subset F_{n+1}(x) \times Q^{n+1}$  of extension dimension  $e\text{-dim}\widehat{H}_n(x) \leq [L]$  such that the pair  $H_n(x) \subset \widehat{H}_n(x)$  is  $[L]$ -connected. Since  $F_{n+1}$  is strongly l.s.c., any point  $x \in X$  has a neighborhood  $\mathcal{O}_n(x)$  such that the product  $\mathcal{O}_n(x) \times \widehat{H}_n(x)$  is contained in  $\Gamma_{F_{n+1}} \times Q^{n+1}$ . Since  $X$  is paracompact, we can choose neighborhoods  $\mathcal{O}_n(x)$  in such a way that the covering  $\mathcal{O}_n = \{\mathcal{O}_n(x)\}_{x \in X}$  strongly star refines  $\omega_{n-1}$ . Let  $\omega_n = \{W_\lambda^n\}_{\lambda \in \Lambda_n}$  be a locally finite open cover of  $X$  refining  $\mathcal{O}_n$ . For every  $\lambda \in \Lambda_n$  we fix a point  $x_\lambda$  such that  $W_\lambda^n \subset \mathcal{O}_n(x_\lambda)$  and put  $M_\lambda^n = \widehat{H}_n(x_\lambda)$ . For every  $\lambda \in \Lambda_n$  we fix  $\alpha(\lambda) \in \Lambda_{n-1}$  such that  $\text{St}(W_\lambda^n, \mathcal{O}_n) \subset W_{\alpha(\lambda)}^{n-1}$  and put  $K_\lambda^n = M_{\alpha(\lambda)}^{n-1}$ .

To show that the pair  $G_{n-1} \subset G_n$  is stably  $[L]$ -connected, we prove that the pair  $G_{n-1}(W_\lambda^n) \subset \cap\{G_n(x) \mid x \in W_\lambda^n\}$  is  $[L]$ -connected for any  $W_\lambda^n \in \omega_n$ . By the construction of  $G_n$ , the set  $K_\lambda^n$  is contained in  $\cap\{G_n(x) \mid x \in W_\lambda^n\}$ . We

know that the pair  $H_{n-1}(x_{\alpha(\lambda)}) \subset \widehat{H}_{n-1}(x_{\alpha(\lambda)}) = M_{\alpha(\lambda)}^{n-1} = K_{\lambda}^n$  is  $[L]$ -connected. Therefore it is enough to show the following inclusion:

$$G_{n-1}(W_{\lambda}^n) = \bigcup \{K_{\beta}^{n-1} \mid W_{\lambda}^n \cap W_{\beta}^{n-1} \neq \emptyset\} \subset \cup \{M_{\nu}^{n-2} \mid x_{\alpha(\lambda)} \in W_{\nu}^{n-2}\} = H_{n-1}(x_{\alpha(\lambda)})$$

which follows from the fact that  $W_{\lambda}^n \cap W_{\beta}^{n-1} \neq \emptyset$  implies  $x_{\alpha(\lambda)} \in W_{\alpha(\beta)}^{n-2}$  (note that  $M_{\alpha(\beta)}^{n-2} = K_{\beta}^{n-1}$ ). By the choice of  $\alpha(\lambda)$  we have  $W_{\lambda}^n \subset \mathcal{O}_{n-1}(x_{\alpha(\lambda)})$ . Then  $W_{\lambda}^n \cap W_{\beta}^{n-1} \neq \emptyset$  implies  $\mathcal{O}_{n-1}(x_{\alpha(\lambda)}) \cap W_{\beta}^{n-1} \neq \emptyset$  and  $x_{\alpha(\lambda)} \in \mathcal{O}_{n-1}(x_{\alpha(\lambda)}) \subset \text{St}(W_{\beta}^{n-1}, \mathcal{O}_{n-1}) \subset W_{\alpha(\beta)}^{n-2}$ .  $\square$

**Definition 2.6.** For a space  $Z$  a pair of spaces  $V \subset U$  is said to be  $Z$ -connected if for every closed subspace  $A \subset Z$  any mapping of  $A$  into  $V$  can be extended to a mapping of  $Z$  into  $U$ .

**Definition 2.7.** A pair  $F \subset H$  of multivalued mappings from  $X$  to  $Y$  is called *stably  $Z$ -connected* if every point  $x \in X$  has a neighborhood  $O_x$  such that the pair  $F(O_x) \subset \cap_{z \in O_x} H(z)$  is  $Z$ -connected.

We say that the pair  $F \subset H$  is called *stably  $Z$ -connected with respect to a covering  $\omega \in \text{cov} X$* , if for any  $W \in \omega$  the pair  $F(W) \subset \cap_{x \in W} H(x)$  is  $Z$ -connected.

A filtration  $\{F_i\}$  of multivalued mappings is called *stably  $Z$ -connected* if every pair  $F_i \subset F_{i+1}$  is stably  $Z$ -connected.

**Theorem 2.8.** *Let  $F: X \rightarrow Y$  be a multivalued mapping of paracompact  $C$ -space  $X$  to a topological space  $Y$ . If  $F$  admits infinite stably  $X$ -connected filtration of multivalued mappings, then  $F$  has a singlevalued continuous selection.*

*Proof.* Let  $\{F_i\}_{i=-1}^{\infty}$  be the given filtration of  $F$ . Let  $\{\omega_i\}_{i=-1}^{\infty}$  be a sequence of coverings of  $X$  such that  $\omega_{i+1}$  refines  $\omega_i$  and the pair  $F_i \subset F_{i+1}$  is stably  $X$ -connected with respect to the covering  $\omega_i$ . Since  $X$  is paracompact  $C$ -space, there exists a locally finite closed cover  $\Sigma$  of  $X$  of the form  $\Sigma = \cup_{i=0}^{\infty} \sigma_i$  such that  $\sigma_i$  is discrete collection refining  $\omega_i$ . Define  $\Sigma_n = \cup_{i=0}^n \sigma_i$ . We will construct a continuous selection  $f$  of  $F$  extending it successively over the sets  $\Sigma_n$ .

First, we construct  $f_0: \Sigma_0 \rightarrow Y$ . We define  $f_0$  separately on every element  $s$  of the discrete collection  $\sigma_0$ : take a point  $p \in F_{-1}(s)$  and put  $f_0(s) = p$ . Since the set  $s$  refines  $\omega_0$ , then  $p \in F_0(x)$  for any  $x \in s$  and therefore  $f_0$  is a selection of  $F_0|_{\Sigma_0}$ .

Suppose that we already constructed  $f_n$  — a continuous selection of  $F_n|_{\Sigma_n}$ . Let us define  $f_{n+1}$  on arbitrary element  $Z$  of discrete collection  $\sigma_{n+1}$ . Since  $\Sigma$  is locally finite, the set  $A = Z \cap \Sigma_n$  is closed in  $X$ . Since  $f_n$  is a selection of  $F_n$ , then  $f_n(A)$  is contained in  $F_n(Z)$ . Since the pair  $F_n(Z) \subset \cap_{x \in Z} F_{n+1}(x)$  is

$X$ -connected, we can extend  $f_n|_A$  to a mapping  $f'_n: Z \rightarrow \bigcap_{x \in Z} F_{n+1}(x)$ . Clearly,  $f'_n$  is a selection of  $F_{n+1}|_Z$ . We define  $f_{n+1}$  on the set  $Z$  as  $f'_n$ .

Finally, we define  $f$  to be equal to  $f_n$  on the set  $\Sigma_n$ .  $\square$

**Theorem 2.9.** *Let  $L$  be a finite CW-complex and  $F: X \rightarrow Y$  be a multivalued mapping of paracompact  $C$ -space  $X$  of extension dimension  $\text{e-dim} X \leq [L]$  to a topological space  $Y$ . If  $F$  admits infinite fiberwise  $[L]_c$ -connected filtration of strongly l.s.c. multivalued mappings, then  $F$  has a singlevalued continuous selection.*

*Proof.* By Theorem 2.5, the mapping  $F': X \rightarrow Y \times Q$  defined as  $F'(x) = F(x) \times Q$  contains a stably  $[L]$ -connected filtration of multivalued mappings. By Theorem 2.8  $F'$  has a singlevalued continuous selection  $f'$ . Then the mapping  $f = \text{pr}_Y \circ f'$  is a singlevalued continuous selection of  $F$ .  $\square$

### 3. HUREWICZ THEOREM

The proof of the following theorem is similar to the proof of Theorem 2.4 from [3].

**Theorem 3.1.** *Let  $f: X \rightarrow Y$  be a closed mapping of  $k$ -space  $X$  onto paracompact  $C$ -space  $Y$ . Suppose that  $\text{e-dim} Y \leq [M]$  for a finite CW-complex  $M$ . If for every point  $y \in Y$  and for every compactum  $Z$  with  $\text{e-dim} Z \leq [M]$  we have  $\text{e-dim}(f^{-1}(y) \times Z) \leq [L]$  for some CW-complex  $L$ , then  $\text{e-dim} X \leq [L]$ .*

*Proof.* Suppose  $A \subset X$  is closed and  $g: A \rightarrow L$  is a map. We are going to find a continuous extension  $\tilde{g}: X \rightarrow L$  of  $g$ . Let  $K$  be the cone over  $L$  with a vertex  $v$ . We denote by  $C(X, K)$  the space of all continuous maps from  $X$  to  $K$  equipped with the compact-open topology. We define a multivalued map  $F: Y \rightarrow C(X, K)$  as follows:

$$F(y) = \{h \in C(X, K) \mid h(f^{-1}(y)) \subset K \setminus \{v\} \text{ and } h|_A = g\}.$$

*Claim.*  $F$  admits continuous singlevalued selection.

If  $\varphi: Y \rightarrow C(X, K)$  is a continuous selection for  $F$ , then the mapping  $h: X \rightarrow K$  defined by  $h(x) = \varphi(f(x))(x)$  is continuous on every compact subset of  $X$  and because  $X$  is a  $k$ -space,  $h$  is continuous. Since  $\varphi(f(x)) \in F(f(x))$  for every  $x \in X$ , we have  $h(X) \subset K \setminus \{v\}$ . Now if  $\pi: K \setminus \{v\} \rightarrow L$  denotes the natural retraction, then  $\tilde{g} = \pi \circ h: X \rightarrow L$  is the desired continuous extension of  $h$ .

*Proof of the claim.* We are going to apply Theorem 2.9 to infinite filtration  $F \subset F \subset F \subset \dots$ . To do this, we have to show that  $F$  is strongly l.s.c. and that the pair  $F(y) \subset F(y)$  is  $[M]_c$ -connected for every point  $y \in Y$ .

First, we show that  $F$  is strongly l.s.c. Let  $y_0 \in Y$  and  $P \subset F(y_0)$  be compact. We have to find a neighborhood  $V$  of  $y_0$  in  $Y$  such that  $P \subset F(y)$  for every  $y \in V$ . For every  $x \in X$  define a subset  $P(x) = \{h(x) \mid h \in P\}$  of  $K$ . Since  $P \subset C(X, K)$  is compact and  $X$  is a  $k$ -space, by the Ascoli theorem, each

$P(x)$  is compact and  $P$  is evenly continuous. This easily implies that the set  $W = \{x \in X \mid P(x) \subset K \setminus \{v\}\}$  is open in  $X$  and, obviously,  $f^{-1}(y_0) \subset W$ . Since  $f$  is closed, there exists a neighborhood  $V$  of  $y_0$  in  $Y$  with  $f^{-1}(V) \subset W$ . Then, according to the choice of  $W$  and the definition of  $F$ , we have  $P \subset F(y)$  for every  $y \in V$ .

Fix an arbitrary point  $y \in Y$ . Let us prove that the pair  $F(y) \subset F(y)$  is  $[M]_c$ -connected. Consider a pair of compacta  $B \subset Z$  where  $\text{e-dim}Z \leq [M]$  and a mapping  $\varphi: B \rightarrow F(y)$ . Since  $B \times X$  is a  $k$ -space (as a product of a compact space and a  $k$ -space), the map  $\psi: B \times X \rightarrow K$  defined as  $\psi(b, x) = \varphi(b)(x)$  is continuous. Extend  $\psi$  to a set  $Z \times A$  letting  $\psi(z, a) = g(a)$ . Clearly,  $\psi$  takes the set  $Z \times f^{-1}(y) \cap (Z \times A \cup B \times X)$  into  $K \setminus \{v\} \cong L \times [0, 1)$ . Since  $\text{e-dim}(Z \times f^{-1}(y)) \leq [L]$ , we can extend  $\psi$  over the set  $Z \times f^{-1}(y)$  to take it into  $K \setminus \{v\}$ . Finally extend  $\psi$  over  $Z \times X$  as a mapping into  $AE$ -space  $K$ . Now define an extension  $\tilde{\varphi}: Z \rightarrow F(y)$  of the mapping  $\varphi$  by the formula  $\tilde{\varphi}(z)(x) = \psi(z, x)$ .  $\square$

**Corollary 3.2** (cf. Theorem 2.25 from [6]). *Let  $f: X \rightarrow Y$  be a mapping of finite-dimensional compacta where  $\text{e-dim}Y = [M]$  for finite CW-complex  $M$ . If for some CW-complex  $L$  we have  $\text{e-dim}(f^{-1}(y) \times Y) \leq [L]$  for every point  $y \in Y$ , then  $\text{e-dim}X \leq [L]$ .*

*Proof.* By Theorem 6.3 from [5] for any compactum  $Z$  with  $\text{e-dim}Z \leq \text{e-dim}Y$  we have  $\text{e-dim}(f^{-1}(y) \times Z) \leq [L]$ . Thus, we can apply Theorem 3.1  $\square$

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