A STUDY ON THE GLOBAL REGULARITY FOR A MODEL OF 
THE 3D AXISYMMETRIC NAVIER-STOKES EQUATIONS

LIZHENG TAO AND JIAHONG WU

Abstract. This paper investigates the global regularity issue concerning a model equation proposed by Hou and Lei [6] to understand the stabilizing effects of the nonlinear terms in the 3D axisymmetric Navier-Stokes and Euler equations. Two major results are obtained. The first one establishes the global regularity of a generalized version of their model with a fractional Laplacian when the fractional power satisfies an explicit condition. This condition is exactly the same as in the case of the 3D generalized Navier-Stokes equations and is due to the balance between a more regular nonlinearity and a less effective (five-dimensional) Laplacian. The second result assesses a finite-time singularity in a quantity associated with certain solution of the inviscid counterpart of their model.

1. Introduction

The global regularity issue concerning the 3D axisymmetric Navier-Stokes and Euler equations has recently attracted a lot of attention and much progress has been made (see e.g. [1],[2],[3],[4],[5],[7],[8],[10]). The results presented here were motivated by recent work of T. Hou and his collaborators on two models for the axisymmetric Navier-Stokes and Euler equations ([4],[7],[8]).

In several recent papers ([3],[4],[5],[6]), Hou, Lei and Li proposed two systems of equations for study in order to understand the stabilizing effects of the nonlinear terms in the 3D axisymmetric Navier-Stokes and Euler equations. We shall briefly summarize their derivation of these model equations. The incompressible 3D axisymmetric Navier-Stokes equations can be written as

\[
\begin{align*}
\frac{\partial}{\partial t} u^r - \frac{(u^\theta)^2}{r} = &- p_r + \nu \left( \partial_{rr} + \frac{1}{r} \partial_r + \partial_{zz} - \frac{1}{r^2} \right) u^r, \\
\frac{\partial}{\partial t} u^\theta + \frac{u^r u^\theta}{r} = &\nu \left( \partial_{rr} + \frac{1}{r} \partial_r + \partial_{zz} - \frac{1}{r^2} \right) u^\theta, \\
\frac{\partial}{\partial t} u^z = &- p_z + \nu \left( \partial_{rr} + \frac{1}{r} \partial_r + \partial_{zz} \right) u^z, \\
\partial_r u^r + \frac{1}{r} u^r + \partial_z u^z = &0,
\end{align*}
\]

where \( u^r, u^\theta \) and \( u^z \) are the cylindrical coordinates of the velocity field \( \mathbf{u} \), and

\[
\frac{\partial}{\partial t} = \partial_t + u^r \partial_r + u^z \partial_z.
\]

2000 Mathematics Subject Classification. 35Q53, 35B35, 35B65, 76D03.

Key words and phrases. 3D axisymmetric Navier-Stokes.
When $\nu = 0$, these equations reduce to the axisymmetric Euler equations. The corresponding vorticity $\omega = \nabla \times u$ obey

$$
\begin{align*}
\frac{\partial}{\partial t} \omega^r &= \nu \left( \rho_{rr} + \frac{1}{r} \rho_r + \rho_{zz} - \frac{1}{r^2} \right) \omega^r + (\omega^r \rho_r + \omega^z \rho_z)u^r, \\
\frac{\partial}{\partial t} \omega^\theta &= \nu \left( \rho_{rr} + \frac{1}{r} \rho_r + \rho_{zz} - \frac{1}{r^2} \right) \omega^\theta + (\omega^r \rho_r + \omega^z \rho_z)u^\theta + \frac{u^r \omega^\theta}{r}, \\
\frac{\partial}{\partial t} \omega^z &= \nu \left( \rho_{rr} + \frac{1}{r} \rho_r + \rho_{zz} \right) \omega^z + (\omega^r \rho_r + \omega^z \rho_z)u^z.
\end{align*}
$$

(1.2)

Noticing that $u^r$ and $u^z$ can be represented by $\psi^\theta$, $\omega^r$ and $\omega^z$ by $u^\theta$ and $\omega^\theta$ and $\psi^\theta$ are related by

$$
- \left( \rho^2 + \frac{1}{r} \rho_r + \rho^2 - \frac{1}{r^2} \right) \psi^\theta = \omega^\theta,
$$

(1.3)

the axisymmetric Navier-Stokes equations reduce to a system of equations for the swirl components $\psi^\theta$, $u^\theta$ and $\omega^\theta$. By substituting the expansions of $\psi^\theta$, $u^\theta$ and $\omega^\theta$ near $r = 0$ and keeping the leading order terms, Hou and Li derived an one-dimensional model that approximates the Navier-Stokes equations along the symmetric axis [6]. This model has some interesting properties. In particular, the nonlinear terms have a very special structure and appear to have depletion mechanism that prevents a finite-time singularity.

By substituting the new variables

$$
u_1 = \frac{u^\theta}{r}, \quad \omega_1 = \frac{\omega^\theta}{r}, \quad \psi_1 = \frac{\psi^\theta}{r}
$$

in the swirl component equations of (1.1), (1.2) and in (1.3), and dropping the convection terms, Hou and Lei [4] obtained the following system of model equations

$$
\begin{align*}
\partial_t u_1 &= \nu \left( \rho_{rr} + \frac{3}{r} \rho_r + \rho_{zz} \right) u_1 + 2 \rho_z \psi_1 u_1, \\
\partial_t \omega_1 &= \nu \left( \rho_{rr} + \frac{3}{r} \rho_r + \rho_{zz} \right) u_1 + \rho_z (u_1^2), \\
- \left( \rho_{rr} + \frac{3}{r} \rho_r + \rho_{zz} \right) \psi_1 &= \omega_1.
\end{align*}
$$

(1.4)

Clearly this system of equations is self-contained. When the convection terms are added back to this system of equations, the 3D axisymmetric Navier-Stokes equations can be recovered. Even without the convection terms, these equations possess many similarities as the 3D axisymmetric Navier-Stokes equations. As demonstrated in [4] and [5], regularity criteria of the Prodi-Serrin type and of the Beal-Kato-Majda type still hold for this system of equations.

Our attention is focused on the open problem of whether classical solutions of (1.4) are global in time. The issue is investigated here from two different perspectives. First, we generalize this model to include dissipation given by a fractional Laplacian. For this purpose, we need to interpret these equations as a system of equations in 5-dimensional space. To be more precise, we set $y = (y_1, y_2, y_3, y_4, z) \in \mathbb{R}^5$ and write $\Delta_y$ for the 5D
Laplacian, namely
\[ \Delta_y = \sum_{j=1}^{4} \partial_{y_j y_j} + \partial_{zz}. \]

If a function \( f = f(y) \) is axisymmetric about the \( z \)-axis, then
\[ \Delta_y f = \left( \partial_{rr} + \frac{3}{r} \partial_{r} + \partial_{zz} \right) f. \]

Identifying \( u_1, \omega_1 \) and \( \psi_1 \) as 5D axisymmetric functions, we can write the equations in (1.4) as
\[
\begin{align*}
\partial_t u_1 &= \nu \Delta_y u_1 + 2 \partial_z \psi_1 u_1, \\
\partial_t \omega_1 &= \nu \Delta_y \omega_1 + \partial_z (u_1^2), \\
-\Delta_y \psi_1 &= \omega_1.
\end{align*}
\]

Replacing \( \Delta_y \) by the fractional Laplacian \( -(-\Delta_y)^\alpha \) for a parameter \( \alpha > 0 \) in the first two equations, we obtain the generalized Hou-Lei model
\[
\begin{align*}
\partial_t u_1 &= -\nu (-\Delta_y)^\alpha u_1 + 2 \partial_z \psi_1 u_1, \\
\partial_t \omega_1 &= -\nu (-\Delta_y)^\alpha \omega_1 + \partial_z (u_1^2), \\
(-\Delta_y) \psi_1 &= \omega_1.
\end{align*}
\]

More generally, for any integer \( n \geq 3 \), we can consider the following equations of \( n+2 \)-dimensional axisymmetric functions \( u_1, \omega_1 \) and \( \psi_1 \),
\[
\begin{align*}
\partial_t u_1 &= -\nu (-\Delta_{n+2})^\alpha u_1 + 2 \partial_z \psi_1 u_1 \\
\partial_t \omega_1 &= -\nu (-\Delta_{n+2})^\alpha \omega_1 + \partial_z (u_1^2), \\
(-\Delta_{n+2}) \psi_1 &= \omega_1,
\end{align*}
\]

where \( \Delta_{n+2} \) denotes the Laplacian operator in \( \mathbb{R}^{n+2} \). We study the initial-value problems of these generalized Hou-Lei equations with the initial data
\[
\begin{align*}
u_1(x,0) &= \nu_{10}(x), \quad \omega_1(x,0) = \omega_{10}(x), \quad \psi_1(x,0) = \psi_{10}(x).
\end{align*}
\]

The first part of this paper establishes the global regularity of (1.5) for \( \alpha \geq \frac{5}{4} \) and that of (1.6) for \( \alpha \geq \frac{1}{2} + \frac{n}{4} \). We remark that the condition on \( \alpha \) is exactly the same as the condition for the generalized Navier-Stokes equations (see e.g. [11]). This is not completely expected. (1.5) is obtained by dropping the nonlinear terms and the remaining nonlinearity is supposed to behave more regular than that of the generalized 3D Navier-Stokes equations. However, the dissipation in (1.5) is given by a five-dimensional Laplacian and is less effective than the 3D Laplacian in controlling the nonlinear term. Sobolev type inequalities lose its effectiveness as the dimension of the space gets large. It is this balance between a less singular nonlinearity and a weaker dissipation that results in the exactly same requirement on \( \alpha \).

The global regularity results can be stated as the following theorems. In the these theorems and in the rest of this paper, \( \|f\|_q \) with \( 1 \leq q \leq \infty \) denotes the norm in the Lebesgue space \( L^q(\mathbb{R}^5) \), \( \|f\|_{H^k} \) denotes the norm in the space \( H^k(\mathbb{R}^5) \) and \( \|f\|_{k,q} \) the norm in the Sobolev space \( W^{k,q}(\mathbb{R}^5) \).
Theorem 1.1. Consider the generalized 3D model (1.5). Assume that the initial data $(u_{10}, \omega_{10}, \psi_{10})$ in (1.7) satisfies

$$u_{10} \in H^1(\mathbb{R}^5), \quad \psi_{10} \in H^2(\mathbb{R}^5) \quad \text{and} \quad \omega_{10} = -\Delta_y \psi_{10}.$$  

When $\alpha \geq \frac{5}{2}$, the solution $(u_1, \omega_1, \psi_1)$ emanating from $(u_{10}, \omega_{10}, \psi_{10})$ remains bounded in $H^1(\mathbb{R}^5) \times L^2(\mathbb{R}^5) \times H^2(\mathbb{R}^5)$ for all time. More precisely, we have, for any $0 \leq t < \infty$,

$$
\left( \|u_1\|_{H^1(\mathbb{R}^5)}^2 + 2\|\omega_1\|_{L^2}^2 \right) + \nu \int_0^t \left( \|\Lambda_y^\alpha u_1\|_{L^2}^2 + \|\Lambda_y^{1+\alpha}(u_1, \psi_1)\|_{L^2}^2 + 2\|\Lambda_y^{\alpha} \omega_1\|_{L^2}^2 \right) \, dt \leq C,
$$

where $\Lambda_y = (-\Delta_y)^{1/2}$ and $C$ is a constant depending on $\|u_{10}\|_{H^1}$, $\|\omega_1\|_2$ and $\|\psi_{10}\|_{H^2}$ only.

A similar global result holds for the general system of equations given by (1.6).

Theorem 1.2. Consider the generalized model (1.6) with the initial data given by (1.7). Assume that $(u_{10}, \omega_{10}, \psi_{10})$ satisfies

$$u_{10} \in H^1(\mathbb{R}^{n+2}), \quad \psi_{10} \in H^2(\mathbb{R}^{n+2}) \quad \text{and} \quad \omega_{10} = -\Delta_{n+2} \psi_{10}.$$  

If

$$\alpha \geq \frac{1}{2} + \frac{n}{4},$$

then any solution of (1.6) emanating from $(u_{10}, \omega_{10}, \psi_{10})$ remains bounded in $H^1(\mathbb{R}^{n+2}) \times L^2(\mathbb{R}^{n+2}) \times H^2(\mathbb{R}^{n+2})$ for all time.

The second part of this paper is to show that certain solutions of the inviscid counterpart of (1.4) develop a finite-time singularity. More precisely, we consider

$$
\begin{aligned}
\partial_t u &= 2u \partial_z \psi, \quad (r, z) \in \Omega, \\
\partial_t \left( \partial_{rr} + \frac{3}{r} \partial_r + \partial_{zz} \right) (-\psi) &= \partial_z \left( u^2 \right), \quad (r, z) \in \Omega, \\
u u(r, z, 0) &= u_0(r, z), \quad \psi(r, z, 0) = \psi_0(r, z),
\end{aligned}
$$

(1.8)

where $\Omega = [0, \infty) \times \mathbb{T}$ with $\mathbb{T} = [0, 1]$ denoting the one-dimensional torus. We show that, for some initial data $(u_0, \psi_0)$ and a smooth test function $\phi(r, z)$, there is $T^* < \infty$ such that

$$
\iint_{\Omega} \phi(r, z) u^2(r, z, t) \, r \, dr \, dz \to \infty \quad \text{as} \quad t \to T^*
$$

More precisely, we have the following theorem.

Theorem 1.3. Consider the initial-value problem (1.8) on $\Omega$. Let $u_0 \in H^m(\Omega)$ and $\psi_0 \in H^{m+1}(\Omega)$ with $m > \frac{3}{2}$. For $\lambda > \pi^2$, define the test function on $\Omega$ by

$$
\phi(r, z) = \sin(\pi z) \sum_{m=1}^\infty \frac{\left( \lambda \right)}{4} \frac{r^{2m}}{m!(m-1)!}.
$$

Set

$$
A = \iint_{\Omega} \phi \log(u_0^2) \, r \, dr \, dz \quad \text{and} \quad B = 4 \iint_{\Omega} \phi \psi_0 \, r \, dr \, dz.
$$
When either $A > 0, B > 0$ or $A < 0, B \leq 0$ or $A < 0, B > 0$, the solution of (1.8) satisfies, for some $T^* < \infty$,

$$\int_{\Omega} \int \phi(r, z) u^2(r, z, t) r dr dz \to \infty \quad \text{as } t \to T^*. $$

We remark that (1.8) does possess a local in time solution. In fact, the following proposition holds.

**Proposition 1.4.** Let $u_0 \in H^m(\Omega)$ and $\psi_0 \in H^{m+1}(\Omega)$ with $m > \frac{5}{2}$. We seek solutions of (1.8) that also satisfy, for any integer $k \geq 0$ and for any $z \in \mathbb{T}$,

$$\partial^k_r u(r, z) \to 0, \quad \partial^k_r \psi(r, z) \to 0 \quad \text{as } r \to \infty. $$

Then, there exists $T > 0$ such that (1.8) has a unique solution $(u, \psi)$ on $[0, T)$ satisfying

(1.9) \quad $u \in L^\infty([0, T); H^m(\Omega)), \quad \psi \in L^\infty([0, T); H^{m+1}(\Omega)).$

This study is motivated by a recent work of Hou, Shi and Wang ([8]). One of their major results in [8] is the demonstration of a finite-time singularity in the system of equations

(1.10) \quad \begin{align*}
    u_t = 2u_1 \partial_z \psi, \quad &-\Delta \psi_t = \partial_z(u^2)
\end{align*}

where $u$ and $\psi$ are functions of $(x, y, z, t)$ with $(x, y)$ in the periodic box $\mathbb{T}^2$ and $z$ in a finite interval or a half-line and $\Delta$ denotes the 3D Laplacian. They imposed a mixed Dirichlet-Neumann boundary condition. (1.10) is inviscid version of (1.4), but the axisymmetry is not imposed in (1.10). Our purpose here is to keep the axisymmetry and still establish the finite-time singularity.

The rest of this paper is divided into two sections. The second section proves Theorems 1.1 and 1.2 while the third section presents the proof of Theorem 1.3.

## 2. Proofs of Theorems 1.1 and 1.2

This section presents the proofs of the global regularity results stated in Theorem 1.1 and Theorem 1.2.

**Proof of Theorem 1.1.** Multiplying the first equation in (1.5) by $u_1$, the second by $2\psi_1$, integrating over $y \in \mathbb{R}^5$ and performing several integration by parts, we obtain

$$\frac{1}{2} \frac{d}{dt} \int_{\mathbb{R}^5} (u_1^2 + 2|\nabla_y \psi_1|^2) \, dy + \nu \int_{\mathbb{R}^5} (|\Lambda_y^\alpha u_1|^2 + 2|\Lambda_y^{1+\alpha} \psi_1|^2) \, dy = 0,$$

where $\Lambda_y = (-\Delta_y)^{\frac{3}{2}}$. Integrating in time yields

(2.1) \quad \int_{\mathbb{R}^5} (u_1^2 + 2|\nabla_y \psi_1|^2) \, dy + 2\nu \int_0^t \int_{\mathbb{R}^5} (|\Lambda_y^\alpha u_1|^2 + 2|\Lambda_y^{1+\alpha} \psi_1|^2) \, dy \, dt

$$= \int_{\mathbb{R}^5} (u_{10}^2 + 2|\nabla_y \psi_{10}|^2) \, dy.$$
To obtain further bounds, we multiply the first equation in (1.5) by $\Delta_y u_1$, integrate over $y \in \mathbb{R}^5$ to obtain

$$
(2.2) \quad \frac{1}{2} \frac{d}{dt} \int (|\nabla_y u_1|^2 + 2|\omega_1|^2) \ dy + \nu \int (|\Lambda_y^{1+\alpha} u_1|^2 + 2|\Lambda_y^0 \omega_1|^2) \ dy = J_1 + J_2,
$$

where

$$
J_1 = \int 2\partial_z \psi_1 u_1 \Delta_y u_1 \ dy, \quad J_2 = \int 2\omega_1 \partial_z u_1^2 \ dy.
$$

We estimate $J_1$ and $J_2$. By Hölder’s inequality,

$$
(2.3) \quad |J_1| \leq C \|\Delta_y u_1\|_2 \|\partial_z \psi_1\|_4 \|u_1\|_4.
$$

By the Gagliardo-Nirenberg type inequality, for $\alpha \geq 1$,

$$
(2.4) \quad \|\Delta_y u_1\|_2 \leq C \|u_1\|^\frac{\alpha+1}{2+\alpha} \|\Lambda_y^{1+\alpha} u_1\|^\frac{2}{2+\alpha},
$$

we obtain

$$
(2.5) \quad \|u_1\|_4 \leq C \|u_1\|^\frac{a}{2} \|\nabla_y u_1\|^b \|\Lambda_y\alpha u_1\|^c \|\Lambda_y^{1+\alpha} u_1\|^d,
$$

where the indices $a, b, c, d \in [0, 1]$ and satisfy

$$
(2.6) \quad a + b + c + d = 1, \quad \frac{1}{4} = \frac{a}{2} + b \left(\frac{1}{2} - \frac{1}{5}\right) + c \left(\frac{1}{2} - \frac{\alpha}{5}\right) + d \left(\frac{1}{2} - \frac{1 + \alpha}{5}\right).
$$

Writing $a$ and $b$ in terms of $c$ and $d$, we have

$$
(2.7) \quad a = -\frac{1}{4} + (\alpha - 1)c + \alpha d, \quad b = \frac{5}{4} - \alpha c - (1 + \alpha)d.
$$

Similarly,

$$
\|\partial_z \psi_1\|_4 \leq C \|\partial_z \psi_1\|^e \|\nabla_y \partial_z \psi_1\|^f \|\Lambda_y^\alpha \partial_z \psi_1\|^g \|\Lambda_y^{1+\alpha} \partial_z \psi_1\|^h,
$$

(2.8)

$$
(2.9) \quad e + f + g + h = 1, \quad \frac{1}{4} = \frac{e}{2} + f \left(\frac{1}{2} - \frac{1}{5}\right) + g \left(\frac{1}{2} - \frac{\alpha}{5}\right) + h \left(\frac{1}{2} - \frac{1 + \alpha}{5}\right).
$$

Or

$$
(2.10) \quad e = (\alpha - 1)g + \alpha h - \frac{1}{4}, \quad f = \frac{5}{4} - \alpha g - (1 + \alpha)h.
$$

Inserting (2.4), (2.5) and (2.8) in (2.3), we obtain

$$
(2.11) \quad |J_1| \leq C \|u_1\|^\frac{\alpha+1}{2+\alpha} \|\nabla_y \psi_1\|^e \|\nabla_y u_1\|^f \|\omega_1\|^g \|\Lambda_y\alpha u_1\|^c \|\Lambda_y^{1+\alpha} \psi_1\|^h \times \|\Lambda_y^{1+\alpha} u_1\|^\frac{2}{2+\alpha} \|\Lambda_y^0 \omega_1\|^h.
$$

When

$$
\frac{2}{1 + \alpha} + d + h \leq 2,
$$

we apply Young’s inequality with

$$
(2.12) \quad \frac{h}{2} + \frac{1}{1 + \alpha} + \frac{d}{2} + \frac{1}{p} = 1 \quad \text{or} \quad p = \frac{2(\alpha + 1)}{2\alpha - (\alpha + 1)(h + d)}
$$
to obtain
\[
|J_1| \leq \frac{\nu}{2} \| \Lambda_y^\alpha \omega_1 \|_2^2 + \frac{\nu}{2} \| \Lambda_y^{1+\alpha} u_1 \|_2^2 \\
+ C(\nu) \| u_1 \|_2^2 \| \nabla_y \psi_1 \|_2^2 \| \nabla_y \omega_1 \|_2^2 \| \omega_1 \|_2^2 \| \Lambda_y^\alpha u_1 \|_2^2 \| \Lambda_y^{1+\alpha} \psi_1 \|_2^2 ,
\]
where
\[
\gamma_1 = p \left( \frac{\alpha - 1}{\alpha + 1} + a \right), \quad \gamma_2 = p e, \quad \gamma_3 = p b, \quad \gamma_4 = p f, \quad \gamma_5 = p c, \quad \gamma_6 = p g.
\]
When \(\gamma_3 + \gamma_4 \leq 2\) and \(\gamma_5 + \gamma_6 \leq 2\), namely
(2.13) \quad p(b + f) \leq 2 \quad \text{and} \quad p(c + g) \leq 2,
we can apply Young’s inequality again to further bound \(J_1\) by
(2.14) \quad |J_1| \leq \frac{\nu}{2} \| \Lambda_y^\alpha \omega_1 \|_2^2 + \frac{\nu}{2} \| \Lambda_y^{1+\alpha} u_1 \|_2^2 \| \nabla_y \psi_1 \|_2^2 \\
\times \left( \| \nabla_y u_1 \|_2^2 + \| \omega_1 \|_2^2 \right) \left( \| \Lambda_y^\alpha u_1 \|_2^2 + \| \Lambda_y^{1+\alpha} \psi_1 \|_2^2 \right).
Invoking (2.7), (2.10) and (2.12), the conditions in (2.13) can be rewritten as
(2.15) \quad \frac{2(\alpha + 1)}{2\alpha - (\alpha + 1)(d + h)} \left( \frac{5}{2} - \alpha(c + g) - (1 + \alpha)(d + h) \right) \leq 2,
(2.16) \quad \frac{2(\alpha + 1)}{2\alpha - (\alpha + 1)(d + h)} (c + g) \leq 2.
Equivalently,
(2.17) \quad \frac{\alpha + 5}{2\alpha(\alpha + 1)} \leq (c + g) + (d + h) \leq \frac{2\alpha}{\alpha + 1}.
When \(\alpha \geq \frac{5}{4}\),
\[
\frac{\alpha + 5}{2\alpha(\alpha + 1)} \leq \frac{2\alpha}{\alpha + 1}
\]
and we can select suitable \(c, g, d, h\) so that (2.17) holds and thus (2.13) holds. Some special choices of the indices \(a, b, c, d\) and \(e, f, g, h\) are
\[
a = 0, \quad b = \frac{4}{9}, \quad c = \frac{4}{9}, \quad d = \frac{1}{9}, \quad e = 0, \quad f = \frac{4}{9}, \quad g = \frac{4}{9}, \quad h = \frac{1}{9}
\]
in the case \(\alpha = \frac{5}{4}\), and
\[
a = e = 0, \quad b = f = \frac{4\alpha^2 + 3\alpha - 5}{4\alpha(\alpha + 1)}, \quad c = g = \frac{1}{\alpha + 1}, \quad d = h = \frac{5 - 3\alpha}{4\alpha(\alpha + 1)}
\]
in the case of \(\alpha \geq \frac{5}{4}\).

We now bound \(J_2\). By the third equation in (1.5), \(J_2\) can be written as
\[
J_2 = -4 \int u_1 \partial_z u_1 \Delta_y \psi_1 dy.
\]
For any \(p, q \in [1, \infty]\) and \(\frac{1}{p} + \frac{1}{q} = \frac{1}{2}\), we have, by Hölder’s inequality,
(2.18) \quad |J_2| \leq \| u_1 \|_p \| \partial_z u_1 \|_q \| \omega_1 \|_2.
Furthermore, by the Gagliardo-Nirenberg type inequalities
\[
\|u_1\|_p \leq C \|u_1\|_2^{a_1} \|\nabla_y u_1\|_2^{b_1} \|\Lambda^\alpha u_1\|_2^{c_1} \|\Lambda^{1+\alpha} u_1\|_2^{d_1},
\]
(2.19) \[
\|\partial_z u_1\|_q \leq C \|\nabla_y u_1\|_2^{b_2} \|\Lambda^\alpha u_1\|_2^{c_2} \|\Lambda^{1+\alpha} u_1\|_2^{d_2}
\]
with the indices satisfying
\[
a_1 + b_1 + c_1 + d_1 = 1, \quad b_2 + c_2 + d_2 = 1,
\]
\[
\frac{1}{p} = \frac{a_1}{2} + b_1 \left(\frac{1}{2} - \frac{1}{5}\right) + c_1 \left(\frac{1}{2} - \frac{\alpha}{5}\right) + d_1 \left(\frac{1}{2} - \frac{1+\alpha}{5}\right),
\]
\[
\frac{1}{q} - \frac{1}{5} = b_2 \left(\frac{1}{2} - \frac{1}{5}\right) + c_2 \left(\frac{1}{2} - \frac{\alpha}{5}\right) + d_2 \left(\frac{1}{2} - \frac{1+\alpha}{5}\right),
\]
we obtain
\[
\|u_1\|_p \|\partial_z u_1\|_q \leq C \|u_1\|_2^{a_1} \|\nabla_y u_1\|_2^{b_3} \|\Lambda^\alpha u_1\|_2^{c_3} \|\Lambda^{1+\alpha} u_1\|_2^{d_3},
\]
(2.20) where \(b_3 = b_1 + b_2, c_3 = c_1 + c_2\) and \(d_3 = d_1 + d_2\). Clearly
\[
a_1 + b_3 + c_3 + d_3 = 2,
\]
(2.21) \[
\frac{a_1}{2} + b_3 \frac{3}{10} + c_3 \frac{5 - 2\alpha}{10} + d_3 \frac{3 - 2\alpha}{10} = \frac{3}{10}.
\]
(2.22) Inserting (2.20) in (2.18) and applying Young’s inequality, we obtain
\[
|J_2| \leq \frac{\nu}{2} \|\Lambda^{1+\alpha} u_1\|_2^2 + C(\nu) \|u_1\|_2^{2a_1 \frac{2}{2-a_3}} \|\nabla_y u_1\|_2^{2b_3 \frac{2}{2-b_3}} \|\Lambda^\alpha u_1\|_2^{2c_3 \frac{2}{2-c_3}} \|\omega_1\|_2^{\frac{2}{2-c_3}}.
\]
If
\[
\frac{2c_3}{2 - d_3} \leq 2, \quad \frac{2b_3}{2 - d_3} + \frac{2}{2 - d_3} \leq 2,
\]
(2.23) a further application of Young’s inequality implies
\[
|J_2| \leq \frac{\nu}{2} \|\Lambda^{1+\alpha} u_1\|_2^2 + C(\nu) \|u_1\|_2^{2a_1 \frac{2}{2-a_3}} \|\Lambda^\alpha u_1\|_2^{2c_3 \frac{2}{2-c_3}} \left(\|\nabla_y u_1\|_2^2 + \|\omega_1\|_2^2\right).
\]
(2.24) When \(\alpha \geq \frac{5}{4}\), we can choose suitable \(a_1, b_2, c_3\) and \(d_3\) so that they satisfy (2.21), (2.22) and (2.23). In fact, these conditions are equivalent to
\[
a_1 + c_3 = 2 - (b_3 + d_3),
\]
\[
(b_3 + d_3) + \alpha(c_3 + d_3) = \frac{7}{2},
\]
\[
c_3 + d_3 \leq 2, \quad b_3 + d_3 \leq 1
\]
and all of them are obviously satisfied if we set
\[
a_1 = 0, \quad b_3 = 2 - \frac{5}{2\alpha}, \quad c_3 = 1 \quad \text{and} \quad d_3 = \frac{5}{2\alpha} - 1.
\]
Combining (2.2), (2.14) and (2.24), we find that
\[
\frac{d}{dt} \left(\|\nabla_y u_1\|_2^2 + 2\|\omega_1\|_2^2\right) + \nu \left(\|\Lambda^{1+\alpha} u_1\|_2^2 + 2\|\Lambda^\alpha \omega_1\|_2^2\right)
\]
\[
\leq C(\nu) \|u_1\|_2^{2a_1 \frac{2}{2-a_3}} \|\nabla_y u_1\|_2^{2b_3 \frac{2}{2-b_3}} \left(\|\Lambda^\alpha u_1\|_2^2 + \|\Lambda^{1+\alpha} \psi_1\|_2^2\right) \left(\|\nabla_y u_1\|_2^2 + 2\|\omega_1\|_2^2\right)
\]
\[
+ C(\nu) \|u_1\|_2^{2a_1 \frac{2}{2-a_3}} \|\Lambda^\alpha u_1\|_2 \left(\|\nabla_y u_1\|_2^2 + 2\|\omega_1\|_2^2\right).
\]
It then follows from Gronwall’s inequality and (2.1) that
\[
\left( \| \nabla_y u_1 \|^2 + 2 \| \omega_1 \|^2 \right) + \nu \int_0^t \left( \| \Lambda^{1+\alpha} y u_1 \|^2 + 2 \| \Lambda^\alpha y \omega_1 \|^2 \right) \, dt \leq C,
\]
where \( C \) is a constant depending on the norms of the initial data, namely \( \| u_{10} \|_2 + \| \nabla_y u_{10} \|_2, \| \nabla_y \psi_{10} \|_2 \) and \( \| \omega_{10} \|_2 \). When the initial data are more regular, the solution of (1.5) can be shown to be more regular. In particular, smooth data yield smooth solutions. This completes the proof of Theorem 1.1.

\[ \square \]

**Proof of Theorem 1.2.** The proof is similar to the proof of Theorem 1.1. The estimates in the proof of Theorem 1.1 remain valid although the associated indices should be suitably modified. For example, (2.6),(2.7),(2.9),(2.10),(2.15),(2.16) and (2.17) should be changed to (2.25),(2.26),(2.27),(2.28),(2.29),(2.30) and (2.31), respectively, where the new equations are given by

\[
a + b + c + d = 1,
\]

\[
\frac{1}{4} = \frac{a}{2} + b \left( \frac{1}{2} - \frac{1}{n+2} \right) + c \left( \frac{1}{2} - \frac{\alpha}{n+2} \right) + d \left( \frac{1}{2} - \frac{1 + \alpha}{n+2} \right),
\]

\[
a = -\frac{n - 2}{4} + (\alpha - 1)c + \alpha d,
\]

\[
\frac{1}{4} = \frac{e + f + g + h}{4} = 1,
\]

\[
e + f + g + h = 1,
\]

\[
\frac{1}{4} = \frac{e}{2} + f \left( \frac{1}{2} - \frac{1}{n+2} \right) + g \left( \frac{1}{2} - \frac{\alpha}{n+2} \right) + h \left( \frac{1}{2} - \frac{1 + \alpha}{n+2} \right),
\]

\[
e = -\frac{n - 2}{4} + (\alpha - 1)g + \alpha h,
\]

\[
f = \frac{n + 2}{4} - \alpha g - (1 + \alpha)h,
\]

\[
\frac{2(\alpha + 1)}{2\alpha - (\alpha + 1)(d + h)} \cdot \left( \frac{n + 2}{2} - \alpha(c + g) - (1 + \alpha)(d + h) \right) \leq 2,
\]

\[
\frac{2(\alpha + 1)}{2\alpha - (\alpha + 1)(d + h)} (c + g) \leq 2,
\]

\[
\frac{n(\alpha + 1) + 2 - 2\alpha}{2\alpha(\alpha + 1)} \leq (c + g) + (d + h) \leq \frac{2\alpha}{\alpha + 1}.
\]

When \( \alpha \geq \frac{1}{2} + \frac{n}{4} \),

\[
\frac{n(\alpha + 1) + 2 - 2\alpha}{2\alpha(\alpha + 1)} \leq \frac{2\alpha}{\alpha + 1}
\]

and the indices \( a, b, c, d, e, f, g \) and \( h \) can be selected so that all the estimates in the proof of Theorem 1.1 remain valid. A special set of indices is

\[ a = e = 0, b = f = \frac{4\alpha^2 + (6 - n)\alpha - n - 2}{4\alpha(\alpha + 1)}, c = g = \frac{1}{\alpha + 1}, d = h = \frac{(n - 6)\alpha + n + 2}{4\alpha(\alpha + 1)} \]

We omit further details and this completes the proof of Theorem 1.2. \[ \square \]
This section proves the finite-time blowup result presented in Theorem 1.3. We first supply the proof of the local well-posedness of Proposition 1.4.

**Proof of Proposition 1.4.** It suffices to establish local (in time) bounds for \( u \in H^m(\Omega) \) and \( \psi \in H^{m+1}(\Omega) \), where \( \Omega = [0, \infty) \times \mathbb{T} \). As explained in the introduction, the equations in (1.8) can be identified as the equations of axisymmetric functions \((u, \psi)\) in \(\mathbb{R}^4 \times \mathbb{T}\), namely

\[
(3.1) \quad \begin{cases} 
\partial_t u = 2u \partial_z \psi; \\
\partial_t(-\Delta \psi) = \partial_z \left( u^2 \right),
\end{cases}
\]

where \( y = (y_1, y_2, y_3, y_4, z) \in \mathbb{R}^4 \times \mathbb{T} \). Multiplying the first equation in (3.1) by \( 2u \) and the second by \( 2\psi \) and integrating over \( \mathbb{R}^4 \times \mathbb{T} \), we obtain after integration by parts,

\[
\frac{d}{dt} \int_{\mathbb{R}^4 \times \mathbb{T}} (u^2 + 2|\nabla_y \psi|^2) \ dy = \frac{d}{dt} \int_{\Omega} (u^2 + 2\partial_r^2 \psi + 2\partial_z^2 \psi) r^3 \ dr dz = 0.
\]

Therefore,

\[
\int_{\Omega} (u^2 + 2\partial_r^2 \psi + 2\partial_z^2 \psi) r^3 \ dr dz = \int_{\Omega} (u_0^2 + 2\partial_0^2 \psi + 2\partial_0^2 \psi) r^3 \ dr dz.
\]

Let \( \gamma \) be a multi-index with \( |\gamma| \leq m \). Taking the inner product of \( D^\gamma u \) with \( D^\gamma \psi \) of the first equation in (3.1) and of \( D^\gamma \psi \) with \( D^\gamma \psi \) of the second equation, and integrating by parts, we obtain

\[
\frac{d}{dt} \int_{\mathbb{R}^4 \times \mathbb{T}} |D^\gamma (u \partial_\gamma \psi) D^\gamma u| \ dy \leq \| D^\gamma u \|_{L^2} \| D^\gamma (u \partial_\gamma \psi) \|_{L^2}
\]

\[
\leq C \| D^\gamma u \|_{L^2} (\| u \|_{H^m} \| \partial_\gamma \psi \|_{L^\infty} + \| u \|_{L^\infty} \| \psi \|_{H^{m+1}}).
\]

\[
\int_{\mathbb{R}^4 \times \mathbb{T}} |D^\gamma (u^2) D^\gamma \partial_\gamma \psi| \ dy \leq \| D^\gamma \partial_\gamma \psi \|_{L^2} \| D^\gamma (u^2) \|_{L^2}
\]

\[
\leq C \| D^\gamma \nabla_y \psi \|_{L^2} \| u \|_{H^m} \| u \|_{L^\infty}.
\]

Since \( m > \frac{5}{2} \), we have the following Sobolev inequalities

\[
\| u \|_{L^\infty(\mathbb{R}^5)} \leq C \| u \|_{H^m(\mathbb{R}^5)}, \quad \| \partial_\gamma \psi \|_{L^\infty(\mathbb{R}^5)} \leq C \| \psi \|_{H^{m+1}(\mathbb{R}^5)}.
\]

It then follows that

\[
\frac{d}{dt} \int_{\mathbb{R}^4 \times \mathbb{T}} (|D^\gamma u|^2 + |D^\gamma \nabla \psi|^2) \ dy \leq C \left( \| D^\gamma u \|_{L^2} \| u \|_{H^m} \| \psi \|_{H^{m+1}} + \| D^\gamma \nabla_y \psi \|_{L^2} \| u \|_{H^m}^2 \right).
\]

Summing over \( |\gamma| = 1 \) to \( |\gamma| = m \), we obtain

\[
\frac{d}{dt} (\| u \|_{H^m}^2 + \| \psi \|_{H^{m+1}}^2) \leq C \| u \|_{H^m}^2 \| \psi \|_{H^{m+1}}.
\]

This estimate allows us to obtain a local bound for \( \| u \|_{H^m} \) and \( \| \psi \|_{H^{m+1}} \). This completes the proof of Proposition 1.4. \( \square \)
In the proof of Proposition 1.4, we have used the calculus inequality stated in the following lemma. This inequality can be found in [9, p.334]).

**Lemma 3.1.** Let $1 < p < \infty$ and let $s > 0$. Let $J$ denote the differential operator $J = (I - \Delta)^{1/2}$. Then

$$
\|J^s(fg)\|_{L^p} \leq C (\|f\|_{L^{p_1}} \|g\|_{W^{s,p_2}} + \|f\|_{W^{s,p_3}} \|g\|_{L^{p_4}})
$$

where $p_2, p_3 \in (1, \infty)$ and $1/p = 1/p_1 + 1/p_2 = 1/p_3 + 1/p_4$.

We now turn to the proof of Theorem 1.3. It involves a test function $\phi(r,z)$ of the form $\phi(r,z) = \sin(\pi z) f(r)$ with $f$ solving an eigenvalue problem of the following lemma.

**Lemma 3.2.** Let $\lambda > 0$. Then there exists a smooth function $f = f(r)$ on $[0, \infty)$ satisfying

$$(3.2) \quad f'' - \frac{1}{r}f' - \lambda f = 0, \quad r \in (0, \infty),$$

$$(3.3) \quad f(0) = 0, \quad \lim_{r \to 0} rf'(r) = 0.$$

**Proof.** We seek a series solution of the form $f(r) = \sum_{k=0}^{\infty} a_k r^k$. Inserting it in (3.2) yields

$$
-\frac{a_1}{r} - \sum_{k=2}^{\infty} ((k^2 - 2k)a_k - \lambda a_{k-2}) r^{k-2} = 0.
$$

Taking into account of (3.3), we find

$$
a_0 = a_1 = a_{2m-1} = 0, \quad a_{2m} = \frac{\lambda}{4m(m-1)} a_{2m-2}, \quad m = 2, 3, 4, \cdots.
$$

Thus, the solution is given by

$$(3.4) \quad f(r) = a \sum_{m=1}^{\infty} \frac{\lambda^{m-1}}{4^{m-1}m!(m-1)!} r^{2m} \quad \text{with} \quad a > 0.
$$

Clearly the series given by (3.4) converges for all $r \in [0, \infty)$ and is smooth. \qed

We are now ready to prove Theorem 1.3.

**Proof of Theorem 1.3.** According to Proposition 1.4, there exists $T > 0$ such that (1.8) has a unique local solution $(u, \psi)$ on $[0, T)$ satisfying

$$(3.5) \quad u \in L^\infty([0, T); H^m(\Omega)), \quad \psi \in L^\infty([0, T); H^{m+1}(\Omega)).$$

We prove the finite-time singularity by contradiction. Suppose that $(u, \psi)$ remains in the regularity class for all time. Then all manipulations below are justified.

Multiplying the second equation in (1.8) by $r \partial_z \phi$ and integrating over $\Omega$, we have

$$
(3.6) \quad \partial_t \iint \partial_z \phi \left( -\partial_r \psi - \frac{3}{r} \partial_r \psi - \partial_{zz} \psi \right) r dr dz = \iint \partial_z \phi \partial_z (u^2) r dr dz.
$$
For the sake of clarity, we divide the integral on the left into two parts. Integrating by parts and applying the boundary conditions on $\psi$ and $\phi$, we obtain

$$\int \int (-\partial_{rr}\psi) \partial_z \phi \, r \, dr \, dz = \int \int \partial_{zrr} \psi \, \phi \, r \, dr \, dz$$

$$= - \int \int \partial_{zr} \psi (\phi + r \partial_r \phi) \, dr \, dz$$

$$= \int \int \partial_z \psi (2 \partial_r \phi + r \partial_{rr} \phi) \, dr \, dz.$$

All boundary terms vanish due to the boundary conditions on $\psi$ and $\phi$. Similarly,

$$\int \int \partial_z \phi \left( -\frac{3}{r} \partial_r \psi - \partial_{zz} \psi \right) \, r \, dr \, dz = \int \int \partial_z \psi (-3 \partial_r \phi + r \partial_{zz} \phi) \, dr \, dz.$$

Inserting these equations in (3.6) and integrating by parts for the term on the right of (3.6), we have

$$\partial_t \int \int \partial_z \psi \left( \partial_{rr} \phi - \frac{1}{r} \partial_r \phi + \partial_{zz} \phi \right) \, r \, dr \, dz = - \int \int u^2 \partial_{zz} \phi \, r \, dr \, dz.$$

Invoking the following properties for $\phi$

$$\partial_{rr} \phi - \frac{1}{r} \partial_r \phi = \lambda \phi, \quad \partial_{zz} \phi = -\pi^2 \phi,$$

we find

(3.7) $$\partial_t \int \int \phi \partial_z \psi \, r \, dr \, dz = \frac{\pi^2}{\lambda - \pi^2} \int \int \phi u^2 \, r \, dr \, dz.$$

Multiplying the first equation in (1.8) by $\phi \, r$ and integrating on $\Omega$, we have

(3.8) $$\partial_t \int \int \phi \ln(u^2) \, r \, dr \, dz = 4 \int \int \phi \partial_z \psi \, r \, dr \, dz.$$

Combining (3.7) and (3.8), we obtain

$$\partial_{tt} \int \int \phi \ln(u^2) \, r \, dr \, dz = \frac{4\pi^2}{\lambda - \pi^2} \int \int \phi u^2 \, r \, dr \, dz.$$

Integrating twice in time, we have

(3.9) $$\int \int \phi \ln(u^2) \, r \, dr \, dz = \frac{4\pi^2}{\lambda - \pi^2} \int_0^t \int_0^s \int \int \phi u^2 \, r \, dr \, dz \, d\tau \, ds + A \, t + B.$$

Using the fact that $\phi \geq 0$ in $\Omega$, we find that

(3.10) $$\int \int \phi \ln(u^2) \, r \, dr \, dz \leq \int \int \phi \ln^+(u^2) \, r \, dr \, dz \leq \int \int \phi u^2 \, r \, dr \, dz.$$

Therefore, if we set

(3.11) $$F(t) = \frac{4\pi^2}{\lambda - \pi^2} \int_0^t \int_0^s \int \int \phi u^2 \, r \, dr \, dz \, d\tau \, ds + A \, t + B,$$

we then combine (3.9) and (3.10) to obtain

$$F''(t) \geq \frac{4\pi^2}{\lambda - \pi^2} F(t).$$
Noticing that $F'(t) > 0$, multiplying by $F'(t)$ and integrating in time, we obtain

$$F'(t) \geq \left( \frac{8\pi^2}{3(\lambda - \pi^2)} F^3(t) + A^2 - \frac{8\pi^2B^3}{3(\lambda - \pi^2)} \right)^{1/2}$$

Since $F'(t) > 0$, $F(t)$ is an increasing function $t \geq 0$. If $\lim_{t \to \infty} F(t) = C_0 < \infty$ then

$$\int_B^{F(t)} \frac{dF}{\sqrt{\frac{8\pi^2}{3(\lambda - \pi^2)} F^3(t) + A^2 - \frac{8\pi^2B^3}{3(\lambda - \pi^2)}}} = t.$$

Letting $t \to \infty$ in the equation above would yield a contradiction since the integral on the left remains bound. Let $T^*$ be the blowup time. It then follows from (3.11) that

$$\lim_{t \to T^*} \iint_\Omega \phi u^2 r dr dz = \infty.$$

This completes the proof of Theorem 1.3. □

ACKNOWLEDGEMENTS

This work is partially supported by NSF grant DMS 0907913 and by a Foundation at Oklahoma State University.

REFERENCES


DEPARTMENT OF MATHEMATICS, OKLAHOMA STATE UNIVERSITY, 401 MATHEMATICAL SCIENCES, STILLWATER, OK 74078, USA.
E-mail address: ltao@math.okstate.edu
E-mail address: jiahong@math.okstate.edu