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MAXIMUM PRINCIPLE FOR INFINITE-HORIZON OPTIMAL CONTROL PROBLEMS WITH DOMINATING DISCOUNT

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Abstract. The paper revisits the issue of necessary optimality conditions for infinite-horizon optimal control problems. It is proved that the normal form maximum principle holds with an explicitly specified adjoint variable if an appropriate relation between the discount rate, the growth rate of the solution and the growth rate of the objective function is satisfied. The main novelty is that the result applies to general non-stationary systems and the optimal objective value needs not be finite (in which case the concept of overtaking optimality is employed). In two important particular cases it is shown that the current-value adjoint variable is characterized as the unique bounded solution of the adjoint equation. The results in this paper are applicable to several economic models for which the known optimality conditions fail.

Keywords: infinite horizon, Pontryagin maximum principle, transversality conditions.

AMS (MOS) subject classification: 49J15, 49K15, 91B62

1 Introduction

Infinite-horizon optimal control problems arise in many fields of economics, in particular in models of economic growth. Typically, the initial state is fixed and the terminal state (at infinity) is free in such problems, while the utility functional to be maximized is defined as an improper integral of the discounted instantaneous utility on the time interval $[0, \infty)$. The last circumstance gives rise to specific mathematical features of the problems. To be specific, let $x_*(\cdot)$ be an optimal trajectory and $(\psi^0, \psi(\cdot))$ be a pair of adjoint variables corresponding to $x_*(\cdot)$ according to the Pontryagin maximum principle. Although the state at infinity is not constrained, such problems could be abnormal (i.e. $\psi^0 = 0$) and the "standard" transversality conditions of the form $\lim_{t\to\infty} \psi(t) = 0$ or $\lim_{t\to\infty} \langle \psi(t), x_*(t)\rangle = 0$ may fail. Examples demonstrating pathologies of these types are well known (see [4, 7, 13, 15, 18]).

In the end of 1970s it was suggested in [6] that a normal form version of the Pontryagin maximum principle involving even a stronger "transversality" condition holds if the discount rate ρ is sufficiently large, and this condition provides bounds (in appropriate L-space) for $\psi(\cdot)$ rather only the asymptotics at infinity. Such a stronger "transversality" condition was proved in [6] for linear autonomous control systems.

Recently, the result in [6] was extended in [3, 4] for nonlinear autonomous systems. Moreover, it is proved in [3, 4] that if the discount rate ρ is sufficiently large then the adjoint variable $\psi(\cdot)$ that satisfies the conditions of the maximum principle admits an explicit representation similar to the classical Cauchy formula for the solutions of systems of linear differential equations. In the linear case this Cauchy-type representation of $\psi(\cdot)$ implies the integral "transversality" condition suggested in [6] and an even stronger exponential pointwise estimate for $\psi(\cdot)$ (see [4, 5] for more details).

The requirement for the discount rate $\rho \geq 0$ to be sufficiently large is expressed in [3, 4, 6] in the form of the following inequality:

$$\rho > (r + 1)\lambda, \tag{1}$$

where $r \geq 0$ and $\lambda \in R^1$ are parameters characterizing the growth of the instantaneous utility and trajectories of the control system, respectively (see [3, 4, 6] for precise definitions of the parameters r and λ). Condition (1) requires that the discount factor ρ "dominates" the parameters r and λ. In the sequel we refer to conditions of this type as *dominating discount* conditions.

Note that the approaches used in [6] and [3, 4] for establishing additional characterizations of the adjoint variable $\psi(\cdot)$ are different. The approach used in [6] is based on methods of functional and non-smooth analysis. This approach essentially exploits the linearity of the control system under consideration. The method of finite-horizon approximations used in [3, 4] is based on an appropriate *regularization* of the infinite-horizon problem, namely on an explicit approximation by a family of standard finite-horizon problems. Then optimality conditions for the infinite-horizon problem are obtained by taking the limit in the conditions of the maximum principle for the approximating problems with respect to the perturbation parameters. As it is demonstrated in [3, 4] this approach is applicable to a broad class of nonlinear control systems, however sometimes under rather restrictive assumptions of boundedness and convexity akin to the perturbation methodology, in general.

The contribution of the present paper is twofold. First we extend the version of the Pontryagin maximum principle for infinite-horizon optimal control problems with dominating discount established in [3, 4] to a more general class of non-autonomous problems and relax the assumptions. Second, we adopt the classical needle variations technique [17] to the case of infinite-horizon problems. Thus, the approach in present paper essentially differs from the ones used in [3, 4, 6]. The needle variations technique is a standard tool in the optimal control theory. The advantage of this technique is that as a rule it produces (if applicable) the most general versions of the Pontryagin maximum principle. Nevertheless, application of needle variations technique is not so straightforward in the case of infinite-horizon problems (see discussion of some difficulties that arise in Chapter 4 in [17]).

Another important feature of our main result is that it is applicable also for problems where the objective value may be infinite. In this case the notion of overtaking optimality is adapted (see [7]). In contrast to the known results, the maximum principle that we obtain has a normal form, that is, the multiplier of the objective function in the associated Hamiltonian can be taken equal to one.

In addition, we mention that the approach presented in this paper seems to be appropriate also for obtaining transversality conditions for optimal control problems on infinite horizon for distributed parameter systems – a challenging issue that is yet undeveloped (although such conditions are used in a number of economic papers without strict proofs). This is demonstrated in [10] for a very specific distributed problem with age-structure.

The paper is organized as follows. In Section 2 we state the problem and formulate our main result. In Section 3 we present the proof. In Sections 4 and 5 we elaborate the main result in two important particular cases: for systems with one-sided Lipschitz dynamics and for systems with regular linearization. An illustrative economic example is given in Section 6.

2 Statement of the problem and main result

Let G be a nonempty open convex subset of R^n and U be an arbitrary nonempty set in R^m. Let

$$f : [0,\infty) \times G \times U \mapsto R^n \quad \text{and} \quad g : [0,\infty) \times G \times U \mapsto R^1.$$

Consider the following optimal control problem (P):

$$J(x(\cdot), u(\cdot)) = \int_0^\infty e^{-\rho t} g(t, x(t), u(t))\, dt \to \max\,, \tag{2}$$

$$\dot{x}(t) = f(t, x(t), u(t)), \qquad x(0) = x_0, \tag{3}$$

$$u(t) \in U. \tag{4}$$

Here $x_0 \in G$ is a given initial state of the system and $\rho \in R^1$ is a "discount" rate (which could be even negative).

Assumption (A1): *The functions* $f : [0,\infty) \times G \times U \mapsto R^n$ *and* $g : [0,\infty) \times G \times U \mapsto R^1$ *together with their partial derivatives* $f_x(\cdot,\cdot,\cdot)$ *and* $g_x(\cdot,\cdot,\cdot)$ *are continuous in* (x, u) *on* $G \times U$ *for any fixed* $t \in [0,\infty)$*, and measurable and locally bounded in* t*, uniformly in* (x, u) *in any bounded set.* [1]

In what follows we assume that the class of *admissible controls* in problem (P) consists of all measurable locally bounded functions $u : [0,\infty) \mapsto U$. Then for any initial state $x_0 \in G$ and any admissible control $u(\cdot)$ plugged in the right-hand side of the control system (3) we obtain the following Cauchy problem:

$$\dot{x}(t) = f(t, x(t), u(t)), \qquad x(0) = x_0. \tag{5}$$

Due to assumption $(A1)$ this problem has a unique solution $x(\cdot)$ (in the sense of Carathéodory) which is defined on some time interval $[0, \tau]$ with $\tau > 0$ and takes values in G (see e.g.

[1] The local boundedness of these functions of t, x and u (take $\phi(\cdot,\cdot,\cdot)$ as a representative) means that for every $T > 0$ and for every bounded set $Z \subset G \times U$ there exists M such that $\|\phi(t, x, u)\| \le M$ for every $t \in [0, T]$ and $(x, u) \in Z$.

[11]). This solution is uniquely extendible to a maximal interval of existence in G and is called *admissible trajectory* corresponding to the admissible control $u(\cdot)$.

If $u(\cdot)$ is an admissible control and the corresponding admissible trajectory $x(\cdot)$ exists on $[0,T]$ in G, then the integral

$$J_T(x(\cdot),u(\cdot)) := \int_0^T e^{-\rho t} g(t,x(t),u(t))\,dt$$

is finite. This follows from (A1), the definition of admissible control and the existence of $x(\cdot)$ on $[0,T]$.

We will use the following modification of the notion of weakly overtaking optimal control (see [7]).

Definition 1: *An admissible control $u_*(\cdot)$ for which the corresponding trajectory $x_*(\cdot)$ exists on $[0,\infty)$ is locally weakly overtaking optimal (LWOO) if there exists $\delta > 0$ such that for any admissible control $u(\cdot)$ satisfying*

$$\mathrm{meas}\,\{t \geq 0 : u(t) \neq u_*(t)\} \leq \delta$$

and for every $\varepsilon > 0$ and $T > 0$ one can find $T' \geq T$ such that the corresponding admissible trajectory $x(\cdot)$ is either non-extendible to $[0,T']$ in G or

$$J_{T'}(x_*(\cdot),u_*(\cdot)) \geq J_{T'}(x(\cdot),u(\cdot)) - \varepsilon.$$

Notice that the expression $d(u(\cdot),u_*(\cdot)) = \mathrm{meas}\,\{t \in [0,T] : u(t) \neq u_*(t)\}$ generates a metric in the space of the measurable functions on $[0,T]$, $T > 0$, which is suitable to use in the framework of the needle variations technique (see [2]).

In the sequel we denote by $u_*(\cdot)$ an LWOO control and by $x_*(\cdot)$ – the corresponding trajectory.

Assumption (A2): *There exist numbers $\mu \geq 0$, $r \geq 0$, $\kappa \geq 0$, $\beta > 0$ and $c_1 \geq 0$ such that for every $t \geq 0$*

(i) $\|x_*(t)\| \leq c_1 e^{\mu t}$;

(ii) for every admissible control $u(\cdot)$ for which $d(u(\cdot),u_*(\cdot)) \leq \beta$ the corresponding trajectory $x(\cdot)$ exists on $[0,\infty)$ in G and it holds that

$$\|g_x(t,y,u_*(t))\| \leq \kappa\,(1+\|y\|^r) \qquad \text{for every } \; y \in \mathrm{co}\,\{x(t),x_*(t)\}.$$

Assumption (A3): *There are numbers* $\lambda \in R^1$, $\gamma > 0$ *and* $c_2 \geq 0$ *such that for every* $\zeta \in G$ *with* $\|\zeta - x_0\| < \gamma$ *equation (5) with* $u(\cdot) = u_*(\cdot)$ *and initial condition* $x(0) = \zeta$ *(instead of* $x(0) = x_0$*) has a solution* $x(\zeta; \cdot)$ *on* $[0, \infty)$ *in* G *and*

$$\|x(\zeta; t) - x_*(t)\| \leq c_2 \|\zeta - x_0\| \, e^{\lambda t}.$$

Finally, we introduce the following *dominating discount* condition.

Assumption (A4):

$$\rho > \lambda + r \max\{\lambda, \mu\}.$$

For an arbitrary $\tau \geq 0$ consider the following linear differential equation (the linearization of (5) along $(x_*(\cdot), u_*(\cdot))$):

$$\dot{y}(t) = f_x(t, x_*(t), u_*(t)) y(t), \quad t \geq 0 \tag{6}$$

with initial condition

$$y(\tau) = y_0. \tag{7}$$

Due to assumption $(A1)$ the partial derivative $f_x(\cdot, x_*(\cdot), u_*(\cdot))$ is measurable and locally bounded. Hence, there is a unique (Carathéodory) solution $y_*(\cdot)$ of the Cauchy problem (6), (7) which is defined on the whole time interval $[0, \infty)$. Moreover,

$$y_*(t) = K_*(t, \tau) y_*(\tau), \quad t \geq 0, \tag{8}$$

where $K_*(\cdot, \cdot)$ is the Cauchy matrix of differential system (6) (see [14]). Recall that

$$K_*(t, \tau) = Y_*(t) Y_*^{-1}(\tau), \quad t, \tau \geq 0,$$

where $Y_*(\cdot)$ is the fundamental matrix solution of (6) normalized at $t = 0$. This means that the columns $y_i(\cdot)$, $i = 1, \ldots, n$, of the $n \times n$ matrix function $Y_*(\cdot)$ are (linearly independent) solutions of (6) on $[0, \infty)$ that satisfy the initial conditions

$$y_i^j(0) = \delta_{i,j}, \quad i, j = 1, \ldots, n,$$

where

$$\delta_{i,i} = 1, \quad i = 1, \ldots, n, \quad \text{and} \quad \delta_{i,j} = 0, \quad i \neq j, \quad i, j = 1, \ldots, n.$$

Analogously, consider the fundamental matrix solution $Z_*(\cdot)$ (normalized at $t = 0$) of the linear adjoint equation

$$\dot{z}(t) = -\left[f_x(t, x_*(t), u_*(t))\right]^* z(t). \tag{9}$$

Then $Z_*^{-1}(t) = [Y_*(t)]^*$, $t \geq 0$.

Define the normal-form Hamilton-Pontryagin function $\mathcal{H} : [0, \infty) \times G \times U \times R^n \mapsto R^1$ for problem (P) in the usual way:

$$\mathcal{H}(t, x, u, \psi) = e^{\rho t} g(t, x, u) + \langle f(t, x, u), \psi \rangle, \quad t \in [0, \infty),\ x \in G,\ u \in U,\ \psi \in R^n.$$

The following theorem presents the main result of the paper – a version of the Pontryagin maximum principle for non-autonomous infinite-horizon problems with dominating discount.

Theorem 1. *Assume that (A1)–(A4) hold. Let $u_*(\cdot)$ be an admissible LWOO control and let $x_*(\cdot)$ be the corresponding trajectory. Then*

(i) *For any $t \geq 0$ the integral*

$$I_*(t) = \int_t^\infty e^{-\rho s} [Z_*(s)]^{-1} g_x(s, x_*(s), u_*(s))\, ds \tag{10}$$

converges absolutely.

(ii) *The vector function $\psi : [0, \infty) \mapsto R^n$ defined by*

$$\psi(t) = Z_*(t) I_*(t), \quad t \geq 0 \tag{11}$$

is (locally) absolutely continuous and satisfies the conditions of the normal form maximum principle, i.e. $\psi(\cdot)$ is a solution of the adjoint system

$$\dot{\psi}(t) = -\mathcal{H}_x(t, x_*(t), u_*(t), \psi(t)) \tag{12}$$

and the maximum condition holds:

$$\mathcal{H}(t, x_*(t), u_*(t), \psi(t)) \overset{a.e.}{=} \sup_{u \in U} \mathcal{H}(t, x_*(t), u, \psi(t)). \tag{13}$$

3 Proof of the main result

First we shall prove that

$$\|Y_*(t)\| \leq c_2 e^{\lambda t}, \quad t \geq 0, \tag{14}$$

where c_2 is the constant in assumption $(A3)$. Indeed due to $(A3)$ for any $\zeta_i \in R^n$: $\zeta_i^j = \delta_{i,j}$, $i, j = 1, \ldots n$, and for all $\alpha \in (0, \gamma)$ we have

$$\|x(x_0 + \alpha \zeta_i; t) - x_*(t)\| \leq \alpha c_2 e^{\lambda t}. \tag{15}$$

Due to the theorem on differentiation of the solution of a differential equation with respect to the initial conditions (see e.g. Chapter 2.5.6 in [1]) we get the following equality

$$x(x_0 + \alpha \zeta_i; t) = x_*(t) + \alpha y_i(t) + \mathrm{o}_i(\alpha, t), \quad i = 1, \ldots, n, \quad t \geq 0.$$

Here the vector functions $y_i(\cdot)$, $i = 1, \dots, n$ are columns of $Y_*(\cdot)$ and for any $i = 1, \dots, n$ we have $\|\mathrm{o}_i(\alpha, t)\|/\alpha \to 0$ as $\alpha \to 0$, uniformly on any finite time interval $[0, T]$, $T > 0$. Then in view of (15) we get

$$\left\| y_i(t) + \frac{\mathrm{o}_i(\alpha, t)}{\alpha} \right\| \leq c_2 e^{\lambda t}, \quad i = 1, \dots, n, \quad t \geq 0.$$

Passing to the limit with $\alpha \to 0$ in the last inequality for an arbitrarily fixed $t \geq 0$ and $i = 1, \dots, n$ we get

$$\|y_i(t)\| \leq c_2 e^{\lambda t}, \quad i = 1, \dots, n, \quad t \geq 0.$$

This implies (14).

Consider integral (10):

$$I_*(t) = \int_t^\infty e^{-\rho s} [Z_*(s)]^{-1} g_x(s, x_*(s), u_*(s))\, ds, \quad t \geq 0.$$

Due to (14) we have

$$\left\| [Z_*(s)]^{-1} \right\| = \| [Y_*(s)]^* \| = \| Y_*(s) \| \leq c_2 e^{\lambda s}, \quad s \geq 0. \tag{16}$$

Further, due to $(A2)(ii)$ applied to $y = x_*(t)$ and $(A2)(i)$ we have

$$\|g_x(s, x_*(s), u_*(s))\| \leq \kappa \left(1 + \|x_*(s)\|^r\right) \leq \kappa \left(1 + (c_1 e^{\mu s})^r\right), \quad s \geq 0,$$

Combining this inequality with (16) we get the following estimate for the integrand in (10):

$$\begin{aligned} \left\| e^{-\rho s} [Z_*(s)]^{-1} g_x(s, x_*(s), u_*(s)) \right\| &\stackrel{a.e.}{\leq} e^{-\rho s} c_2 e^{\lambda s} \kappa \left(1 + c_1^r e^{\mu r s}\right) \\ &\leq c_3\, e^{-(\rho - \lambda - r\mu) s}, \quad s \geq 0, \end{aligned}$$

where $c_3 \geq 0$ is a constant.

Then due to the dominating discount condition $(A4)$ we get that for any $t \geq 0$ the improper integral (10) converges absolutely. This proves claim (i) of Theorem 1.

Further, due to the proved properties of the integral $I_*(\cdot)$ and the definition of the matrix function $Z_*(\cdot)$, the vector function $\psi : [0, \infty) \mapsto R^n$ defined by (11) is locally absolutely continuous and by direct differentiation we get that it satisfies the adjoint system (12). The first assertion in (ii) is proved.

Now let us prove the maximum condition (13) by using a simple needle variation of the control $u_*(\cdot)$ [17].

Let us fix an arbitrary $v \in V$. Denote by $\Omega(v)$ the set of all $\tau > 0$ which are Lebesgue points of each of the measurable functions $f(\cdot, x_*(\cdot), u_*(\cdot))$, $g(\cdot, x_*(\cdot), u_*(\cdot))$, $f(\cdot, x_*(\cdot), v)$, $g(\cdot, x_*(\cdot), v)$. This means (see [16]) that for every $\tau \in \Omega(v)$ and each of these functions of t (take $\varphi(\cdot)$ as a representative)

$$\lim_{\alpha \to 0} \frac{1}{\alpha} \int_{\tau - \alpha}^{\tau} \varphi(t)\, dt = \varphi(\tau).$$

Note that almost every $\tau \in [0,\infty)$ belongs to $\Omega(v)$.

Let us fix an arbitrary $\tau \in \Omega(v)$. For any $0 < \alpha \le \tau$ define

$$u_\alpha(t) = \begin{cases} u_*(t), & t \notin (\tau-\alpha,\tau], \\ v, & t \in (\tau-\alpha,\tau]. \end{cases}$$

If $\alpha < \beta$ (see $(A2)$) then meas $\{t \ge 0 : u_\alpha(t) \ne u_*(t)\} < \beta$. Then according to the first part of $(A2)(ii)$ the trajectory $x_\alpha(\cdot)$ corresponding to $u_\alpha(\cdot)$ exists on $[0,\infty)$.

Due to $\tau \in \Omega(v)$ and $(A1)$ we have

$$x_\alpha(\tau) - x_*(\tau) = \alpha\,[f(\tau,x_*(\tau),v) - f(\tau,x_*(\tau),u_*(\tau))] + o_1(\alpha),$$

where here and further $\mathrm{o}_i(\alpha)$ denotes a function of α that satisfies $\mathrm{o}_i(\alpha)/\alpha \to 0$ as $\alpha \to 0$. Note that $\mathrm{o}_i(\alpha)$ may depend on v and τ (which are fixed in the present consideration), but not on t, unless this is explicitly indicated in the notation.

Let $y(\cdot)$ be the solution on the time interval $[0,\infty)$ of the Cauchy problem

$$\dot{y}(t) = f_x(t,x_*(t),u_*(t))\,y(t), \tag{17}$$

with the initial condition

$$y(\tau) = f(\tau,x_*(\tau),v) - f(\tau,x_*(\tau),u_*(\tau)). \tag{18}$$

Then

$$x_\alpha(\tau) = x_*(\tau) + \alpha y(\tau) + \mathrm{o}_1(\alpha)$$

and

$$x_\alpha(t) = x_*(t) + \alpha y(t) + \mathrm{o}_2(\alpha,t), \quad t \ge \tau, \tag{19}$$

where for arbitrary $T > \tau$ we have $\mathrm{o}_2(\alpha,t)/\alpha \to 0$ as $\alpha \to 0$, uniformly with respect to $t \in [\tau,T]$.

Let us prove that the following estimate holds:

$$\left\| \frac{\mathrm{o}_2(\alpha,t)}{\alpha} \right\| \le c_4 e^{\lambda t}, \quad t \ge \tau, \tag{20}$$

where constant c_4 is independent of α and t.

Consider on $[0,\tau]$ the Cauchy problem

$$\dot{x}(t) = f(t,x(t),u_*(t)), \qquad x(\tau) = x_\alpha(\tau). \tag{21}$$

Due to the theorems on continuous dependence and differentiability of the solution of a differential equation with respect to the initial conditions (see e.g. chapters 2.5.5 and 2.5.6 in [1]) for all sufficiently small $\alpha > 0$ the solution $\tilde{x}_\alpha(\cdot)$ of the Cauchy problem (21) is defined on $[0,\tau]$, $\|\tilde{x}_\alpha(0) - x_0\| < \gamma$ (see $(A3)$), and

$$\tilde{x}_\alpha(0) = x_0 + \alpha y(0) + \mathrm{o}_3(\alpha), \tag{22}$$

where $y(\cdot)$ is the solution of the Cauchy problem (17), (18).

Let us extend $\tilde{x}_\alpha(\cdot)$ on $[\tau,\infty)$ as $\tilde{x}_\alpha(t)=x_\alpha(t)$ for $t>\tau$. Then $\tilde{x}_\alpha(\cdot)$ is the solution of the Cauchy problem

$$\dot{x}(t)=f(t,x(t),u_*(t)),\qquad x(0)=\tilde{x}_\alpha(0).$$

Due to $(A3)$ and (22) we have

$$\|\tilde{x}_\alpha(t)-x_*(t)\|\le c_2\|\alpha y(0)+\mathrm{o}_3(\alpha)\|e^{\lambda t},\quad t\in[0,\infty).$$

Then from (19) we obtain

$$\left\|y(t)+\frac{\mathrm{o}_2(\alpha,t)}{\alpha}\right\|\le c_2\left\|y(0)+\frac{\mathrm{o}_3(\alpha)}{\alpha}\right\|e^{\lambda t}\le c_5e^{\lambda t},\quad t\ge\tau, \tag{23}$$

where c_5 is independent of α and t. From this, due to (14) we get

$$\left\|\frac{\mathrm{o}_2(\alpha,t)}{\alpha}\right\|\le\|y(t)\|+\left\|y(t)+\frac{\mathrm{o}_2(\alpha,t)}{\alpha}\right\|\le c_4e^{\lambda t},\quad t\ge\tau,$$

where c_4 is independent of α and t. Estimate (20) is proved.

Since the admissible trajectory $x_\alpha(\cdot)$ exists on $[0,\infty)$ for all sufficiently small $\alpha>0$, Definition 1 implies the following: there exists $\delta>0$ such that for all sufficiently small $\alpha\in(0,\delta)$, $\varepsilon=\alpha^2$ and any natural number k one can find $T_k'=T_k'(\alpha)\ge k$ such that

$$J_{T_k'}(x_\alpha(\cdot),u_\alpha(\cdot))-J_{T_k'}(x_*(\cdot),u_*(\cdot))\le\alpha^2.$$

Hence, using that $\tau\in\Omega(v)$ and (19), we have

$$\begin{aligned}
&J_{T_k'}(x_\alpha(\cdot),u_\alpha(\cdot))-J_{T_k'}(x_*(\cdot),u_*(\cdot))\\
&=\int_{\tau-\alpha}^{\tau}e^{-\rho t}\left[g(t,x_\alpha(t),v)-g(t,x_*(t),u_*(t))\right]dt\\
&+\int_{\tau}^{T_k'}e^{-\rho t}\left[g(t,x_\alpha(t),u_*(t))-g(t,x_*(t),u_*(t))\right]dt\\
&=\alpha e^{-\rho\tau}\left[g(\tau,x_*(\tau),v)-g(\tau,x_*(\tau),u_*(\tau))\right]+\mathrm{o}_4(\alpha)\\
&+\alpha\int_{\tau}^{T_k'}e^{-\rho t}\left\langle\int_0^1 g_x(t,x_*(t)+s(x_\alpha(t)-x_*(t)),u_*(t))\,ds,\;y(t)\right\rangle dt\\
&+\int_{\tau}^{T_k'}e^{-\rho t}\left\langle\int_0^1 g_x(t,x_*(t)+s(x_\alpha(t)-x_*(t)),u_*(t))\,ds,\;\mathrm{o}_2(\alpha,t)\right\rangle dt\le\alpha^2.
\end{aligned}\tag{24}$$

Consider the integrals

$$I_{1,k}(\alpha) = \int_\tau^{T'_k} e^{-\rho t} \left\langle \int_0^1 g_x(t, x_*(t) + s(x_\alpha(t) - x_*(t)), u_*(t))\, ds,\ y(t) \right\rangle dt$$

and

$$I_{2,k}(\alpha) = \int_\tau^{T'_k} e^{-\rho t} \left\langle \int_0^1 g_x(t, x_*(t) + s(x_\alpha(t) - x_*(t)), u_*(t))\, ds, \mathrm{o}_2(\alpha, t) \right\rangle dt$$

in (24).

Using (8) we obtain that

$$I_{1,k}(\alpha)$$

$$= \left\langle \int_\tau^{T'_k} e^{-\rho t} \, [K_*(t,\tau)]^* \int_0^1 g_x(t, x_*(t) + s(x_\alpha(t) - x_*(t)), u_*(t))\, ds\, dt,\ y(\tau) \right\rangle$$

$$= \left\langle Z_*(\tau) \int_\tau^{T'_k} e^{-\rho t} \, [Z_*(t)]^{-1} \int_0^1 g_x(t, x_*(t) + s(x_\alpha(t) - x_*(t)), u_*(t))\, ds\, dt,\ y(\tau) \right\rangle,$$

where due to $(A2)$, (16) and (23) the integrand in the outer integral can be estimated as follows:

$$\left\| e^{-\rho t} \, [Z_*(t)]^{-1} \int_0^1 g_x(t, x_*(t) + s(x_\alpha(t) - x_*(t)), u_*(t))\, ds \right\|$$

$$\le e^{-\rho t} c_2 e^{\lambda t} \kappa \left(1 + (\|x_*(t)\| + \|\alpha y(t) + \mathrm{o}_2(\alpha, t)\|)^r\right)$$

$$\le e^{-\rho t} c_2 e^{\lambda t} \kappa \left(1 + \left(c_1 e^{\mu t} + \alpha c_5 e^{\lambda t}\right)^r\right)$$

$$\le c_6 e^{-(\rho - \lambda) t} e^{r \max\{\lambda, \mu\} t}. \tag{25}$$

Here the constant c_6 does not depend on α and k. According to $(A4)$ the right-hand side is bounded. Hence taking a limit in the outer integral as $k \to \infty$ we get

$$\lim_{k\to\infty} \int_\tau^{T'_k} \left\{ e^{-\rho t} \, [Z_*(t)]^{-1} \int_0^1 g_x(t, x_*(t) + s(x_\alpha(t) - x_*(t)), u_*(t))\, ds \right\} dt$$

$$= \int_\tau^{\infty} \left\{ e^{-\rho t} \, [Z_*(t)]^{-1} \int_0^1 g_x(t, x_*(t) + s(x^v_{\alpha,\tau}(t) - x_*(t)), u_*(t))\, ds \right\} dt.$$

Hence, there is a limit

$$I_1(\alpha) = \lim_{k\to\infty} I_{1,k}(\alpha)$$

$$= \left\langle Z_*(\tau) \int_\tau^{\infty} e^{-\rho t} \, [Z_*(t)]^{-1} \int_0^1 g_x(t, x_*(t) + s(x_\alpha(t) - x_*(t)), u_*(t))\, ds\, dt,\ y(\tau) \right\rangle.$$

As shown above, the integrand in the outer integral is uniformly bounded by an integrable exponent. Then the limit in $\alpha \to 0$ also exists:

$$\begin{aligned}
\lim_{\alpha\to 0}\lim_{k\to\infty} I_{1,k}(\alpha) &= \lim_{\alpha\to 0} I_1(\alpha) \\
&= \left\langle Z_*(\tau)\int_\tau^\infty \left\{e^{-\rho t}\,[Z_*(t)]^{-1}\,g_x(t,x_*(t),u_*(t))\right\}\,dt,\; y(\tau)\right\rangle \\
&= \langle \psi(\tau), f(\tau,x_*(\tau),v) - f(\tau,x_*(\tau),u_*(\tau))\rangle\,. \qquad (26)
\end{aligned}$$

For the second integral in (24) we have

$$\frac{I_{2,k}(\alpha)}{\alpha} = \int_\tau^{T'_k} e^{-\rho t}\left\langle \int_0^1 g_x(t,x_*(t)+s(x_\alpha(t)-x_*(t)),u_*(t))\,ds,\,\frac{o_2(\alpha,t)}{\alpha}\right\rangle\,dt.$$

Here due to $(A2)$, (23) and (20) we obtain, in a similar way as (25), that the integrand is bounded by an integrable exponent. Hence, there is a limit

$$\lim_{k\to\infty}\frac{I_{2,k}(\alpha)}{\alpha} = \int_\tau^{\infty} e^{-\rho t}\left\langle \int_0^1 g_x(t,x_*(t)+s(x_\alpha(t)-x_*(t)),u_*(t))\,ds,\,\frac{o_2(\alpha,t)}{\alpha}\right\rangle\,dt,$$

and due to the Lebesgue theorem we get

$$\lim_{\alpha\to 0}\lim_{k\to\infty}\frac{I_{2,k}(\alpha)}{\alpha} \qquad (27)$$

$$= \lim_{\alpha\to 0}\int_\tau^{\infty} e^{-\rho t}\left\langle \int_0^1 g_x(t,x_*(t)+s(x_\alpha(t)-x_*(t)),u_*(t))\,ds,\,\frac{o_2(\alpha,t)}{\alpha}\right\rangle\,dt = 0.$$

Now dividing the last inequality in (24) by $\alpha > 0$ we obtain that

$$e^{-\rho\tau}\left[g(\tau,x_*(\tau),v) - g(\tau,x_*(\tau),u_*(\tau))\right] + I_{1,k}(\alpha) + \frac{I_{2,k}(\alpha)}{\alpha} + \frac{o_4(\alpha)}{\alpha} \le \alpha.$$

Taking the limit in this inequality first with $k \to \infty$ and then with $\alpha \to 0$ and using (26) and (27) we obtain that

$$\begin{aligned}
e^{-\rho\tau}&\left[g(\tau,x_*(\tau),v) - g(\tau,x_*(\tau),u_*(\tau))\right] \\
&+ \langle \psi(\tau), f(\tau,x_*(\tau),v) - f(\tau,x_*(\tau),u_*(\tau))\rangle \le 0.
\end{aligned}$$

For the fixed $v \in U$, this inequality holds for every $\tau \in \Omega(v)$. Let U^d be a countable and dense subset of U. From the above inequality we have

$$\mathcal{H}(t,x_*(t),u_*(t),\psi(t)) \ge \mathcal{H}(t,x_*(t),v,\psi(t)) \quad \text{for every } v \in U^d$$

and for every $t \in \cap_{v\in U^d}\Omega(v)$, that is, for almost every t. Due to the continuity of the Hamiltonian with respect to u the last inequality implies the maximum condition (13). This completes the proof of Theorem 1.

4 One-sided Lipschitz and dissipative systems

Without targeting generality we consider problem (P) under the following conditions, in addition to $(A1)$:

$$\langle f(t,x,u) - f(t,y,u), x-y\rangle \le \lambda \|x-y\|^2 \qquad \forall\, t \ge 0,\ x,y \in G,\ u \in U, \tag{28}$$

$$\|g_x(t,x,u)\| \le c_7 \qquad \forall\, t \ge 0,\ x \in G,\ u \in U, \tag{29}$$

$$\rho > \lambda, \tag{30}$$

where $\lambda \in R^1$ and $c_7 \ge 0$ are constants. In addition, we assume that for the LWOO solution $(u_*(\cdot), x_*(\cdot))$ (if such exists) there is $\gamma > 0$ such that

$$x_*(t) + \gamma e^{\lambda t}\mathcal{B} \subset G, \tag{31}$$

where $\mathcal{B}$ is the unit ball in R^n.

Condition (28) is often called *one-sided Lipschitz condition.* In contrast to the standard Lipschitz condition (with respect to x), which obviously implies (28), it may be fulfilled with a negative constant λ. An example is $f(x) = -x$. In the case $\lambda < 0$ condition (28) is also referred to as *dissipativity* or as *strong monotonicity.*

Clearly, due to (29) assumption $(A2)$(ii) is fulfilled with $r = 0$, therefore $(A2)$(i) is not needed at all, since $r \max\{\lambda, \mu\} = 0$. (In fact, in order to verify that the growth of the optimal $x_*(\cdot)$ does not matter in the case $r = 0$ one has to go trough the proof of Theorem 1). The dominating discount condition $(A4)$ is also fulfilled; it takes the form (30). The next lemma claims that $(A3)$ also holds true. Although the proof is standard we present it for completeness.

Lemma 1. *If* $(A1)$*, (28)–(31) are fulfilled then* $(A3)$ *is also fulfilled with* $c_2 = 1$ *and* γ *from (31). Moreover,* $\|K_*(t,\tau)\| \le e^{\lambda(t-\tau)}$ *for every* t*,* $\tau \in [0,\infty)$*.*

Proof. Let us fix an arbitrary ζ such that $\|\zeta - x_0\| < \gamma$. For all t for which the corresponding solution $x(\zeta;t)$ still exists in G we denote $\Delta(t) = \|x(\zeta;t) - x_*(t)\|$. Due to the uniqueness of the solution of (3) we have that $\Delta(t) > 0$. Then

$$\dot{\Delta}(t)\Delta(t) = \frac{1}{2}\frac{d}{dt}(\Delta(t))^2 = \langle \dot{x}(\zeta;t) - \dot{x}_*(t), x(\zeta;t) - x_*(t)\rangle$$

$$\langle f(t, x(\zeta;t), u_*(t)) - f(t, x_*(t), u_*(t)), x(\zeta;t) - x_*(t)\rangle \le \lambda(\Delta(t))^2.$$

Hence,

$$\dot{\Delta}(t) \le \lambda\Delta(t), \quad \Delta(0) = \|\zeta - x_0\|.$$

Then $\Delta(t) \le e^{\lambda t}\|\zeta - x_0\| \le \gamma e^{\lambda t}$ and due to (31) this holds on $[0,\infty)$. This proves the first part of the lemma.

Exactly in the same way one can prove that for any $\tau \in [0,\infty)$, if $\|\zeta - x^*(\tau)\| \le \gamma e^{\lambda\tau}$, then the solution $x(\tau,\zeta;\cdot)$ of (3) with $u(\cdot) = u_*(\cdot)$ and condition $x(\tau,\zeta;\tau) = \zeta$ exists on $[\tau,\infty)$ in G and $\|x(\tau,\zeta;t) - x^*(t)\| \le \|\zeta - x^*(\tau)\| e^{\lambda(t-\tau)}$ for all $t \ge \tau$. The second claim of the lemma follows from this fact exactly in the same way as inequality (14) in the beginning of the proof of Theorem 1. The lemma is proved. □

Then we obtain the following corollary of Theorem 1.

Corollary 1. *If* $(A1)$, *(28)–(31) are fulfilled then the claims of Theorem 1 hold true with the additional property that* $\psi(\cdot)$ *is the unique solution of the adjoint equation (12) for which the corresponding current value adjoint variable* $\xi(t) := e^{\rho t}\psi(t)$, $t \ge 0$, *is bounded.*

Proof. To prove the additional claim in the corollary we first estimate, using (11), (29) and Lemma 1,

$$\begin{aligned}
\|\xi(t)\| &\le c_7 \int_t^\infty e^{-\rho(s-t)} \left\| Z_*(t)\,[Z_*(s)]^{-1} \right\| ds \\
&= c_7 \int_t^\infty e^{-\rho(s-t)} \left\| [(Y_*(t))^*]^{-1}\,(Y_*(s))^* \right\| ds \\
&= c_7 \int_t^\infty e^{-\rho(s-t)} \|(K_*(s,t))^*\|\, ds \\
&\le c_7 \int_0^\infty e^{-(\rho-\lambda)(s-t)}\, ds = \frac{c_7}{\rho-\lambda}.
\end{aligned}$$

Now let $\psi(\cdot)$ and $\tilde{\psi}(\cdot)$ be two solutions of the adjoint equation (12) such that there is a constant $c_8 \ge 0$ for which $\|e^{\rho t}\psi(t)\| \le c_8$ and $\|e^{\rho t}\tilde{\psi}(t)\| \le c_8$ for every $t \ge 0$. Then

$$\frac{d}{dt}(\psi(t) - \tilde{\psi}(t)) = -(f_x(t, x_*(t), u_*(t)))^*\,(\psi(t) - \tilde{\psi}(t)),$$

hence, from the Cauchy formula we have for any $\tau > 0$

$$\psi(0) - \tilde{\psi}(0) = [Z_*(\tau)]^{-1}(\psi(\tau) - \tilde{\psi}(\tau)).$$

Then

$$\|\psi(0) - \tilde{\psi}(0)\| \le \|Y_*(\tau)\| \|\psi(\tau) - \tilde{\psi}(\tau)\| \le e^{\lambda\tau}\|\psi(\tau) - \tilde{\psi}(\tau)\| \le 2c_8 e^{-(\rho-\lambda)\tau}.$$

Since τ can be taken arbitrarily large the dominating discount condition $(A4)$ (see (30)) implies that $\psi(0) - \tilde{\psi}(0) = 0$, which completes the proof of the corollary. □

5 Systems with regular linearization

First we recall a few facts from the stability theory of linear systems (see [8, 9] for more details).

Consider a linear differential system

$$\dot{y}(t) = A(t)y(t), \tag{32}$$

where $t \in [0, \infty)$, $y \in R^n$, and all components of the real $n \times n$ matrix function $A(\cdot)$ are bounded measurable functions.

Let $y(\cdot)$ be a nonzero solution of system (32). Then, the number

$$\tilde{\lambda} = \limsup_{t \to \infty} \frac{1}{t} \ln \|y(t)\|$$

is called *characteristic Lyapunov exponent* or, briefly, *characteristic exponent* of the solution $y(\cdot)$. The characteristic exponent $\tilde{\lambda}$ of any nonzero solution $y(\cdot)$ of system (32) is finite. The set of characteristic exponents corresponding to all nonzero solutions of system (32) is called *spectrum* of system (32). The spectrum of system (32) always consists of at most n different numbers.

The solutions of the system of linear differential equations (32) form a finite-dimensional linear space of dimension n. Any basis of this space, i.e., any n linearly independent solutions $y_1(\cdot), \dots, y_n(\cdot)$, is called *fundamental system* of solutions of system (32). A fundamental system of solutions $y_1(\cdot), \dots, y_n(\cdot)$ is said to be *normal* if the sum of the characteristic exponents of these solutions $y_1(\cdot), \dots, y_n(\cdot)$ is minimal among all fundamental systems of solutions of (32).

It turns out that a normal fundamental system of solutions of (32) always exists. If $y_1(\cdot), \dots, y_n(\cdot)$ is a normal fundamental system of solutions, then the characteristic exponents of the solutions $y_1(\cdot), \dots, y_n(\cdot)$ cover the entire spectrum of system (32). This means that for any number $\tilde{\lambda}$ in the spectrum $\tilde{\lambda}_1, \dots, \tilde{\lambda}_l$ of system (32), there exists a solution of this system from the set $y_1(\cdot), \dots, y_n(\cdot)$ that has this number as its characteristic exponent. Note that different solutions $y_j(\cdot)$ and $y_k(\cdot)$ in the fundamental system $y_1(\cdot), \dots, y_n(\cdot)$ may have the same characteristic exponent. In this case denote by n_s the multiplicity of the characteristic exponent $\tilde{\lambda}_s$, $s = 1, \dots, l$, of the spectrum of differential system (32). Any normal fundamental system contains the same number n_s of solutions to (32) with characteristic number $\tilde{\lambda}_s$, $1 \le s \le l$, $l \le n$, from the Lyapunov spectrum of (32).

Let

$$\sigma = \sum_{s=1}^{l} n_s \tilde{\lambda}_s$$

be the sum of all numbers $\tilde{\lambda}_1, \dots, \tilde{\lambda}_l$ of the spectrum of differential system (32), counted with their multiplicities n_s, $s = 1, \dots, l$.

The linear system (32) is said to be *regular* if

$$\sigma = \liminf_{t\to\infty} \frac{1}{t}\int_0^t \operatorname{trace} A(s)\, ds,$$

where trace$A(s)$ is the sum of all elements of $A(s)$ that lie on the principal diagonal. If system (32) is regular, then, for any $\varepsilon > 0$, its Cauchy matrix $K(\cdot,\cdot)$ satisfies the following inequality:

$$\|K(s,t)\| \le c(\varepsilon) e^{\bar{\lambda}(s-t)+\varepsilon s} \quad \text{for any} \quad t \ge 0 \quad \text{and any} \quad s \ge t, \tag{33}$$

where $\bar{\lambda}$ is the maximal element of the spectrum and the constant $c(\varepsilon) \ge 0$ depends only on ε.

Note that differential system (32) with constant matrix $A(t) \equiv A$, $t \ge 0$, is always regular. In this case the maximal element $\bar{\lambda}$ of the spectrum of differential system (32) equals the maximal real part of the eigenvalues of the matrix A.

Similarly to Corollary 1 we deduce another corollary of Theorem 1.

Corollary 2. *Let assumptions* (A1)–(A4) *be fulfilled and let the linearized system* (6) *be regular. Then the claims of Theorem 1 hold true with the additional property that* $\psi(\cdot)$ *is the unique solution of the adjoint equation* (12) *for which the corresponding current value adjoint variable* $\xi(t) := e^{\rho t}\psi(t)$, $t \ge 0$, *is bounded.*

Proof. Let $\bar{\lambda}_*$ be the maximal element of the spectrum of system (6). Then due to (14) we have $\bar{\lambda}_* \le \lambda$. Hence for any $\varepsilon > 0$ the Cauchy matrix $K_*(\cdot,\cdot)$ of system (6) satisfies the inequality (see (33))

$$\|K_*(s,t)\| \le c(\varepsilon) e^{\lambda(s-t)+\varepsilon s} \quad \text{for any} \quad t \ge 0 \quad \text{and any} \quad s \ge t,$$

where $c(\varepsilon) \ge 0$ is a constant depending only on ε.

Then for a fixed $\varepsilon < \rho - \lambda - r\mu$, using also (11) and (A2)(ii), we obtain the following estimate:

$$\begin{aligned}
\|\xi(t)\| &\le \int_t^\infty e^{-\rho(s-t)} \left\| Z_*(t)\, [Z_*(s)]^{-1} \right\| \|g'_x(s, x_*(s), u_*(s))\|\, ds \\
&\le \kappa \int_t^\infty e^{-\rho(s-t)} \left\| [(Y_*(t))^*]^{-1} (Y_*(s))^* \right\| (1 + \|x_*(s)\|^r)\, ds \\
&\le \kappa \int_t^\infty e^{-\rho(s-t)} \|(K(s,t))^*\| \left(1 + (c_1 e^{\mu s})^r\right) ds \\
&\le \kappa c(\varepsilon) \int_t^\infty e^{-(\rho-\lambda)(s-t)+\varepsilon s} \left(1 + c_1^r e^{\mu r s}\right) ds \\
&\le c_3 c(\varepsilon) \int_0^\infty e^{-(\rho-\lambda-r\mu-\varepsilon)s}\, ds = \frac{c_3 c(\varepsilon)}{\rho - \lambda - r\mu - \varepsilon},
\end{aligned}$$

where $c_3 \geq 0$ is the same constant as in the proof of Theorem 1.

Now let $\psi(\cdot)$ and $\tilde{\psi}(\cdot)$ be two solutions of adjoint equation (12) such that there is a constant $c_9 \geq 0$ for which $\|e^{\rho t}\psi(t)\| \leq c_9$ and $\|e^{\rho t}\tilde{\psi}(t)\| \leq c_9$ for every $t \geq 0$. Then

$$\frac{d}{dt}(\psi(t) - \tilde{\psi}(t)) = -(f_x(t, x_*(t), u_*(t)))^* (\psi(t) - \tilde{\psi}(t)),$$

hence, from the Cauchy formula we have that for any $\tau > 0$

$$\psi(0) - \tilde{\psi}(0) = [Z_*(\tau)]^{-1}(\psi(\tau) - \tilde{\psi}(\tau)).$$

Then

$$\|\psi(0) - \tilde{\psi}(0)\| \leq \|Y_*(\tau)\| \|\psi(\tau) - \tilde{\psi}(\tau)\| \leq e^{\lambda\tau}\|\psi(\tau) - \tilde{\psi}(\tau)\| \leq 2c_9 e^{-(\rho-\lambda)\tau}.$$

Since τ can be taken arbitrarily large the dominating discount condition $(A4)$ implies that $\psi(0) - \tilde{\psi}(0) = 0$, which completes the proof of the corollary. □

6 An illustrative economic example

As an example, consider the following stylized (micro-level) economic problem $(P1)$:

$$J(K(\cdot), I(\cdot)) = \int_0^\infty e^{-dt}\left[e^{pt}(K(t))^\sigma - \frac{b}{2}(I(t))^2\right] dt \to \max$$

under the constraints

$$\dot{K}(t) = -\nu K(t) + I(t), \quad K(0) = K_0,$$
$$I(t) \geq 0.$$

Here $K(t)$ is interpreted as the capital stock at time t, $I(t)$ is the investment, $\nu > 0$ is the depreciation rate, $K_0 > 0$ is a given initial capital stock, $d \geq 0$ is the discount rate, $p \geq 0$ is the (exogenous) exponential rate of technological advancement, $bI^2(t)$ $(b > 0)$ is the cost of investment $I(t)$, and $\sigma \in (0, 1]$ defines the "production function". We put $G = (0, \infty)$. The optimality of an admissible control $I_*(\cdot)$ in problem $(P1)$ is understood in the sense of Definition 1.

Since an infinite growth is possible in this model, we introduce the "detrended" variables

$$x(t) = e^{-\alpha t}K(t), \quad u(t) = e^{-\alpha t}I(t), \quad \text{with} \quad \alpha = \frac{p}{2-\sigma}.$$

In the new variables the model takes the form of the following problem $(P2)$:

$$J(x(\cdot), u(\cdot)) = \int_0^\infty e^{-(d-2\alpha)t}\left[(x(t))^\sigma - \frac{b}{2}(u(t))^2\right] dt \to \max$$

subject to

$$\dot{x}(t) = -(\nu + \alpha)x(t) + u(t), \quad x(0) = K_0.$$

$$u(t) \geq 0.$$

Here as above $G = (0, \infty)$ and we are looking for an admissible LWOO control $u_*(\cdot)$ in problem $(P2)$.

It is clear that for this problem the optimal trajectory (if it exists) is uniformly positive (even if $p = 0$), so that without loss of generality we may take $G = (\eta, \infty)$, where η is a sufficiently small positive number. Then conditions $(A1)$, $(A2)$(ii) and $(A3)$ are fulfilled with $r = 0$, $\kappa = \sigma/\eta^{1-\sigma}$, $\lambda = -(\nu + \alpha)$, $c_2 = 1$ and any small enough $\gamma > 0$, therefore $(A2)$(i) is not needed at all, since $r \max\{\lambda, \mu\} = 0$ (see discussion in the beginning of Section 4). Further, since here $\rho = d - 2\alpha$ and $r = 0$, the dominating discount condition $(A4)$ reads as

$$d + \nu > \alpha \quad \left(= \frac{p}{2 - \sigma} \right). \tag{34}$$

We assume that this condition is fulfilled.

Notice that the "discount rate" ρ can be non-positive, namely if

$$\frac{d}{2} \leq \alpha < d + \nu.$$

In this case a solution with a finite objective value does not exist, although a LWOO solution exists, as it will be shown below.

Denote

$$M = \frac{4}{b} \frac{\sigma}{\eta^{1-\sigma}} \left(\frac{1}{d + \nu - \alpha} + 2 \right).$$

We shall show that if an admissible control $u(\cdot)$ takes values greater than M on a set of positive measure, then $u(\cdot)$ is not optimal in the problem $(P2^T)$, which is $(P2)$ considered on a finite horizon $[0, T]$.

Take an arbitrary admissible control $u(\cdot)$ for which the set

$$\mathfrak{M}_M = \{t > 0 : u(t) > M\}$$

is of positive measure. Let $\tau \in \mathfrak{M}_M$ be a Lebesgue point of the measurable functions $u(\cdot)$ and $u^2(\cdot)$. Almost all points $\tau \in \mathfrak{M}_M$ are such.

For an arbitrary $0 < \varepsilon \leq \tau$ define a simple needle variation $u_\varepsilon(\cdot)$ of $u(\cdot)$ as follows (see Section 3):

$$u_\varepsilon(t) = \begin{cases} u(t), & t \notin (\tau - \varepsilon, \tau], \\ 0, & t \in (\tau - \varepsilon, \tau]. \end{cases}$$

If $\varepsilon \leq \delta$ then $\text{meas}\{t \geq 0 : u_\varepsilon(t) \neq u(t)\} \leq \delta$. Denote by $x(\cdot)$ and $x_\varepsilon(\cdot)$ the admissible trajectories corresponding to $u(\cdot)$ and $u_\varepsilon(\cdot)$ respectively.

The direct calculations give that for any $t > \tau$ we have

$$x_\varepsilon(t) = x(t) - e^{-(\nu+\alpha)(t-\tau)} \int_{\tau-\varepsilon}^{\tau} u(s)\, ds$$

$$= x(t) - e^{-(\nu+\alpha)(t-\tau)} \left(u(\tau)\varepsilon + \mathrm{o}_1(\varepsilon) \right)$$

where $o_1(\varepsilon)$ depends only on ε and $o_1(\varepsilon)/\varepsilon \to 0$ as $\varepsilon \to 0$. This implies that for any $t > \tau$

$$(x(t))^\sigma - (x_\varepsilon(t))^\sigma \le \frac{\sigma}{\eta^{1-\sigma}} e^{-(\nu+\alpha)(t-\tau)} \left(u(\tau)\varepsilon + o_1(\varepsilon)\right).$$

Hence (in view of (34)), for any $T > \tau$ and $\varepsilon \in (0,1]$ so small that $e^{|d-2\alpha|\varepsilon} \le 2$ we get

$$J_T(x(\cdot), u(\cdot)) - J_T(x_\varepsilon(\cdot), u_\varepsilon(\cdot)) = \int_0^T e^{-(d-2\alpha)t} \left((x(t))^\sigma - (x_\varepsilon(t))^\sigma\right) \, dt$$

$$- \frac{b}{2} \int_{\tau-\varepsilon}^{\tau} e^{-(d-2\alpha)t} (u(t))^2 \, dt \le \frac{\sigma e^{(\nu+\alpha)(\tau-\varepsilon)}}{\eta^{1-\sigma}} \int_{\tau-\varepsilon}^{\tau} e^{-(d+\nu-\alpha)t} \, dt \int_{\tau-\varepsilon}^{\tau} u(t) \, dt$$

$$+ \frac{\sigma e^{(\nu+\alpha)\tau}}{\eta^{1-\sigma}} \int_\tau^T e^{-(d+\nu-\alpha)t} \left(u(\tau)\varepsilon + o_1(\varepsilon)\right) dt \; - \; \frac{b}{4} e^{-(d-2\alpha)\tau} \left(\varepsilon (u(\tau))^2 + o_2(\varepsilon)\right)$$

$$\le e^{(\alpha-d)\tau} \left[\frac{\sigma}{\eta^{1-\sigma}} \left(\frac{1}{d+\nu-\alpha} + 2\varepsilon \right) \left(\varepsilon u(\tau) + o_3(\varepsilon)\right) \; - \; \frac{b}{4} \left(\varepsilon (u(\tau))^2 + o_2(\varepsilon)\right) \right],$$

where $o_2(\varepsilon)$ and $o_3(\varepsilon)$ result from the Lebesgue property of $(u(\cdot))^2$ and $u(\cdot)$ at τ. Due to the definition of M the last expression is strictly negative for all sufficiently small $\varepsilon > 0$. Thus the control $u(\cdot)$ is not optimal in problem $(P2^T)$.

Let us show that an optimal control in problem $(P2^T)$ exists. If $\{u_k^T(\cdot)\}$, $k = 1, 2, \ldots$, is a maximizing sequence of admissible controls in the problem $(P2^T)$ then there are constants $C_k \ge 0$ such that $\|u_k^T(t)\| \le C_k$, $t \in [0, T]$. Considering problems $(P2_k^T)$, $k = 1, 2, \cdots$, with additional control constraints $\|u(t)\| \le C_k$. By a standard weak convergence argument (in the space of measurable and square integrable controls $u(\cdot)$ on $[0, T]$) we obtain existence of optimal controls $u_{k,*}^T(\cdot)$ in each problem $(P2_k^T)$, and all $u_{k,*}^T(\cdot)$ are bounded by M. As far as the sequence $\{u_k^T(\cdot)\}$, $k = 1, 2, \ldots$ is maximizing in $(P2_T)$ and the controls $u_{k,*}^T(\cdot)$ are optimal in $(P2_k^T)$, the latter form also a maximizing sequence in $(P2_T)$. Then taking if necessary a subsequence we obtain that the sequence $\{u_{k,*}^T(\cdot)\}$, $k = 1, 2, \cdots$, converges weakly to an admissible control $u_*^T(\cdot)$ which is optimal in problem $(P2^T)$.

Now let us sketch the prove of existence of a LWOO control $u_*(\cdot)$ in problem $(P2)$. Select a sequence $\{T_k\}$, $k = 1, 2, \ldots$, such that $T_k < T_{k+1}$ and $T_k \to \infty$ as $k \to \infty$, and consider the corresponding sequence of problems $(P2^{T_k})$, $k = 1, 2, \ldots$, on finite time intervals $[0, T_k]$. As it is shown above for any $k = 1, 2, \ldots$ there is an optimal admissible control $u_*^{T_k}(\cdot)$ in problem $(P2^{T_k})$ on $[0, T_k]$, and all these controls $u_*^{T_k}(\cdot)$ are bounded by the same constant M. Let us extend $u_*^{T_k}(\cdot)$, $k = 1, 2, \ldots$, as constant to $[0, \infty)$. Then passing if necessary to a subsequence we define an admissible control $u_*(\cdot)$ in problem $(P2)$ as the weak limit of sequence $\{u_*^{T_k}(\cdot)\}$, $k = 1, 2, \ldots$, as $k \to \infty$ on arbitrary finite time interval $[0, T]$, $T > 0$. It turns out that the so constructed $u_*(\cdot)$ is a LWOO control in $(P2)$. If $d - 2\alpha > 0$ then this fact can be proved directly by assuming the contrary (in this case the integral utility functional is finite and $u_*(\cdot)$ is optimal in $(P2)$ in the usual sense). If $d - 2\alpha \leq 0$ then one can prove LWOO optimality of control $u_*(\cdot)$ assuming the contrary and using the relations of the Pontryagin maximum principle for the optimal controls $u_k(\cdot)$ in problems $(P2^{T_k})$ on $[0, T_k]$, $k = 1, 2, \dots$, and inequality (34).

The current value adjoint system for problem $(P2)$ reads as

$$\dot{\xi}(t) = (\rho + \lambda)\xi(t) - \sigma x^{\sigma-1} = (\nu + d - \alpha)\xi - \sigma x^{\sigma-1}.$$

According to Corollary 1 (or 2 as far as both Corollaries 1 and 2 are applicable in this example) this equation has a unique bounded solution $\xi(\cdot)$ and the LWOO optimal control satisfies (due to (13)) $u_*(t) = \frac{1}{b}\xi(t)$. Thus we come up with the following system of equations determining the LWOO optimal solution:

$$\begin{aligned} \dot{x}(t) &= -(\nu + \alpha)x(t) + \frac{1}{b}\xi(t), \qquad x(0) = K_0, \\ \dot{\xi}(t) &= -\sigma x(t)^{\sigma-1} + (\nu + d - \alpha)\xi(t), \qquad \xi(\cdot) \text{ is bounded.} \end{aligned}$$

According to Corollary 1 (or 2) this specific "boundary value problem" has a unique solution. This property makes it possible to apply standard methods of investigation, based on the fact that $(x(0), \xi(0)) = (K_0, \xi(0))$ must belong to the stable invariant manifold of the above system (see e.g. [12]). In the particular case $\sigma = 1$ the solution is explicit:

$$\xi(t) = \frac{1}{\nu + d - p},$$

hence the LWOO optimal control for the original problem is

$$I(t) = \frac{e^{pt}}{b(\nu + d - p)},$$

provided that $d + \nu > p$. We stress again that in the case $d + \nu < p$ a WLOO solution does not exist, and that in the case $\frac{d}{2} \leq p < d + \nu$ the LWOO solution produces infinite objective value, thus it has no "classical" meaning.

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