

Reduction mod ℓ of Theta Series of Level ℓ^n

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Abstract

It is proved that the theta series of an even lattice whose level is a power of a prime ℓ is congruent modulo ℓ to an elliptic modular form of level 1. The proof uses arithmetic and algebraic properties of lattices rather than methods from the theory of modular forms. The methods presented here may therefore be especially pleasing to those working in the theory of quadratic forms, and they admit generalizations to more general types of theta series as they occur e.g. in the theory of Siegel or Hilbert modular forms.

1 Statement of Results

Let ℓ be a prime. We assume throughout that $\ell \geq 5$. It is well-known that every modular form of level ℓ^n is congruent modulo ℓ to a modular form of level one [Serre, Théorème 5.4]. This fact applies in particular to theta series associated to quadratic forms whose level is a power of ℓ . The purpose of this note is to prove a slightly more precise statement and to discuss various consequences. Though the main result is actually a statement about modular forms, the proof presented here works only for theta series. The virtue of this method of proof, however, is that it admits generalizations to more general types of theta series. We shall pursue this elsewhere. In this article we shall prove the following theorem.

Main Theorem. *Let $\underline{L} = (L, b)$ be an even integral lattice whose level is a power of ℓ , and let $e(\underline{L})$ be the sum of the elementary divisors of \underline{L} . Then there exists a modular form f of level 1 and weight $e(\underline{L})/2$ and with integral Fourier coefficients such that*

$$\theta_{\underline{L}} := \sum_{x \in L} q^{\frac{1}{2}b(x,x)} \equiv f \pmod{\ell}$$

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Here we are using standard terminology. By a lattice $\underline{L} = (L, b)$, we understand a free \mathbb{Z} -module L of finite rank equipped with a symmetric positive definite bilinear form b . We call it integral if $b(x, x)$ is an integer for all x in L , and we call it even, if $b(x, x)$ is an even integer for all x in L . Note that in this article the word lattice refers always to what is sometimes called more precisely *positive definite lattice*. The elementary divisors of an even \underline{L} of rank r are the r elementary divisors of the Gram matrix $G = (b(x_i, x_j))_{i,j}$, where the x_i run through a \mathbb{Z} -basis of L , and the level of \underline{L} is the smallest natural number l such that lG^{-1} is an integral matrix with even integers on its diagonal. Of course, the elementary divisors and the level do not depend on the particular choice of the x_i .

The congruence stated in the theorem has to be understood in the naive sense that the difference of the series on both sides of the congruence, viewed as formal power series in q with coefficients in \mathbb{Z} , lies in $\ell\mathbb{Z}[[q]]$. Here, as usual, modular forms as functions of a variable z in the complex upper half plane are identified with the formal power series obtained by expanding them in powers of $q = \exp(2\pi iz)$.

Note that $e(\underline{L})$, for an even \underline{L} as in the main theorem, is divisible by 4. In fact, the rank r of the underlying \mathbb{Z} -module L is even since the determinant $d = \det(G)$ of its associated Gram matrix G is odd. Moreover, using, for any integer $n \geq 0$, the congruence $\ell^n \equiv 1 + n(\ell - 1) \pmod{2(\ell - 1)}$ and the fact that d equals the product of the elementary divisors of \underline{L} , one finds that

$$\frac{e(\underline{L})}{2} \equiv \begin{cases} \frac{r}{2} \pmod{\ell - 1} & \text{if } d \text{ is a perfect square,} \\ \frac{r+\ell-1}{2} \pmod{\ell - 1} & \text{otherwise.} \end{cases}$$

But $(-1)^{\frac{r}{2}}d \equiv 1 \pmod{4}$, and hence $\frac{r}{2}$ is even unless d is not a perfect square and $d \equiv \ell \equiv -1 \pmod{4}$.

The simplest examples for the main theorem are provided by binary quadratic forms. If $[a, b, c]$ denotes a positive definite integral binary form (in Gauss notation) of discriminant $-\ell = b^2 - 4ac$ then by the theorem

$$\theta_{[a,b,c]} = \sum_{x,y \in \mathbb{Z}} q^{ax^2+bx+cy^2}$$

is congruent modulo ℓ to a modular form of level 1 and weight $\frac{\ell+1}{2}$. Note-

worthy examples are

$$\begin{aligned}
\theta_{[1,1,2]} &\equiv E_4 \equiv 1 + 2q + 4q^2 + \cdots \pmod{7} \\
\theta_{[2,1,3]} &\equiv E_4^3 - 720\Delta \equiv 1 + 2q^2 + 2q^3 + 2q^4 + \cdots \pmod{23} \\
\theta_{[2,1,4]} &\equiv E_4^4 - 960E_4\Delta \equiv 1 + 2q^2 + 2q^4 + \cdots \pmod{31} \\
\theta_{[3,1,4]} &\equiv E_4^6 - 1440E_4^3\Delta + 125280\Delta^2 \equiv 1 + 2q^3 + 2q^4 + \cdots \pmod{47} \\
\theta_{[4,3,5]} &\equiv E_4^9 - 2160E_4^6\Delta + 965520E_4^3\Delta^2 - 27302400\Delta^3 \\
&\equiv 1 + 2q^4 + 2q^5 + \cdots \pmod{71}
\end{aligned}$$

Here and in the following, for an even positive integer k , we use

$$\begin{aligned}
E_k &= 1 - \frac{2k}{B_k} \sum_{n \geq 1} \sigma_{k-1}(n) q^n, \\
\Delta &= \frac{E_4^3 - E_6^2}{12^3} = q \prod_{n \geq 1} (1 - q^n)^{24},
\end{aligned}$$

with the Bernoulli numbers $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_6 = \frac{1}{42}$, \dots . Note that the modular forms on the right are the *extremal modular forms* of the respective weights, i.e. the modular forms f_k of weight k (here divisible by 4) whose Fourier expansion is of the form $f_k \equiv 1 \pmod{q^{\lfloor \frac{k}{12} \rfloor + 1}}$. It is well known that for $2k = 8, 24, 32, 48$, these extremal modular forms are equal to the theta series of even unimodular lattices. An obvious explanation for a congruence modulo ℓ between two theta series associated to lattices \underline{L} and \underline{M} is the existence of an automorphism σ of \underline{L} whose order is a power of ℓ and such that \underline{M} is isomorphic to the *fixed lattice* $\underline{L}^\sigma = (L^\sigma, b')$, where L^σ is the submodule of all x in L which are fixed by σ and where we use b' for the restriction of b to $L^\sigma \times L^\sigma$ (cf. Theorem 1 below). And indeed, it is known [N-S1] that the even unimodular lattices E_8 , the Leech lattice, Λ_{RM} and P_{48q} , whose theta series are equal to f_4 , f_{12} , f_{16} and f_{24} , have automorphisms of order 7, 23, 31 and 47, respectively. (However, some of the other lattices which have theta series equal to f_{16} or f_{24} do not have such automorphisms). Though the congruence for $\theta_{[4,3,5]}$ does not prove that an extremal lattice of dimension 72, if it existed, would have an automorphism of order 71, it supports such a speculation. There are exactly 55475 even unimodular lattices of dimension 72 which have an automorphism of order 71¹

¹The even 72-dimensional lattices having an automorphism of order 71 can be downloaded from <http://data.countnumber.de>. A report on the computation of these lattices will be published elsewhere.

We discuss some consequences of the main theorem. For this let $\Theta(\ell^\infty)$ be the $\mathbb{Z}_{(\ell)}$ -algebra generated by the theta series $\theta_{\underline{L}}$, where \underline{L} runs through all even lattices whose level is a power of ℓ . Here and in the sequel we use $\mathbb{Z}_{(\ell)}$ for the localization of \mathbb{Z} at ℓ , i.e. for the ring of rational numbers of the form $\frac{r}{s}$ with integers r, s and s not divisible by ℓ . We have a natural filtration given by the subalgebras $\Theta(\ell^n)$ generated by those $\theta_{\underline{L}}$, where the level of \underline{L} divides ℓ^n . Moreover, let M_k be the $\mathbb{Z}_{(\ell)}$ -module of modular forms of level 1 and weight k whose Fourier coefficients are in $\mathbb{Z}_{(\ell)}$, and let M be the $\mathbb{Z}_{(\ell)}$ -algebra generated by all these modular forms. Then M is the direct sum of the M_k , and $M = \mathbb{Z}_{(\ell)}[E_4, E_6]$. If $\mathbb{F}_\ell = \mathbb{Z}/\ell\mathbb{Z}$ denotes the field with ℓ elements we have a natural map

$$\mathbb{Z}_{(\ell)}[[q]] \rightarrow \mathbb{F}_\ell[[q]] \cong \mathbb{Z}_{(\ell)}[[q]]/\ell\mathbb{Z}_{(\ell)}[[q]], \quad f \mapsto \tilde{f}$$

which is defined by reducing each coefficient of f modulo ℓ . Identifying modular forms and theta series with power series in q we can therefore rewrite the statement of the main theorem in the (weaker) form

$$\widetilde{\Theta(\ell^n)} \subseteq \widetilde{M}.$$

If $\ell = 5$ then $\widetilde{E_4} = 1$, and by the main theorem $\widetilde{E_6} = \widetilde{\theta_{\underline{L}}}$ for every quaternary lattice \underline{L} of level 5 and determinant 25. One may e.g. take the lattice \underline{F} defined by the quaternary form

$$F = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 0 & 1 \\ 1 & 0 & 4 & 2 \\ 1 & 1 & 2 & 4 \end{pmatrix}.$$

We thus find

$$\widetilde{M} = \widetilde{\Theta(5^\infty)} = \mathbb{F}_\ell[\widetilde{\theta_{\underline{F}}}] .$$

Similarly, if $\ell = 7$, then $\widetilde{E_6} = 1$ and, by the main theorem, $\widetilde{E_4} = \widetilde{\theta_{[1,2,8]}}$. We conclude

$$\widetilde{M} = \widetilde{\Theta(7^\infty)} = \mathbb{F}_\ell[\widetilde{\theta_{[1,1,2]}}] .$$

If $\ell = 11$, then $\widetilde{\theta_{[1,1,3]}} = \widetilde{E_6}$. Since $\widetilde{E_4}\widetilde{E_6} = \widetilde{E_{10}} = 1$ we find here

$$\widetilde{M} = \widetilde{\Theta(11^\infty)} = \mathbb{F}_\ell[\widetilde{\theta_{[1,1,3]}}, 1/\widetilde{\theta_{[1,1,3]}}]$$

More generally, it is not hard to deduce from the main theorem:

Corollary 1. *In the notations of the preceding paragraphs one has*

$$\begin{aligned} \widetilde{\Theta(1)} &= \widetilde{\Theta(\ell)} = \widetilde{\Theta(\ell^2)} = \dots = \widetilde{\Theta(\ell^\infty)} = \widetilde{M} \quad \text{for } \ell \equiv 3 \pmod{4}, \\ \widetilde{\Theta(1)} &\subsetneq \widetilde{\Theta(\ell)} = \widetilde{\Theta(\ell^2)} = \dots = \widetilde{\Theta(\ell^\infty)} = \widetilde{M} \quad \text{for } \ell \equiv 1 \pmod{4}. \end{aligned}$$

In particular, $\widetilde{\Theta}(\ell^\infty)$ is a finitely generated algebra over \mathbb{F}_ℓ of transcendence degree 1.

Corollary 2. $\widetilde{\Theta}(\ell^\infty)$ is a $\mathbb{Z}/(\ell - 1)\mathbb{Z}$ -graded algebra:

$$\widetilde{\Theta}(\ell^\infty) = \bigoplus_{t \bmod \ell - 1} \widetilde{\Theta}(\ell^\infty)^t,$$

where $\widetilde{\Theta}(\ell^\infty)^t$ is the \mathbb{F}_ℓ -subspace generated by all $\widetilde{\theta}_F$ with $\frac{e(F)}{2} \equiv t \pmod{\ell - 1}$.

Proof of Corollaries 1 and 2. It is well-known that $M = \mathbb{Z}_{(\ell)}[E_4, E_6]$ and that \widetilde{M} is isomorphic to $\mathbb{F}_\ell[X, Y]/(A - 1)$ via the map $p(X, Y) \mapsto p(\widetilde{E}_4, \widetilde{E}_6)$, where A denotes the polynomial such that $E_{\ell-1} = A(E_4, E_6)$ (see [Sw-D, Theorem 2]). Moreover, $E_4 = \theta_{E_8}$, where E_8 is the unique irreducible root lattice of dimension 8, in particular, E_4 is in $\Theta(1)$.

For the proof of Corollary 1 it thus suffices to show that (i) \widetilde{E}_6 is in $\widetilde{\Theta}(1)$ if $\ell \equiv 3 \pmod{4}$, and that, for $\ell \equiv 1 \pmod{4}$, (ii) \widetilde{E}_6 is in $\widetilde{\Theta}(\ell)$, and (iii) there exists a θ in $\widetilde{\Theta}(\ell)$, which is not in $\widetilde{\Theta}(1)$.

Using the fact that every (positive definite) even unimodular lattice has rank divisible by 8, that $E_4^{k-l}\Delta^l$ ($0 \leq l \leq \lfloor \frac{k}{3} \rfloor$) is a $\mathbb{Z}_{(\ell)}$ -basis of M_{4k} , and that, for the theta series θ_{Leech} associated to Leech's lattice, we have $\theta_{\text{Leech}} = E_4^3 - 720\Delta$, we find

$$\Theta(1) = \bigoplus_{k \geq 0} M_{4k} = \mathbb{Z}_{(\ell)}[\theta_{\text{Leech}}, \theta_{E_8}]$$

(provided ℓ does not divide $720 = 2^4 \cdot 3^2 \cdot 5$).

From this (i) follows immediately since $\widetilde{E}_6 = \widetilde{E}_6 \widetilde{E}_{\ell-1}$ is in $\widetilde{M}_{\ell+5}$, and since, for $\ell \equiv 3 \pmod{4}$, we have $M_{\ell+5} \subseteq \Theta(1)$.

For (iii) we use another result of Swinnerton-Dyer [Sw-D, Theorem 2], namely

$$\widetilde{M} = \bigoplus_{t \bmod \ell - 1} \widetilde{M}^t,$$

where \widetilde{M}^t is the sum of all \widetilde{M}_k with $k \equiv t \pmod{\ell - 1}$. Now, if \underline{L} is an even rank 4 lattice of level ℓ and determinant ℓ^2 , the series $\theta_{\underline{L}}$ is in $\Theta(\ell)$ and, by the main theorem, $\widetilde{\theta}_{\underline{L}}$ is in $\widetilde{M}^{\ell+1} = \widetilde{M}^2$. But then $\widetilde{\theta}_{\underline{L}}$ is not in $\widetilde{\Theta}(1)$ since, by the preceding decompositions of $\Theta(1)$ and \widetilde{M} , the space $\widetilde{\Theta}(1)$, for $\ell \equiv 1 \pmod{4}$, equals the sum of those \widetilde{M}^t where t is divisible by 4.

By the main theorem $\widetilde{\Theta}(\ell^\infty)^t$ is contained in \widetilde{M}^t . Corollary 2 follows therefore from the decomposition of \widetilde{M} of in the preceding paragraph.

The proof of (ii) is more difficult. Let \underline{L} be an even lattice of rank 12 with level ℓ and whose determinant is a perfect square $\geq \ell^4$, say, equal to ℓ^{2n} (one may take the threefold direct sum of a suitable even quaternary lattice). Then $\theta_{\underline{L}}$ is a modular form of weight 6 on $\Gamma_0(\ell)$ with trivial character. We may therefore consider its trace

$$\theta(z) := \sum_{A \in \Gamma_0(\ell) \backslash \mathrm{SL}(2, \mathbb{Z})} \theta_{\underline{L}}(Az)(cz+d)^{-6} = \theta_{\underline{L}}(z) + \sum_{t \bmod \ell} \theta_{\underline{L}}(-1/(z+t))(z+t)^{-6},$$

which is a modular form of level 1, and equals hence a multiple of E_6 . Applying Poisson's summation formula to obtain

$$\theta_{\underline{L}}(-1/z) z^{-6} = -\ell^{-n} \sum_{x \in L^\sharp} e^{\pi i z b(x,x)},$$

one finds

$$\theta = \theta_{\underline{L}} - \ell^{1-n} \sum_{\substack{x \in L^\sharp \\ b(x,x) \in \mathbb{Z}}} q^{\frac{1}{2}b(x,x)},$$

in particular, $\theta = (1 - \ell^{1-n})E_6$. Here L^\sharp denotes the set of all y in $\mathbb{Q} \otimes L$ such that $b(y, x)$ is integral for all x in L (and where of course, b has to be bilinearly extended to $\mathbb{Q} \otimes L$). From this we deduce

$$E_6 \equiv \sum_{\substack{x \in L^\sharp \\ b(x,x) \in \mathbb{Z}}} q^{\frac{1}{2}b(x,x)} \pmod{\ell}.$$

But the right hand side can be rewritten as

$$\sum_{\substack{u \in \mathbb{P}(L^\sharp/L) \\ \underline{b}(u,u)=0}} \theta_{\underline{L}_u} - (|\{u \in \mathbb{P}(L^\sharp/L) : \underline{b}(u, u) = 0\}| - 1) \theta_{\underline{L}},$$

where $\mathbb{P}(L^\sharp/L)$ denotes the set of 1-dimensional subspaces of the \mathbb{F}_ℓ -vector space L^\sharp/L , where $\underline{b} : L^\sharp/L \times L^\sharp/L \rightarrow \mathbb{Q}/\mathbb{Z}$ denotes the bilinear form induced by b , and where, for u in $\mathbb{P}(L^\sharp/L)$, we use \underline{L}_u for the lattice with underlying module $\{x \in L^\sharp : x+L \in u\}$ and the corresponding restriction of b as bilinear form. Note that \underline{L}_u , for $\underline{b}(u, u) = 0$, is an even integral lattice of level ℓ (here we use $\ell \neq 2$). We conclude that \widetilde{E}_6 is indeed an element of $\widetilde{\Theta}(\ell)$. \square

There is a final, almost trivial consequence of the main theorem which might be noteworthy. Namely, if $\underline{L} = (L, b)$ is an even lattice and σ an automorphism of \underline{L} , then we may consider the fixed lattice \underline{L}^σ . It is easy to see that $\theta_{\underline{L}}$ and $\theta_{\underline{L}^\sigma}$ are congruent modulo ℓ if the order of σ is a power

of ℓ (cf. Theorem 1 below). If, furthermore, the level of \underline{L}^σ is a power of ℓ then we may apply the main theorem to conclude that $\theta_{\underline{L}}$ is the reduction modulo ℓ of a modular form f of level 1. (For a discussion of the level of \underline{L}^σ in general see Lemma 1 in section 2). By the discussion following the main theorem we know that the weight k of f is congruent modulo $\frac{\ell-1}{2}$ to $\frac{r}{2}$, where r is the rank of \underline{L}^σ , and that r is even. The characteristic polynomial of σ is of the form $(t-1)^r \phi_{l^{n_1}}(t) \dots \phi_{l^{n_t}}(t)$ (where ϕ_h is the h -th cyclotomic polynomial), and hence the rank n of \underline{L} is congruent modulo $\ell-1$ to r . In particular, n is even. We have therefore proved:

Corollary 3. *Let \underline{L} be an even lattice of rank n which possesses an automorphism σ such that its order and the level of the fixed lattice \underline{L}^σ are powers of ℓ . Then there exists a modular form of level 1, weight $k \equiv \frac{n}{2} \pmod{\frac{\ell-1}{2}}$ with integral Fourier coefficients such that $\theta_{\underline{L}} \equiv f \pmod{\ell}$.*

2 Proof of the Main Theorem

The proof of the main theorem is suggested by two observations, which we formulate here as Theorems 1 and 2. The first theorem is well-known (however, we do not know any precise reference).

Theorem 1. *Let \underline{L} be an even integral lattice which possesses an automorphism σ whose order is a power of ℓ , and let \underline{L}^σ be the sublattice of elements fixed by σ . Then*

$$\theta_{\underline{L}} \equiv \theta_{\underline{L}^\sigma} \pmod{\ell}.$$

Proof. For a nonnegative integer n let X and X^σ denote the set of all x in \underline{L} respectively x in \underline{L}^σ such that $b(x, x) = 2n$, where b is the bilinear form of \underline{L} . We have to show $|X| \equiv |X^\sigma| \pmod{\ell}$. But this is an immediate consequence of the orbit formula

$$|X| = \sum_x [\langle \sigma \rangle : \text{Stab}(x)].$$

Here x runs through a complete set of representatives for the orbits in $\langle \sigma \rangle \backslash X$ and $\text{Stab}(x)$ denotes the subgroup of elements in $\langle \sigma \rangle$ fixing x . \square

The second theorem concerns the Weil representation of an even lattice with automorphism of ℓ -power order. For a given even lattice $\underline{L} = (L, b)$ of level s and rank $2k$ we let $O_{\underline{L}} = \mathbb{Z}[\zeta, 1/\chi_{\underline{L}}]$. Here ζ is a primitive s -th root of unity and

$$\chi_{\underline{L}} = \sum_{\rho \in \text{Det}(L)} \exp(2\pi i \underline{q}(\rho)),$$

where $\text{Det}(\underline{L}) = L^\sharp/L$ is the *determinant module of \underline{L}* , and where we use \underline{q} for the map (finite quadratic form) $\underline{q} : \text{Det}(\underline{L}) \rightarrow \mathbb{Q}/\mathbb{Z}$ induced by $x \mapsto \frac{1}{2}b(x, x)$. Thus $O_{\underline{L}}$ is a subring of the cyclotomic field $\mathbb{Q}(\zeta)$. Note that one has $\chi_{\underline{L}} = e^{\pi i k/2} |\text{Det}(\underline{L})|^{\frac{1}{2}}$ (this identity is sometimes called Milgram's theorem). We let $W_{\underline{L}}$ be the $O_{\underline{L}}$ -submodule of $O_{\underline{L}}[[q^{\frac{1}{s}}]]$ spanned by the series $\theta_\rho := \sum_{x \in \rho} q^{\frac{1}{2}b(x, x)}$, where ρ runs through $\text{Det}(\underline{L})$. It is well-known [Kl] that $(\theta, A) \mapsto \theta|_k A$ defines a right action of $\text{SL}(2, \mathbb{Z})$ on $W_{\underline{L}}$ (provided k is integral). Here we view the elements of $W_{\underline{L}}$ as functions of a variable z in the complex upper half plane by setting $q = \exp(2\pi iz)$, and we use $(f|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix})(z) = f\left(\frac{az+b}{cz+d}\right) (cz+d)^{-k}$.

Finally, if σ denotes an automorphism of the (even) lattice \underline{L} , then, by linear extension, σ acts naturally on $\mathbb{Q} \otimes_{\mathbb{Z}} L$ and on $\text{Det}(\underline{L})$. We then have

Theorem 2. *Let \underline{L} be an even lattice of rank $2k$ which possesses an automorphism whose order is a power of ℓ . Suppose that $\text{Det}(\underline{L})^\sigma = 0$. Then k is even integral, and one has*

$$\theta_{\underline{L}}|_k A \equiv \theta_{\underline{L}} \pmod{\ell W_{\underline{L}}}$$

for all A in $\text{SL}(2, \mathbb{Z})$.

Proof. The action of $\text{SL}(2, \mathbb{Z})$ on $W_{\underline{L}}$ induces an action on the quotient $W_{\underline{L}}/\ell W_{\underline{L}}$, and the theorem states that $\theta_{\underline{L}} + \ell W_{\underline{L}}$ is invariant under this action. It suffices to show this invariance for the generators $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ of $\text{SL}(2, \mathbb{Z})$. The invariance under T is trivial. For showing the invariance under S we use the formula

$$\theta_{\underline{L}}|_k S = \chi_{\underline{L}}^{-1} \sum_{\rho \in \text{Det}(\underline{L})} \theta_\rho.$$

(This formula follows from Poisson's summation formula, see [Kl] for details.) Under this action of σ on the determinant group $\text{Det}(\underline{L})$ we have $\theta_{\sigma(\rho)} = \theta_\rho$. Hence we can rewrite the preceding identity in the form

$$\theta_{\underline{L}}|_k S = \chi_{\underline{L}}^{-1} \sum_{\rho} [\langle \sigma \rangle : \text{Stab}(\rho)] \theta_\rho,$$

where ρ runs here through a complete set of representatives for the orbits in $\langle \sigma \rangle \backslash \text{Det}(\underline{L})$, and where, for each such ρ , we use $\text{Stab}(\rho)$ for its stabilizer in $\langle \sigma \rangle$. Similarly, we have

$$\chi_{\underline{L}} = \sum_{\rho} [\langle \sigma \rangle : \text{Stab}(\rho)] e^{2\pi i \underline{q}(\rho)}.$$

The theorem follows now from the fact that 0 is the only element in $\text{Det}(\underline{L})$ fixed by σ .

Note that we have in particular proved $\chi_{\underline{L}} \equiv 1 \pmod{\ell\mathbb{Z}[\zeta]}$, and the same argument implies $|\text{Det}(\underline{L})| \equiv 1 \pmod{\ell}$. On the other hand, we have $\chi_{\underline{L}}^2 = e^{\pi ik} |\text{Det}(\underline{L})|$. We thus recognize that the rank $2k$ of \underline{L} must indeed be divisible by 4 as claimed. \square

The idea of proof of the main theorem is now apparent. Given a lattice of ℓ -power level we construct a lattice \widehat{L} and an automorphism σ of ℓ -power order such that L is isomorphic to the fixed lattice \widehat{L}^σ . Accordingly to Theorem 2 one might expect then that the theta series of \widehat{L} is congruent modulo ℓ to a modular form of level 1, provided some additional assumptions on \widehat{L} and the automorphism σ hold true. Following this idea we can indeed find a proof of the main theorem. We postpone the proof of the following theorem, which relies on a purely algebraic property of quadratic forms, to the Appendix.

Theorem 3. *For every even lattice \underline{L} whose level is a power of ℓ there exists an even lattice \widehat{L} which possesses an automorphism σ of ℓ -power order such that the sublattice of \widehat{L} fixed by σ is isomorphic to \underline{L} . The lattice \widehat{L} can be chosen so that its rank equals $e(\underline{L})$ and such that its level is not divisible by ℓ or any prime $p \equiv -1 \pmod{\ell}$.*

Finally, we still need a lemma which assures that a lattice \widehat{L} as in the preceding theorem satisfies the hypothesis $\text{Det}(\widehat{L})^\sigma = 0$ of Theorem 2.

Lemma 1. *Let σ be an automorphism of $\underline{L} = (L, b)$ whose order is a power of ℓ . There are canonical embeddings of $(L^\sharp)^\sigma / L^\sigma$ into $\text{Det}(\underline{L}^\sigma)$ and $\text{Det}(\underline{L})^\sigma$. The images under these embeddings are subgroups whose index is a power of ℓ , respectively. In particular, if ℓ does not divide the determinant of \underline{L} , then $(L^\sharp)^\sigma / L^\sigma$ can be identified with $\text{Det}(\underline{L})^\sigma$.*

Proof. Let ℓ^n denote the order of σ . We set $V = \mathbb{Q} \otimes_{\mathbb{Z}} L$ and extend b to a bilinear form on V . For a finitely generated \mathbb{Z} -submodule M of V we use M^* for the set of y in $\mathbb{Q}M$ such that $b(y, M) \subset \mathbb{Z}$. (We have of course $L^* = L^\sharp$ with L^\sharp as already used before.) Then $\text{Det}(\underline{L}^\sigma)$ can be identified with $(L^\sigma)^* / L^\sigma$. The natural embeddings of the theorem are given by the inclusion of $(L^\sharp)^\sigma / L^\sigma$ in $(L^\sigma)^* / L^\sigma$ and by the natural map $x + L^\sigma \mapsto x + L$.

If y is in $(L^\sigma)^*$ then $\sigma(y) = y$, hence $\ell^n y = s(y)$, where $s = \sum_{\tau \in \langle \sigma \rangle} \tau$. But

$$b(s(y), L) = b(y, s(L)) \subseteq b(y, L^\sigma) \subseteq \mathbb{Z}.$$

We conclude that $\ell^n y$ is in $(L^\sharp)^\sigma$.

Similarly, if $y + L$ is in $\text{Det}(\underline{L})^\sigma$, then $\ell^n y \equiv s(y) \pmod{L}$, but $s(y)$ is in $(L^\sharp)^\sigma$. \square

Proof of the Main Theorem. Given a lattice \underline{L} whose level is a power of ℓ we choose a lattice $\widehat{\underline{L}}$ of rank $2k$ and level s equipped with an automorphism σ as in Theorem 3. We choose $\widehat{\underline{L}}$ such that $2k = e(\underline{L})$ and s is not divisible by ℓ . By Theorem 1 the series $\theta_{\underline{L}}$ is congruent to $\theta := \theta_{\widehat{\underline{L}}}$ modulo ℓ . Since ℓ does not divide the determinant of $\widehat{\underline{L}}$ and since the determinant of \underline{L} is a power of ℓ we conclude from the Lemma 1 that $\text{Det}(\underline{L})^\sigma = 0$. By Theorem 2 k is even and we have $\theta|_k A \equiv \theta \pmod{\ell O_{\underline{L}}[[q^{1/s}]]}$ for all $A \in \Gamma$. It is well-known that θ is a modular form on $\Gamma_0(s)$ with a real character. Because of the last congruence the character is trivial. The form $g := \sum_A \theta|_k A$, with A running through a complete set of representatives for $\Gamma_0(s) \backslash \Gamma$, is thus a modular form on $\Gamma = \text{SL}(2, \mathbb{Z})$. But $g \equiv n\theta \pmod{\ell O_{\underline{L}}[[q]]}$, where n denotes the index of $\Gamma_0(s)$ in Γ . Note that $n = s \prod_{p|s} (1 + \frac{1}{p})$.

If we write g in the form $g = \sum c_{a,b} E_4^a \Delta^b$ or $g = \sum c_{a,b} E_4^a E_6^b \Delta^b$ (with a, b running over all nonnegative integers such that $4a + 12b = k$ in the first sum and $4a + 12b = k - 6$ in the second sum), we see that the coefficients $c_{a,b}$ are in $O_{\underline{L}}$, and that they are in fact congruent modulo $\ell O_{\underline{L}}$ to rational integers (since g is congruent modulo $\ell O_{\underline{L}}$ to $n\theta$). Replacing the $c_{a,b}$ by these integers we can assume that g has coefficients in \mathbb{Z} . But then $g \equiv n\theta \pmod{\ell \mathbb{Z}[[q]]}$ (since $\mathbb{Z}[\frac{1}{\ell}] \cap \ell O_{\underline{L}} = \mathbb{Z}$).

If we finally choose $\widehat{\underline{L}}$ such that s does not contain any primes congruent to -1 modulo ℓ , then n is invertible modulo ℓ and we have proved the theorem. \square

Appendix

In this section we prove Theorem 3. We shall say that a lattice $\underline{L} = (L, b)$ can be *diagonalized over* a subring R of \mathbb{Q} if $R \otimes_{\mathbb{Z}} L$ contains an orthogonal R -basis, i.e. an R -basis x_i such that $b(x_i, x_j) = 0$ for all $i \neq j$. (Here and in the following we use the same letter b for the bilinear extension of b to $R \otimes L$ as for b itself.) It is easy to see that every lattice can be diagonalized over $\mathbb{Z}_{(\ell)}$.

Lemma 2. *Let \underline{L} be an even lattice whose level is a power of ℓ . Assume that R is a localization of \mathbb{Z} contained in $\mathbb{Z}_{(\ell)}$ such that \underline{L} can be diagonalized over R . Then there exists a lattice $\widehat{\underline{L}}$ which possesses an automorphism σ of ℓ -power order such that the sublattice of $\widehat{\underline{L}}$ fixed by σ is isomorphic to \underline{L} . The lattice $\widehat{\underline{L}}$ can be chosen so that its rank equals $e(\underline{L})$ and such that its level is a unit in R .*

Proof. Let $\underline{L} = (L, b)$, and let e_i ($1 \leq i \leq n$) be an orthogonal R -basis of $R \otimes_{\mathbb{Z}} L$. If a_i is a \mathbb{Z} -basis of L then $(a_i)_i = (e_i)_i M$ with a matrix M in $\text{GL}(n, R)$. Multiplying M by the l.c.m. N of the denominators of its entries (which is a unit in R) and replacing e_i by e_i/N we can assume that L is contained in $H := \bigoplus \mathbb{Z}e_i$. The index $[H : L]$ is an element of the group of units R^* of R . We can therefore find a natural number d in R^* such that $dH \subseteq L$ and such that $d \cdot b(x, x)$ is an even integer for all x in H . Write $b(e_i, e_i) = a_i \ell_i^{\alpha_i}$ with a_i in R^* and an integer $\alpha_i \geq 0$. Note that the $\ell_i^{\alpha_i}$ are the elementary divisors of \underline{L} . Denote by $\widehat{H} = (\widehat{H}, c)$ a lattice of rank $e(\underline{L})$ which possesses an orthonormal basis $e_{i,j}$ ($1 \leq i \leq n, 1 \leq j \leq \ell_i^{\alpha_i}$) such that $c(e_{i,j}, e_{i,j}) = a_i$, and let σ be the automorphism of \widehat{H} , which, for each i , acts as

$$e_{i,1} \mapsto e_{i,2} \mapsto \cdots \mapsto e_{i,\ell_i^{\alpha_i}} \mapsto e_{i,1}.$$

The order of σ is clearly a power of ℓ .

Finally, let \widehat{L} be the sublattice of \widehat{H} whose underlying \mathbb{Z} -module is the set of all $\sum_{i,j} x_{i,j} e_{i,j}$ such that

$$x_{i,1} \equiv x_{i,2} \equiv \cdots \equiv x_{i,\ell_i^{\alpha_i}} \pmod{d}$$

for all i and such that $\sum_i x_{i,1} e_i$ is in L . We leave it to the reader to verify that \widehat{L} is even, that its level is a unit in R , and that \underline{L}^{σ} is isomorphic to \underline{L} . \square

The Theorem 3 is now an immediate consequence of the preceding lemma and the following theorem, whose proof, however, seems to need some deeper facts from algebraic number theory.

Theorem 4. *Let S be the set of all nonzero integers which contain only primes $p \neq \ell$ and $p \not\equiv -1 \pmod{\ell}$ as prime factors, and let $S^{-1}\mathbb{Z}$ the localization of \mathbb{Z} at S (i.e. the set of rational numbers $\frac{r}{s}$ with $r \in \mathbb{Z}$ and $s \in S$). Then every lattice \underline{L} can be diagonalized over $S^{-1}\mathbb{Z}$.*

Proof. Set $R := S^{-1}\mathbb{Z}$. It suffices to show that every integral R -lattice $\underline{M} = (M, b)$ contains an x such that $b(x, x)$ divides $b(y, z)$ (in R) for all y and z in M . Here by integral R -lattice (M, b) we mean a free R -module of finite rank equipped with a (positive definite) symmetric bilinear map $b : M \times M \rightarrow R$.

In fact, if this holds true, and if $\underline{M} = (M, b)$ is an integral R -lattice then choose an element x_1 in M such that $b(x_1, x_1)$ divides all values of b on $M \times M$ and let M_1 be the orthogonal complement of x_1 . Then $M = Rx_1 + M_1$ since, for any y in M , the number $t := b(x_1, y)/b(x_1, x_1)$ is in R and $y - tx_1$ is perpendicular to x_1 . i.e. $y - tx_1$ is in M_1 . Replacing \underline{M} by (M_1, b) we recognize that our claim follows by induction on the rank of \underline{M} .

So let $\underline{M} = (M, b)$ be a R -lattice, and let Rd be the ideal generated by all values of b on $M \times M$. Note that Rd coincides with the ideal generated by all $b(x, x)$ with x in M (since $2b(x, y) = b(x + y, x + y) - b(x, x) - b(y, y)$) and 2, for $\ell \geq 5$ is a unit in R). We want to show the existence of an x in M such that $b(x, x)/d$ is a unit in R .

If M has rank 1 this assumption is trivial. If the rank of \underline{M} is greater than or equal to 2 we can proceed as follows. Choose a y such that $b(y, y) \neq 0$. We can then find a z in M such that $Rb(y, y) + Rb(z, z) = dR$.

Namely, for each prime p dividing $b(y, y)/d$ (in R) which is not a unit in R there is a y_p in M such that p does not divide $b(y_p, y_p)/d$ (since Rd is generated by all values $b(y, y)$). Using the Chinese Remainder Theorem we find a z in M such that $b(z, z) \equiv b(y_p, y_p) \pmod{p}$ for all p in question, in particular, such that $b(x, x)/d$ and $b(z, z)/d$ are relatively prime.

Finally, choose a unit e in R such that $Q(s, t) := \frac{e}{d}b(sy + tz, sy + tz)$ is a positive definite primitive binary quadratic form with integer coefficients. It suffices now to show that Q represents an integer not containing ℓ or a prime $p \equiv -1 \pmod{\ell}$. But this is assured by the subsequent Theorem 5. \square

Theorem 5. *Let $Q(x, y)$ be an integral primitive positive definite binary quadratic form, and let ℓ be a prime, $\ell \geq 5$. Then there exist integers x, y such that $Q(x, y)$ is only divisible by primes $p \not\equiv 0, -1 \pmod{\ell}$.*

Note that the theorem does clearly not hold true for $\ell = 2$. For $\ell = 3$ it does not hold true either: the quadratic form $2x^2 + 3y^2$ represents only numbers $n \equiv 0, -1 \pmod{3}$ and each such n contains at least one prime divisor $p \not\equiv +1 \pmod{3}$.

Proof of Theorem 5. Let $Q(x, y) = ax^2 + bxy + cy^2$, and write $b^2 - 4ac = Df^2$, where D is a fundamental discriminant.

Let $K = Q(\sqrt{D})$, let $\mathfrak{D} = \mathbb{Z} + \mathbb{Z}\omega$ and $\mathfrak{D}_f = \mathbb{Z} + \mathbb{Z}f\omega$, where $\omega = \frac{D + \sqrt{D}}{2}$, and let $M = \mathbb{Z}a + \mathbb{Z}\frac{b + f\sqrt{D}}{2}$. Then $N_Q := \{\frac{1}{a}N(\alpha) : \alpha \in M\}$ is the set of integers represented by Q (we use N for the norm function on numbers or ideals in K). Moreover, $M\mathfrak{D}_f = M$. Replacing Q by an equivalent form, if necessary, we may assume that a and ℓf are relatively prime (since we can find integers x and y such that $Q(x, y)$ is relatively prime to ℓf , e.g. one may take for x the product of all primes in ℓf dividing a but not c , and for y one may take the product of all primes in ℓf not dividing a).

But then $M + \ell f\mathfrak{D}_f = \mathfrak{D}_f$, which in turn implies that

$$N := \left\{ \frac{1}{a}N(\alpha) : \alpha \in M\mathfrak{D} \cap (1 + \ell f\mathfrak{D}) \right\}$$

is a subset of N_Q . Indeed, using $\ell f\mathfrak{D} \subseteq \mathfrak{D}_f$, we have

$$\begin{aligned} M\mathfrak{D} \cap (1 + \ell f\mathfrak{D}) &\subseteq M\mathfrak{D} \cap \mathfrak{D}_f = (M\mathfrak{D} \cap \mathfrak{D}_f)\mathfrak{D}_f \\ &= (M\mathfrak{D} \cap \mathfrak{D}_f)(M + \ell f\mathfrak{D}_f) \subseteq M + \ell fM\mathfrak{D} \subseteq M. \end{aligned}$$

Now $M\mathfrak{D} = \mathbb{Z}a + \mathbb{Z}\omega$ (since a and f are relatively prime), hence $a = N(M\mathfrak{D})$. Therefore N equals the set of norms of all integral ideals in the ideal class $(M\mathfrak{D})^{-1}P \in I/P$, where P is the group of (fractional) ideals generated by the integral principal ideals (α) of K such that $\alpha \equiv 1 \pmod{\ell f}$, and where I is the group of fractional ideals of K generated by all integral ideals relatively prime to ℓf (i.e. I/P is what is usually called the ray class group modulo ℓf).

It remains to show that every ideal class A in $C = I/P$ contains an integral ideal whose norm is in the group of units $(S^{-1}\mathbb{Z})^*$, where $R := S^{-1}\mathbb{Z}$ is the ring introduced in Theorem 4. For the moment, we denote the set of A containing such an ideal by Σ . Note that Σ is a subgroup. It is obviously closed under multiplication. Moreover, if \mathfrak{a} is an integral ideal in a class A in Σ whose norm is in R^* , then A^{-1} contains the integral ideal $\mathfrak{a}^{-1}N(\mathfrak{a})^{\varphi(f\ell)}$ (where φ denotes Euler's φ -function), whose norm is again in R^* . We shall use repeatedly that every ideal class in C contains prime ideals of degree one (as follows e.g. from [Hecke, p. 318]).

We distinguish two cases.

Case 1: $D = -\ell$. Let \mathfrak{p} be a prime ideal of degree one in a given ideal class A in C . For $p = N(\mathfrak{p})$ we then have $\left(\frac{p}{\ell}\right) = \left(\frac{D}{p}\right) = +1$. In particular, $p \not\equiv 0, -1 \pmod{\ell}$ (since $\ell = -D \equiv 3 \pmod{4}$).

Case 2: D contains a prime factor different from ℓ . Here we consider the map $\mathfrak{a} \mapsto \left(\frac{N(\mathfrak{a})}{\ell}\right)$, which induces a group homomorphism of C . The kernel Γ of this homomorphism has index at most 2 in C . In fact, it has index exactly equal to 2: choose a prime p such that $\left(\frac{D}{p}\right) = +1$ and $\left(\frac{p}{\ell}\right) = -1$ (this is possible by Dirichlet's theorem on arithmetic progressions and since D contains a prime different from ℓ). Then p is the norm of a prime ideal which is not in Γ .

If $\ell \equiv 3 \pmod{4}$ then Γ is contained in Σ . Indeed, if A is in Γ , then any prime ideal of degree one in A with norm, say, q satisfies $q \not\equiv 0, -1 \pmod{\ell}$ (since $\left(\frac{q}{\ell}\right) = +1$). But the group Σ is strictly bigger than Γ as can be seen by choosing the prime p of the last paragraph such that $p \not\equiv -1 \pmod{\ell}$ (for fulfilling this and $\left(\frac{p}{\ell}\right) = -1$ at the same time we need $\ell \geq 5$). Since the index of Γ in C is 2 we conclude that its index in Σ is 2 and $\Sigma = C$.

If $\ell \equiv 1 \pmod{4}$ then $C \setminus \Gamma$ is in Σ as can be seen by picking in a given class A in $C \setminus \Gamma$ a prime ideal of degree 1. In fact, its norm q is different from ℓ and satisfies $q \not\equiv -1 \pmod{\ell}$ (since $\left(\frac{q}{\ell}\right) = -1$). Since $\Gamma \neq C$ the set

$C \setminus \Gamma$ is a (the) nontrivial Γ coset, which is contained in Σ , and we again conclude $\Sigma = C$.

This proves the theorem. \square

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