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Counterexamples in Distributed Detection

Moula Cherikh and Paul B. Kantor

***Abstract*—Specific counterexamples violating three very plausible conjectures about optimal design of distributed detection and decision systems are reported. The numerical magnitude of the violation is small in each case.**

***Index Terms*—Distributed detection, parallel detection, detection in series, fusion rules, ROC function, concavity, randomization.**

I. Introduction

There is a considerable literature on the problems of distributed detection and decision in engineering (specifically, radar) contexts, [1]-[8]. The problem is considered to be important because the components of a distributed detection system may amass more data than they can transmit to a fusion node, and must summarize that data by choice of a message drawn from a small set. In most of the literature the case in which there are exactly two states of nature to be discriminated, exactly two possible actions, and exactly two possible messsages is considered.

Although this case avoids many complexities it nonetheless produces a number of analytic challenges. In particular, there are many relations among sensors and fusion rules which appear with such frequency that one is led to suspect the presence of a universal law or theorem. In the course of our work [9], [10] on the completely binary problem, we have encountered a number of such relations. Some of them have led to positive outcomes (theorems), but others have led only to frustration. The purpose of this note is to report three specific counterexamples to entirely plausible conjectures. We hope that any authors who have either proclaimed or assumed the truth of the conjectures thus refuted will excuse us for failing to cite them in this context.

Two of the results concern what may be called parallel fusion. Each of two sensors observes the world, and sends a single binary message to a fusion center, which selects a course of action. The third, and perhaps the most surprising, result concerns the serial situation, in which one sensor sends a message to another sensor which, on the basis of its own observation of the world, and that message, selects an action.

In all of the discussion, we will represent sensors entirely by their performance as summarized by a receiver operating characteristic (ROC) curve. This curve is the upper edge of the boundary of a feasible region in the space of all possible "performances" of the sensor. For the completely binary case, the performance of the composite system can be represented completely by an effective ROC. This is the boundary of the union of several feasible regions for the composite system, corresponding to different choices of the fusion logic.

We present the three counterexamples in order of decreasing "surprise value." In all of our examples we assume that the signals received at the several peripheral sensors are stochastically independent under both states of the world. This is illustrated, for a series arrangement of sensors, in Fig. 1.

II. Two Detectors in Series

In this section, we address a sequencing problem with two detectors in series. It was conjectured in [8] that it is better to sequence the more reliable detector downstream of the poorer one. First, we give some definitions and clarifications of the sequence and the ordering of performance. The functional relationship between the probability of false alarm and the probability of detection is assumed given by the ROC functions $R(\alpha)$, where α is the probability of false alarm and $R(\alpha)$ is the corresponding probability of detection. A detector D_2 is said to be better ("more reliable") than detector D_1 if the ROC function R_2 corresponding to D_2 uniformly dominates the ROC function R_1 corresponding to D_1, i.e.,

$$R_2(\alpha) \geq R_1(\alpha), \qquad \forall \alpha \in [0, 1].$$

With these two detectors, one can design two different systems in series according to whether the better detector is upstream or downstream.

It has been conjectured that the system with the better detector downstream performs better than the one with the better detector upstream. That is, the overall ROC R_0^{12} of the system with the better detector downstream uniformly dominates the overall ROC R_0^{21} of the other system.

$$R_0^{12}(\alpha) \geq R_0^{21}(\alpha), \qquad \forall \alpha \in [0, 1].$$

The two overall ROC's R_0^{12} and R_0^{21} are given by [8, 9]:

$$R_0^{12}(\alpha) = \max_{\substack{x, y \\ \text{s.t.}}} \left\{ R_1\left(\frac{\alpha - x}{y - x}\right) R_2(y) + \left[1 - R_1\left(\frac{\alpha - x}{y - x}\right)\right] R_2(x) \right\}$$
$$0 \leq x \leq \alpha$$
$$\alpha \leq y \leq 1$$

Manuscript received July 12, 1990; revised April 29, 1991. This work was supported in part by SDIO under ONR Contract No. N004-87-C0659.
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IEEE Log Number 9102731.

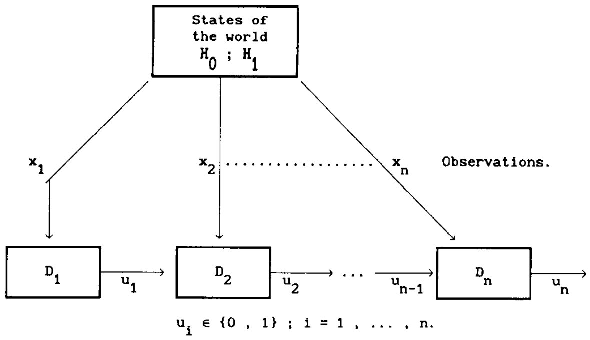

Fig. 1. Detection network in series.

and a similar expression for $R_0^{21}(\alpha)$ with the indices 1 and 2 interchanged everywhere.

The overall ROC R_0^{12} is the "convexified" ROC of the system where detector 1 is upstream of detector 2. The arguments x and y in the previous equation denote the tunings of the upstream detector when the decision of the downstream detector is 0 and 1, respectively, while $(\alpha - x)/(y - x)$ gives the tuning of the downstream detector.

The intuitive argument behind this conjecture is as follows: The better detector somehow provides more information about the state of the world. Whichever detector is upstream has its information reduced to a single binary digit. It seems that "more information is lost" if the better detector suffers this reduction. So, it seems that it would always be better to arrange the system with the better detector downstream.

For many standard distributions (see [8], [9] for more examples) the conjecture is true. However, it is not always true because one cannot prove that the final decision of the design with the better detector downstream is better than the final decision of the other. A counterexample is given by the following ROC functions,

$$R_1(\alpha) = \begin{cases} \frac{8}{5}\alpha, & \text{if } 0 \le \alpha \le \frac{1}{4}, \\ \frac{6}{5}\alpha + \frac{1}{10}, & \text{if } \frac{1}{4} < \alpha \le \frac{1}{2}, \\ \frac{4}{5}\alpha + \frac{3}{10}, & \text{if } \frac{1}{2} < \alpha \le \frac{3}{4}, \\ \frac{2}{5}\alpha + \frac{3}{5}, & \text{if } \frac{3}{4} < \alpha \le 1, \end{cases}$$

and

$$R_2(\alpha) = \begin{cases} \frac{8}{5}\alpha, & \text{if } 0 \le \alpha \le \frac{1}{4}, \\ \frac{7}{5}\alpha + \frac{1}{20}, & \text{if } \frac{1}{4} < \alpha \le \frac{1}{2}, \\ \frac{3}{5}\alpha + \frac{9}{20}, & \text{if } \frac{1}{2} < \alpha \le \frac{3}{4}, \\ \frac{2}{5}\alpha + \frac{3}{5}, & \text{if } \frac{3}{4} < \alpha \le 1. \end{cases}$$

Note that $R_2(\alpha) \ge R_1(\alpha)$ for all $\alpha \in [0, 1]$. We have numerically determined the two overall ROC's R_0^{12} and R_0^{21}. Fig. 2 gives the differences: $R_2(\alpha) - R_1(\alpha)$ and $R_0^{21}(\alpha) - R_0^{12}(\alpha)$ for all $\alpha \in [0, 1]$. As the graph shows there is an interval where $R_0^{21} > R_0^{12}$, showing that the system with the better detector upstream is sometimes better than the one with the better detector downstream, but not always. Although this example disproves the conjecture, the difference in the two overall ROC's is not very large

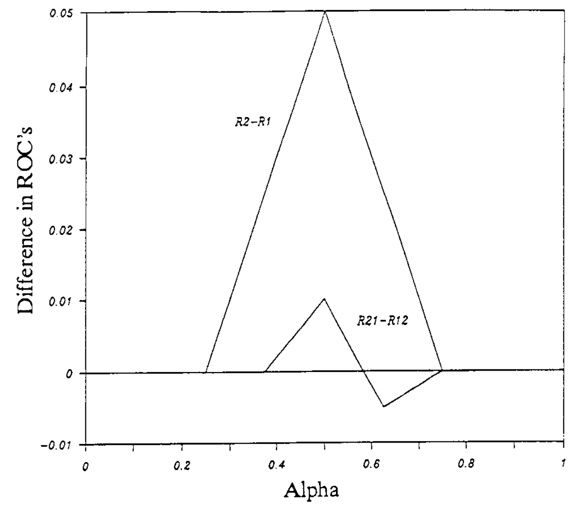

Fig. 2. Series-sequencing counterexample.

$$\left(\max_{\alpha} |R_0^{21}(\alpha) - R_0^{12}(\alpha)| = |R_0^{21}(0.5) - R_0^{12}(0.5)| = 0.775 - 0.765 = 0.01\right).$$

III. Identical Sensors in Parallel Fusion

As shown in [4], [9], optimal fusion rules for the parallel case will be found among the monotonically increasing Boolean functions that do not ignore any of the sensors. The problem of enumerating these laws is so complex that even the number of possible laws is known only for the first few cases. In practice most considerations of the question have restricted analysis to the case of "k-out-of-n" rules. It is known that such rules are not generally optimal, although they may be so for the case of identical sensors. This possibility is enhanced by the observation that only the k-out-of-n rules are manifestly symmetric in the peripheral sensor messages. In any case, for the situation of two sensors in fusion, the possible fusion rules are exhausted by OR (1 out of 2) and AND (2 out of 2).

In many published analyses of that problem [6], [7] and in much of our own work, it is found that whether the optimal rule (for a particular false alarm level α) be AND or OR, the corresponding optimal tunings of the peripheral sensors are the same. This symmetry of the tunings arises so often that one is led to conjecture that it always holds. The argument goes something as described next.

The sensors are the same, so why should it be better to tune one of them differently from the other? How could you decide which one?

It is clear that this is not a rigorous argument by itself. All it requires is that, if the optimum is not symmetric, it can be achieved in two different ways, according to which sensor has which tuning. Efforts to make the argument cleaner and persuasive are doomed to failure because of the existence of counterexamples such as the following. Suppose we have two peripheral detectors with the same

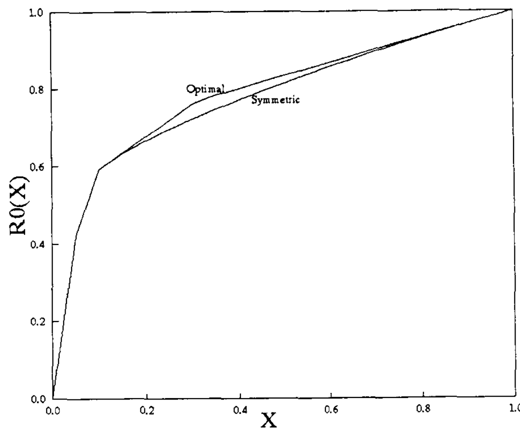

Fig. 3. Example of unsymmetrical R_0^{AND}.

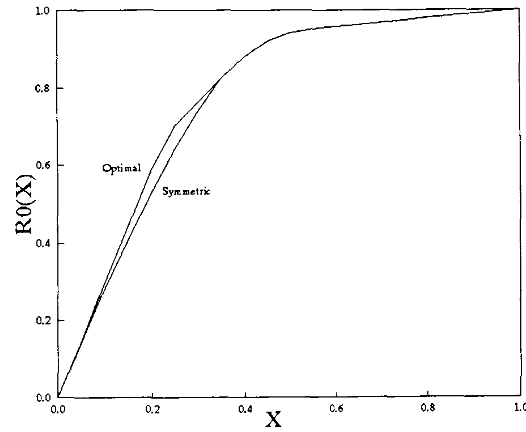

Fig. 4. Example of unsymmetrical R_0^{OR}.

ROC function (using mixed densities) given by

$$R(x) = \begin{cases} 3x, & \text{if } 0 \le x \le \frac{1}{4} - \epsilon, \\ \dfrac{(x-\frac{1}{4})^4}{6\epsilon^3} - \dfrac{(x-\frac{1}{4})^2}{\epsilon} + \dfrac{5}{3}\left(x - \dfrac{1}{4}\right) + \dfrac{3}{4} - \dfrac{\epsilon}{2}, \\ \quad \text{if } \frac{1}{4} - \epsilon < x \le \frac{1}{4} + \epsilon, \\ \frac{1}{3}(x+2), & \text{if } \frac{1}{4} + \epsilon < x \le 1, \end{cases}$$

with $0 \le \epsilon \le \frac{1}{4}$.

We numerically computed the overall optimal ROC for the OR and the AND rules given by

$$R_0^{\text{OR}}(\alpha) = \max_{0 \le x \le \alpha} \left\{ R(x) + (1 - R(x)) R\left(\frac{\alpha - x}{1 - x}\right) \right\}$$

and

$$R_0^{\text{AND}}(\alpha) = \max_{\alpha \le x \le 1} \left\{ R(x) R\left(\frac{\alpha}{x}\right) \right\}, \qquad \forall \alpha \in [0, 1]$$

for the case $\epsilon = 0.1$.

The optimum value of x will be the tuning of the first detector and either $(\alpha - x)/(1 - x)$ or α / x will be the tuning of the second according to the fusion rule. We also find the symmetric ROC's R_0^{AND} and R_0^{OR} corresponding to the AND rule ("2 out of 2" rule) and to the OR rule ("1 out of 2" rule) by requiring that the tunings of the two detectors be the same. That is, $x = \sqrt{\alpha}$ for the AND rule and $x = 1 - \sqrt{1 - \alpha}$ for the OR rule. Figs. 3 and 4 show that the symmetric solutions are not always optimal. Although this example shows that the symmetric solution is not always optimal, extensive numerical experimentation with nicely behaved functions R (strictly concave) suggests that the symmetric is "almost always" the optimal solution. Necessary and/or sufficient conditions to ensure this symmetry have not yet been found.

In [12], [13], two counterexamples for this case are reported, the first reference involves an example with discrete density functions and a different optimality criterion. The second one deals with continuous densities that are nearly discrete concentrated at two points, and where three detectors are used. All these counterexamples seem to occur in situations in which the probability mass of the likelihood ratio at the sensor inputs are concentrated at and around single points. In such situations, sensor diversity (unsymmetry) would be helpful.

IV. Nonconcavity of the Overall ROC

Our third counterexample concerns a conjecture that we have not found claimed or used in the literature and so it may clear up a confusion which existed nowhere but in the minds of the authors. In the construction of a real system, it may be necessary to choose the logic or fusion rule first, and then search for the best tuning. It seemed natural to us that the resulting ROC for each logic, in the case of parallel fusion, should itself be a concave curve. It is the boundary of some feasible region built in a simple way from the feasible regions for the individual sensors. Those regions are convex sets, with concave upper boundaries which are the ROC. In hundreds of examples we found that the ROC curves of the fusion system for specific logics were concave, that is randomization of the tunings is unnecessary. (We remark that, obviously, in most cases where neither logic dominates the other, the overall ROC will not be concave, since it will have a dimple at the point where the curves corresponding to two specific logics cross.) Without trying to argue that this result should be expected to hold universally, we present a counterexample, again using discrete sensors that receive three possible signals where randomization improves the performance of the AND rule:

$$R_1(x) = \begin{cases} 5x, & \text{if } 0 \le x \le \frac{1}{10}, \\ \frac{4}{3}x + \frac{11}{30}, & \text{if } \frac{1}{10} < x \le \frac{4}{10}, \\ \frac{1}{6}x + \frac{5}{6}, & \text{if } \frac{4}{10} < x \le 1, \end{cases}$$

$$R_2(x) = \begin{cases} \frac{47}{20}x, & \text{if } 0 \le x \le \frac{1}{5}, \\ \frac{23}{25}x + \frac{143}{500}, & \text{if } \frac{1}{5} < x \le \frac{9}{20}, \\ \frac{30}{55}x + \frac{25}{55}, & \text{if } \frac{9}{20} < x \le 1. \end{cases}$$

After simple but tedious computations, we get

$$R_0^{\text{AND}}(\alpha) = \begin{cases} \frac{70}{9}\alpha, & \text{if } 0 \le \alpha \le \frac{9}{200}, \\ \frac{30}{11}\alpha + \frac{5}{22}, & \text{if } \frac{9}{200} < \alpha \le \frac{1}{10}, \\ \frac{4}{3}\alpha + \frac{11}{30}, & \text{if } \frac{1}{10} < \alpha \le \frac{1639}{11050}, \\ \frac{621}{300}\alpha + \frac{3861}{15000}, & \text{if } \frac{1639}{11050} < \alpha \le \frac{9}{50}, \\ \frac{27}{22}\alpha + \frac{9}{22}, & \text{if } \frac{9}{50} < \alpha \le \frac{4}{10}, \\ \frac{1}{6}\alpha + \frac{5}{6}, & \text{if } \frac{4}{10} < \alpha \le 1. \end{cases}$$

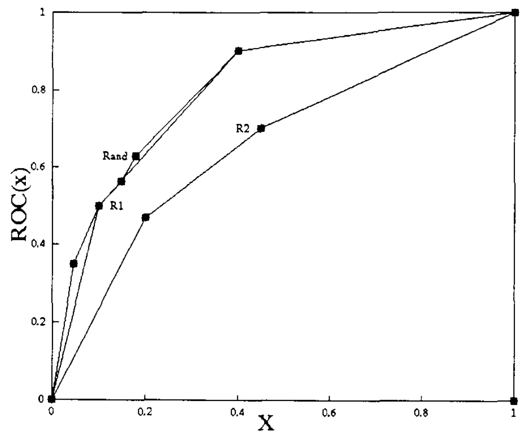

Fig. 5. Example of nonconcavity of R_0^{AND}.

Fig. 5 gives the graph of R_1, R_2, and R_0^{AND}. We see that R_0^{AND} is not a concave function.

V. Discussion and Conclusions

The obvious conclusion is that these are treacherous waters, and something may hold very often, without being true. It may even, as in the case of our first two examples, be supported by a plausible argument, but remain false.

The second example, revealing a breaking of the obvious symmetry, is less surprising, but may have more of a cautionary impact. Failure of this naive symmetry makes us more doubtful about the larger symmetry assumed in the restriction to k-out-of-n rules. That restriction has been of practical importance, in reducing astronomical numbers of possibilities to small numbers of possibilities. It leads us to suspect that the sufficiency of k-out-of-n rules, even for identical sensors, may never be established as a theorem, but will remain a heuristic. The more positive conclusion of our analysis is that the deviations found here, which show certain conjectures to be false, are all numerically small. This holds open the possibility that they are small in every case, so that assuming the conjectures to be true will lead to small numerical errors in the determination of the optimal tuning and fusion of a distributed sensor system.

Acknowledgment

The authors thank the editor and an anonymous referee for helpful comments and [12], [13].

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Jointly Optimal Routing and Scheduling in Packet Radio Networks

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Abstract—**A multihop packet radio network is considered with a single traffic class and given end-to-end transmission requirements. A transmission schedule specifies at each time instant the set of links which are allowed to transmit. The purpose of a schedule is to prevent interference among transmissions from neighboring links. Given amounts of information are residing initially at a subset of the network nodes and must be delivered to a prespecified set of destination nodes. The transmission schedule that evacuates the network in minimum time is specified. The decomposition of the problem into a pure routing and a pure scheduling problem is crucial for the characterization of the optimal transmission schedule.**

Index Terms—**Radio networks, scheduling, routing, throughput, multiple access, delay, protocol, network topology.**

I. Introduction

In this correspondence, we study the problem of joint link activation and route selection in Packet Radio Networks (PRN's). We consider the case of network evacuation, that is the case in which we wish to deliver all packets initially residing at each node of the network to a fixed, common destination node. At each node we assume that there exists a single transceiver. Consequently, to ensure conflict-free transmissions, no two links that share a common node may be activated simultaneously. We also assume that suitable spread-spectrum signaling modulation is used, so that no additional restriction on simultaneous link activation is needed to ensure conflict-free communication, i.e., there is no "hidden terminal" problem [7]. The problem of scheduling link activation in PRN's has been studied extensively under various assumptions [1]-[3], [5], [6]. Hajek and Sasaki in [1] have studied the optimal scheduling problem for given link flow requirements. They derived an algorithm of polynomial time complexity that solves the problem of pure

Manuscript received May 12, 1990; revised May 20, 1991. This work was presented in part at the American Control Conference, Pittsburgh, PA, June 1989.

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IEEE Log Number 9102766.

0018-9448/92$03.00