A generalized DEA model for inputs/outputs estimation

Abdollah Hadi-Vencheh\textsuperscript{a,}\textsuperscript{*}, Ali Asghar Foroughi\textsuperscript{b}

\textsuperscript{a} Department of Mathematics, Islamic Azad University, P.O. Box 81595-158, Khorasgan, Isfahan, Iran
\textsuperscript{b} Department of Mathematics, Qom University, Isfahan Old Road, Qom 37165, Iran

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The first author dedicates his work to the memory of his brother, Mohammad, who passed away so young.

Abstract

In this paper we discuss the question: among a group of decision making units (DMUs), if a DMU changes some of its input (output) levels, to what extent should the unit change outputs (inputs) such that its efficiency index remains unchanged? In order to solve this question we propose a solving method based on Data Envelopment Analysis (DEA) and Multiple Objective Linear Programming (MOLP). In our suggested method, the increase of some inputs (outputs) and the decrease due to some of the other inputs (outputs) are taken into account at the same time, while the other offered methods do not consider the increase and the decrease of the various inputs (outputs) simultaneously. Furthermore, existing models employ a MOLP for the inefficient DMUs and a linear programming for weakly efficient DMUs, while we propose a MOLP which estimates input/output levels, regardless of the efficiency or inefficiency of the DMU. On the other hand, we show that the current models may fail in a special case, whereas our model overcomes this flaw. Our method is immediately applicable to solve practical problems.

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1. Introduction

DEA is a method for assessing the productive efficiency of DMUs such as the branches of a bank or schools which are homogeneous in the sense that they use the same types of resources (inputs) to produce the same kinds of goods or services (outputs). The pioneering work of the economist Farrell [1] provided a non-parametric method for determining the relative efficiency of a DMU. He presented a method of computing the facets of the efficient production frontier based on a set of empirical observed DMUs, rather than estimating the parameters of postulated production functions. The celebrated paper of Charnes et al. [2] introduced DEA, a linear programming technique, to measure Farrell type efficiency.

In recent decades, DEA has rapidly expanded into new applied areas (see [3] for a survey). In addition to its original use in efficiency measurement, DEA is also employed for approximating production possibility sets or

\* Corresponding address: Islamic Azad University, Khorasgan Branch, Department of Mathematics, P.O. Box 81595-158, University Blvd., Arqavanieh, Jey St, Isfahan, Khorasgan, Iran. Tel.: +98 311 5211266; fax: +98 311 5211265.

E-mail address: ahadi@khuisf.ac.ir (A. Hadi-Vencheh).
input/output correspondence [4], recovering shadow prices [5], providing best practice benchmarks [6], monitoring agency problems [7] and so on.

In an inverse optimization problem we need to use an optimal solution to determine parameter values. Indeed, we have a given feasible solution which is not optimal under the current parameter values of the system. We wish to adjust these parameter values as little as possible so that the feasible solution be optimized under the adjusted parameter values. More generally, we wish that after adjusting the parameters as little as possible, the optimal solution should possess some required properties. We refer for such problems and their solution techniques to Refs. [8–10]. On the other hand, Pendharkar [11] proposed a methodology that uses DEA for solving the inverse classification problem. An inverse classification problem involves finding out how predictor attributes of a case can be changed such that the case can be classified into a different and more desirable class. He showed that under the assumption of conditional monotonicity, and convexity of classes, DEA can be used for the inverse classification problem. Cherchye and Puyenbroeck [12] defined an inverse measure based on mix properties of the efficient reference. Recently, Wei et al. [13] introduced inverse DEA, to answer the following question: among a group of DMUs, if we increase certain inputs (outputs) of a particular unit, how much should the outputs (inputs) of the DMU be increased in order that the DMU retains its efficiency level with respect to the other DMUs? Yan et al. [14] discussed inverse DEA on the con ratio DEA model and applied it to the additional resource allocation problem. We deal with a more general case of the above-mentioned question in [13]. In fact in [13] only the increase of inputs (outputs) is considered where a unit may concern the increase of some of inputs (outputs) and decrease of the other inputs (outputs). This question is more general and includes [13,14] as special cases. In this paper, we consider the arbitrary change in inputs (outputs) level and propose a MOLP model for estimating outputs (inputs). In order to estimate the output levels, Wei et al. proposed a MOLP when the DMU under evaluation is inefficient, and introduced a linear programming (LP) model when the DMU is weakly efficient. In this paper we show that this classification is not necessary and our model is able to estimate input/output levels, regardless of the efficiency level of the DMU. As a special case, we consider the case in which inputs (outputs) are simultaneously increased to cover [13]. In fact their model for efficient DMUs is a special case of the relevant MOLP. In addition, we show that the solution proposed by Wei et al. does not guarantee the efficiency result for input estimating. Indeed, their method may fail in a special case, where we propose some solutions to overcome this failure in their model.

The rest is organized as follows: in Section 2 we review DEA models and state our problem; in Section 3 we propose a generalized model for estimating input/output levels; Section 4 investigates some special cases which cover [13] and at the end introduces some solutions to modify [13] for a presented special case failure; we suggest three simple methods to solve the general model in Section 5; a series of real data is presented to substantiate the accuracy and applicability of our model in Section 6; finally conclusions are given in Section 7.

2. Background

Suppose we have \( n \) observations on \( n \) DMUs with input and output vectors \((x_j, y_j)\), for \( j = 1, 2, \ldots, n \). Let \( x_j = (x_{1j}, x_{2j}, \ldots, x_{mj})^T \) and \( y_j = (y_{1j}, y_{2j}, \ldots, y_{sj})^T \), where \( x_j \in \mathbb{R}^m \), \( y_j \in \mathbb{R}^s \), \( x_j > 0 \) and \( y_j > 0 \), for \( j = 1, 2, \ldots, n \). The input matrix \( X \) and output matrix \( Y \) can be represented as

\[
X = [x_1, \ldots, x_j, \ldots, x_n], \quad Y = [y_1, \ldots, y_j, \ldots, y_n]
\]

where \( X \) and \( Y \) are an \((m \times n)\) matrix and an \((s \times n)\) matrix, respectively. As outlined in [15], four different production possibility sets are usually derived from the data set \( \{(x_j, y_j) : j = 1, \ldots, n \} \). We define the most general production possibility set as \( T \) below:

\[
T = \left\{ (x, y) : \sum_{j=1}^{n} x_j \lambda_j \leq x, \sum_{j=1}^{n} y_j \lambda_j \geq y, \lambda \in \Lambda \right\},
\]

where \( \Lambda \) is one of the following:

\[
\Lambda_C = \left\{ \lambda : \lambda \geq 0 \right\},
\]

\[
\Lambda_V = \left\{ \lambda : \sum_{j=1}^{n} \lambda_j = 1, \lambda \geq 0 \right\}.
\]
\[ A_{NI} = \left\{ \lambda : \sum_{j=1}^{n} \lambda_j \leq 1, \lambda \geq 0 \right\}, \]

\[ A_{ND} = \left\{ \lambda : \sum_{j=1}^{n} \lambda_j \geq 1, \lambda \geq 0 \right\}, \]

where \( \lambda = (\lambda_1, \ldots, \lambda_n)^t \in \mathbb{R}^n \). Therefore we obtain four production possibility sets, in which we denote \( T \) with \( T_C, T_V, T_{NI} \) or \( T_{ND} \), when \( \lambda \in A_C, \lambda \in A_V, \lambda \in A_{NI}, \) and \( \lambda \in A_{ND} \), respectively. When a DMU, \( o \in \{1, 2, \ldots, n\} \), is under evaluation, we use an input-oriented DEA model as follows:

\[
(P_I) \min \theta \\
\text{s.t. } (\theta x_o, y_o) \in T,
\]
similarly, one can use an output-oriented model:

\[
(P_O) \max \varphi \\
\text{s.t. } (x_o, \varphi y_o) \in T,
\]
where \( T \) is one of \( T_C, T_V, T_{NI} \) and \( T_{ND} \).

**Definition 1.** The optimal value \( \theta_I(\varphi_O) \) of problem \( P_I(P_O) \) is called the efficiency index of DMU. When \( \theta_I = 1 (\varphi_O = 1) \), we say DMU is (at least) weakly efficient.

Consider the following question: if the outputs change and we desire the efficiency index \( \theta_I \) to remain unchanged, how much should the inputs of the DMU change? To solve our problem, suppose the outputs of DMU are changed from \( y_o \) to \( \beta_o = y_o + \Delta y_o \), where the vector \( \Delta y_o \in \mathbb{R}^t \). We need to estimate the input vector \( \alpha_o \) provided that the efficiency index of DMU \( o \) is still \( \theta_I \). Here

\[
\alpha_o = (\alpha_1, \alpha_2, \ldots, \alpha_m)^t = x_o + \Delta x_o, \quad \Delta x_o \in \mathbb{R}^m.
\]

For convenience, suppose DMU \( n+1 \) represents DMU after changing its inputs and outputs. Hence, to measure the efficiency of DMU \( n+1 \), we use the following model:

\[
(P_I^+) \min \theta \\
\text{s.t. } (\theta x_o, \beta_o) \in T^+,
\]
where

\[
T^+ = \left\{ (x, y) : \sum_{j=1}^{n+1} x_j \lambda_j \leq x, \sum_{j=1}^{n+1} y_j \lambda_j \geq y, \lambda \in A^+ \right\},
\]

\[ x_{n+1} = \alpha_o, y_{n+1} = \beta_o \] and \( A^+ \) is one of \( A_C, A_V, A_{NI} \) and \( A_{ND} \) only with the difference that \( \lambda \in \mathbb{R}^{n+1} \).

**Definition 2.** If the optimal value of problem \( (P_I^+) \) is equal to the optimal value of problem \( (P_I) \), we say that the efficiency is unchanged and we write \( \text{score}(\alpha_o, \beta_o) = \text{score}(x_o, y_o) \).

3. Generalized model

In this section we are going to answer our question. In fact we are looking for inputs that could produce \( \beta_o \) and at the same time preserve the efficiency index. In other words, we find \( \alpha_o \) such that \( (\alpha_o, \beta_o) \in T \) and \( \text{score}(\alpha_o, \beta_o) = \text{score}(x_o, y_o) \). Consider the following MOLP:

\[
(V_I) \min \alpha = (\alpha_1, \ldots, \alpha_m) \\
\text{s.t. } (\theta_I \alpha, \beta_o) \in T,
\]
where \( \beta_o \) and \( T \) are defined as before and \( \theta_I \) is the optimal value of problem \( (P_I) \).
**Definition 3.** Let \((\alpha_o, \lambda)\) be a feasible solution of problem \((V_I)\). If there is no feasible solution \((\alpha^*, \lambda^*)\) of \((V_I)\) such that \(\alpha^*_i \leq \alpha_i\) for all \(i = 1, 2, \ldots, m\) and \(\alpha^*_i < \alpha_i\) for at least one \(i\), then we say \((\alpha_o, \lambda)\) is a strongly efficient solution of \((V_I)\).

**Definition 4.** Suppose \((\alpha_o, \lambda)\) is a feasible solution of problem \((V_I)\). If there is no feasible solution \((\alpha^+, \lambda^+)\) of \((V_I)\) such that \(\alpha^+_i < \alpha_o\), that is, \(\alpha^*_i < \alpha_i\) for all \(i = 1, 2, \ldots, m\), then we say \((\alpha_o, \lambda)\) is a weakly efficient solution of problem \((V_I)\).

For convenience, we say \(\alpha_o\) is a strongly (weakly) efficient solution of problem \((V_I)\).

**Theorem 1.** Let \(\alpha_o\) be a feasible solution of problem \((V_I)\). Then, \(\text{score}(\alpha_o, \beta_o) = \text{score}(x_o, y_o)\) if and only if \(\alpha_o\) is a weakly efficient solution of problem \((V_I)\).

**Proof.** First suppose \(\alpha_o\) is a weakly efficient solution of problem \((V_I)\) and \((\theta^+_I, \lambda^+)\) is an optimal solution of problem \((P_T^+)\). We show that \(\text{score}(\alpha_o, \beta_o) = \text{score}(x_o, y_o)\), in other words \(\theta^+_I = \theta_I\). Since \((\alpha_o, \lambda)\) is a feasible solution of \((V_I)\), therefore \((\theta_I \alpha_o, \beta_o) \in T \subseteq T^+\). Let \(\lambda^+ = (\lambda, 0)^t\); it is easy to see that \((\theta_I, \lambda^+)\) is a feasible solution of problem \((P_T^+)\) and hence \(\theta^+_I \leq \theta_I\). Now assume \(\theta^+_I < \theta_I\). Because \((\theta^+_I, \lambda^+)\) is an optimal solution of problem \((P_T^+)\), we have

\[
\theta^+_I \alpha_o \geq X \vec{\lambda} + \alpha_o \lambda^+ \tag{1}
\]

\[
Y \lambda^+ \geq \beta_o,
\]

\[
\lambda^+ \in \Lambda^+,
\]

where \(\lambda^+ = (\vec{\lambda}, \lambda^+_{n+1})^t\) and \(\vec{\lambda} = (\lambda^+_1, \ldots, \lambda^+_n)^t \in \mathbb{R}^n\). On the other hand \((\alpha_o, \lambda)\) is a feasible solution of \((V_I)\) and \(\theta_I \leq 1\), then

\[
\alpha_o \geq X \lambda. \tag{2}
\]

Now according to (1) and (2),

\[
\theta^+_I \alpha_o \geq X \vec{\lambda} + \alpha_o \lambda^+_{n+1} \geq X \vec{\lambda} + \lambda^+_{n+1} X \lambda = X (\vec{\lambda} + \lambda^+_{n+1} \lambda). \tag{3}
\]

Let \(\tilde{\lambda} = \vec{\lambda} + \lambda^+_{n+1} \lambda\), and write (3) as

\[
\theta^+_I \alpha_o \geq X \tilde{\lambda}. \tag{4}
\]

Using a similar method we have

\[
Y \tilde{\lambda} \geq \beta_o, \tag{5}
\]

and hence

\[
X \tilde{\lambda} \leq \theta^+_I \alpha_o < \theta_I \alpha_o,
\]

\[
Y \tilde{\lambda} \geq \beta_o.
\]

It is easy to see that \(\tilde{\lambda} = (\tilde{\lambda}_1, \ldots, \tilde{\lambda}_n)^t \in \Lambda\). Thus \((\alpha_o, \tilde{\lambda})\) is a feasible solution of problem \((V_I)\) and there exists a \(k < 1\) such that

\[
X \tilde{\lambda} \leq \theta_I (k \alpha_o),
\]

\[
Y \tilde{\lambda} \geq \beta_o;
\]

therefore \((k \alpha_o, \beta_o)\) is a feasible solution of \((V_I)\) and \(k \alpha_o < \alpha_o\). But this is impossible because \(\alpha_o\) is a weakly efficient solution of problem \((V_I)\). Conversely, let \(\theta^+_I = \theta_I\). We show that \(\alpha_o\) is a weakly efficient solution of problem \((V_I)\). By contradiction assume \(\alpha_o\) is not a weakly efficient solution of problem \((V_I)\), so there exists a feasible solution \((\alpha', \lambda')\) of problem \((V_I)\) such that \(\alpha' < \alpha_o\). Since \((\alpha', \lambda')\) is a feasible solution of problem \((V_I)\), we have

\[
X \lambda' \leq \theta_I \alpha' < \theta_I \alpha_o,
\]

\[
Y \lambda' \geq \beta_o.
\]
so there exists a $k < 1$ such that

$$X\lambda' \leq (k\theta_I)\alpha_o,$$

$$Y\lambda' \geq \beta_o.$$  

Now let $\lambda^+ = (\lambda', 0)^T$. It is obvious that $\lambda^+ \in \Lambda^+$. Hence $((k\theta_I)\alpha_o, \beta_o) \in T^+$, that is, $(k\theta_I, \lambda^+)$ is a feasible solution of $(P_o^+)$ and $k\theta_I < \theta_I$. But this is against the assumption that $\theta_I$ is the optimal value of problem $(P_o^+)$. □

Consider the case that the DMU attempts to change input levels from $x_o$ to $\alpha_o = x_o + \Delta x_o, \Delta x_o \in \mathbb{R}^m$, and suppose the efficiency index $\varphi_O$ of the unit remains unchanged. Now the question is: how many outputs should the DMU produce? The following theorem solves this problem.

**Theorem 2.** Let $\beta_o$ be a feasible solution of the following MOLP:

$$(V_O) \max \beta = (\beta_1, \ldots, \beta_s)$$

s.t. $(\alpha_o, \varphi_O \beta) \in T$,

where $\varphi_O$ is given as the optimal value of problem $(P_O)$. Then, $\text{score}(\alpha_o, \beta_o) = \text{score}(x_o, y_o)$ if and only if $\beta_o$ is a weakly efficient solution of problem $(V_O)$.

**Proof.** This is similar to the proof of the Theorem 1 only with some minor modifications. □

4. Some special cases

In this section we consider some special cases. First consider the case that the input levels are increased and estimating the increment of the output levels is desired such that the efficiency index of the DMU remains unchanged. Consider the following MOLP:

$$(V'_O) \max \beta = (\beta_1, \ldots, \beta_s)$$

s.t. $(\theta_I \alpha, \beta_o) \in T$,

$$\beta \geq y_o,$$

where $\varphi_O$ is the optimal value of problem $(P_O)$. Note that the above model is $(V_O)$ only with the difference that the $\beta \geq y_o$ is an additional constraint(s). It is easy to verify that $\beta_o$ is an estimation of the output levels when $\beta_o$ is a weakly efficient solution of $(V'_O)$. In this case the above model is the same as the model obtained by Wei et al. for inefficient DMUs. Besides, we see (in Section 5) that their model for weakly efficient DMUs is a special case of problem $(V'_O)$.

Now consider the case in which DMU$_o$ retains current performance levels, but the output levels are increased and we want to know to what level the inputs of the DMU should be increased. According to [13] if DMU$_o$ is inefficient and $\alpha_o$ is a weakly efficient solution of the following MOLP,

$$(V'_I) \min \alpha = (\alpha_1, \ldots, \alpha_m)$$

s.t. $(\theta_I \alpha, \beta_o) \in T$,

$$\alpha \geq x_o,$$

then, $\alpha_o$ was claimed to be an estimation of the input levels. The following example illustrates that this is not true.

**Example 1.** Consider Table 1.

<table>
<thead>
<tr>
<th>DMU</th>
<th>A</th>
<th>B</th>
</tr>
</thead>
<tbody>
<tr>
<td>Input</td>
<td>$x_1$</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>$x_2$</td>
<td>1</td>
</tr>
<tr>
<td>Output</td>
<td>$y$</td>
<td>2</td>
</tr>
</tbody>
</table>
Here we have 2 DMUs, A and B, with 2 inputs $x_1$ and $x_2$ and one output $y$. We take B into consideration in the BCC model, and solve problem $(P_I)$:

$$\begin{align*}
\min \theta \\
\text{s.t.} \quad \lambda_1 + 2\lambda_2 &\leq 2\theta \\
\lambda_1 + 4\lambda_2 &\leq 4\theta \\
2\lambda_1 + \lambda_2 &\geq 1 \\
\lambda_1 + \lambda_2 &\geq 1 \\
\lambda_1, \lambda_2 &\geq 0.
\end{align*}$$

The optimal solution is $(\lambda_1, \lambda_2) = (1, 0)$ and $\theta_I = \frac{1}{2}$. Now suppose the output is increased from 1 to $\frac{3}{2}$, and we are interested in knowing how many more inputs the unit would require. Therefore we solve problem $(V_I)$:

$$\begin{align*}
\min (\alpha_1, \alpha_2) \\
\text{s.t.} \quad \lambda_1 + 2\lambda_2 &\leq \frac{1}{2}\alpha_1 \\
\lambda_1 + 4\lambda_2 &\leq \frac{1}{2}\alpha_2 \\
2\lambda_1 + \lambda_2 &\geq \frac{3}{2} \\
\alpha_1 &\geq 2 \\
\alpha_2 &\geq 4 \\
\lambda_1 + \lambda_2 &\geq 1 \\
\lambda_1, \lambda_2 &\geq 0.
\end{align*}$$

It can be seen that $(\alpha_1, \alpha_2, \lambda_1, \lambda_2) = (4, 4, 1, 0)$ is a weakly efficient solution of the above problem. Now let $\alpha_B = (4, 4)$ and consider the problem $(P_I^+)$:

$$\begin{align*}
\min \theta \\
\text{s.t.} \quad \lambda_1 + 2\lambda_2 + 4\lambda_3 &\leq 4\theta \\
\lambda_1 + 4\lambda_2 + 4\lambda_3 &\leq 4\theta \\
2\lambda_1 + \lambda_2 + \frac{3}{2}\lambda_3 &\geq \frac{3}{2} \\
\lambda_1 + \lambda_2 + \lambda_3 &\geq 1 \\
\lambda_1, \lambda_2, \lambda_3 &\geq 0.
\end{align*}$$

The optimal solution is $(\theta_I^+, \lambda_1, \lambda_2, \lambda_3) = (\frac{1}{2}, 1, 0, 0)$ and $\theta_I^+ = \frac{1}{4} \neq \frac{1}{2} = \theta_I$. Thus $\text{score}(4, 4, \frac{3}{2}) \neq \text{score}(2, 4, 1)$ and this contradicts the proposed solution in [13] for input estimation. In order to rectify this drawback we present two suggestions as follows:

1. We assume that the output levels are increased and the input levels are changed, then weakly efficient solutions of problem $(V_I)$ are sufficient to estimate inputs.
2. If we are going to estimate the increment in the input levels, then the following theorem can be applied.

**Theorem 3.** Let $\alpha_o$ be a feasible solution of problem $(V_I')$ and satisfy one of the following conditions:

(i) $\alpha_o = x_o$.

(ii) $\alpha_o$ is a weakly efficient solution of $(V_I')$ and $\alpha_o > x_o$.

(iii) $\alpha_o$ is a strongly efficient solution of $(V_I')$.

Then, $\text{score}(\alpha_o, \beta_o) = \text{score}(x_o, y_o)$.

**Proof.** Suppose $\theta_I$ and $\theta_I^+$ are the optimal values of $(P_I)$ and $(P_I^+)$, respectively. We must prove that $\theta_I^+ = \theta_I$. Similarly to **Theorem 1** it is clear that $\theta_I^+ \leq \theta_I$. Now if $\theta_I^+ < \theta_I$ then there exists $\lambda \in A$ such that

$$\begin{align*}
X\lambda &\leq \theta_I^+ x_o, \\
Y\lambda &\geq \beta_o \geq y_o.
\end{align*}$$
Therefore we have:

(i) If $\alpha_o = x_o$, then according to the above constraints, it is obvious that $(\theta_I^+ , \tilde{\lambda})$ is a feasible solution of problem $(P_I)$. Hence $\theta_I \leq \theta_I^+$, which is a contradiction.

(ii) If $\alpha_o$ is a weakly efficient solution of $(V_I')$ and $\alpha_o > x_o$ then $\alpha_o$ is a weakly efficient solution of $(V_I)$. Thus, the proof follows from Theorem 1.

(iii) When $\alpha_o$ is a strongly efficient solution of $(V_I')$, then we have two possibilities:

(iii1) $\alpha_o = x_o$. In this case the proof is the same as case (i).

(iii2) $\alpha_o \geq x_o$. That is, there exists at least one $i, 1 \leq i \leq m$, such that $\alpha_i > x_{io}$. In this case, it is easy to see that there is a $k > 0$ such that $(\alpha_k, \tilde{\lambda})$ is a feasible solution of problem $(V_I')$, where $\alpha_k = \alpha_o - ke_i$ and $e_i$ is the $i$-th unit vector in $\mathbb{R}^m$. But this is against the assumption that $\alpha_o$ is a strongly efficient solution of problem $(V_I')$. □

5. Solving the relevant MOLP

There are several methods that we can apply to solve a MOLP (see e.g. [16]). Here we propose and investigate three simple methods for solving $(V_I)$ or $(V_O)$:

(1) Consider that all inputs are weighted (priced) and the weights (values) are known. Let $w_i, i = 1, \ldots, m$, be the value weight for unit of input $i$. So to solve $(V_I)$ we suggest the following LP model:

$$
\text{min } \sum_{i=1}^{m} w_i \alpha_i \\
\text{s.t. } (\theta_I, \beta_o) \in T.
$$

Example 2. Consider Table 2.

In this table we have 5 DMUs with 2 inputs, $x_1, x_2$, and 2 outputs, $y_1$ and $y_2$. Take B into account in the CCR model:

$$
\text{min } \theta \\
\text{s.t. } 10\lambda_1 + 15\lambda_2 + 20\lambda_3 + 25\lambda_4 + 12\lambda_5 \leq 15\theta \\
20\lambda_1 + 15\lambda_2 + 30\lambda_3 + 15\lambda_4 + 9\lambda_5 \leq 15\theta \\
70\lambda_1 + 100\lambda_2 + 80\lambda_3 + 100\lambda_4 + 90\lambda_5 \geq 100 \\
6\lambda_1 + 3\lambda_2 + 5\lambda_3 + 2\lambda_4 + 8\lambda_5 \geq 3 \\
\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \geq 0.
$$

The optimal value is $\theta = 0.889$. Now suppose the output vector is changed from $y_B = (100, 3)^I$ to $\beta_B = (90, 5)^I$ and let $W = (1, 3)$. To estimate the input vector solve the following LP:

$$
\text{min } \alpha_1 + 3\alpha_2 \\
\text{s.t. } 10\lambda_1 + 15\lambda_2 + 20\lambda_3 + 25\lambda_4 + 12\lambda_5 \leq 0.889\alpha_1 \\
20\lambda_1 + 15\lambda_2 + 30\lambda_3 + 15\lambda_4 + 9\lambda_5 \leq 0.889\alpha_2 \\
70\lambda_1 + 100\lambda_2 + 80\lambda_3 + 100\lambda_4 + 90\lambda_5 \geq 90 \\
6\lambda_1 + 3\lambda_2 + 5\lambda_3 + 2\lambda_4 + 8\lambda_5 \geq 5 \\
\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5 \geq 0.
$$
Table 3

Data of Example 3

<table>
<thead>
<tr>
<th>DMU</th>
<th>A</th>
<th>B</th>
<th>C</th>
<th>D</th>
</tr>
</thead>
<tbody>
<tr>
<td>Input</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$x_1$</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>2</td>
</tr>
<tr>
<td>$x_2$</td>
<td>2</td>
<td>1</td>
<td>10</td>
<td>5</td>
</tr>
<tr>
<td>$x_3$</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>10</td>
</tr>
<tr>
<td>Output</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$y$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
</tbody>
</table>

The optimal solution is $\alpha_1 = 13.498$, $\alpha_2 = 10.124$, $\lambda_5 = 1.000$ and the other variables are equal to zero. Thus the DMU should provide input vector $\alpha_B = (13.498, 10.124)'$ to produce output vector $\beta_B = (90, 5)'$.

(2) Suppose we desire to change inputs into a radial form, that is, $\alpha_o = z X_o$, where $z$ is a parameter. We offer the following LP to be applied for solving $(V_I)$:

$$\min \ z$$

s.t. $(\theta x_o z, \beta_o) \in T$.

Remark 1. If we consider the outputs estimate then the corresponding model is as follows:

$(V''_O) \max \ z$

s.t. $(\alpha_o, \varphi O \beta_o z) \in T$.

Now, when DMU $o$ is weakly efficient then $\varphi_o = 1$. So $(V''_O)$ is converted to

$$\max \ z$$

s.t. $(\alpha_o, \beta_o z) \in T$.

The above model is the same as presented in [13] for weakly efficient DMUs, indeed it is actually a special case of $(V''_O)$.

Example 3. Consider Table 3 with 4 DMUs (A, B, C, D), 3 inputs ($x_1$, $x_2$ and $x_3$) and a single output ($y$).

Consider C in the CCR model:

$$\min \ \theta$$

s.t. $\lambda_1 + \lambda_2 + 2 \lambda_3 + 2 \lambda_4 \leq 2 \theta$

$2 \lambda_1 + \lambda_2 + 10 \lambda_3 + 5 \lambda_4 \leq 10 \theta$

$\lambda_1 + 2 \lambda_2 + 5 \lambda_3 + 10 \lambda_4 \leq 5 \theta$

$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \geq 1$

$\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$.

The optimal solution is $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (1, 0, 0, 0)$ and $\theta_I = \frac{1}{5}$. Suppose the change has a radial form and output level is increased from 1 to 3. In order to estimate the input levels we solve the following LP:

$$\min \ z$$

s.t. $\lambda_1 + \lambda_2 + 2 \lambda_3 + 2 \lambda_4 \leq z$

$2 \lambda_1 + \lambda_2 + 10 \lambda_3 + 5 \lambda_4 \leq 5z$

$\lambda_1 + 2 \lambda_2 + 5 \lambda_3 + 10 \lambda_4 \leq \frac{5}{2}z$

$\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \geq 3$

$\lambda_1, \lambda_2, \lambda_3, \lambda_4 \geq 0$.

The optimal solution is $(\lambda_1, \lambda_2, \lambda_3, \lambda_4) = (3, 0, 0, 0)$ and $z = 3$. So the input levels should be increased from $(2, 10, 5)'$ to $\alpha_C = (6, 30, 15)'$. 
Table 4
Data of key Chinese cities and DEA efficiency scores

<table>
<thead>
<tr>
<th>DMU</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$y_1$</th>
<th>$y_2$</th>
<th>$y_3$</th>
<th>$\psi_0$</th>
<th>$\theta_I$</th>
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<tr>
<td>1</td>
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<td>616961</td>
<td>6785798</td>
<td>1594957</td>
<td>1088699</td>
<td>1.00</td>
<td>1.00</td>
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<tr>
<td>2</td>
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<td>355509</td>
<td>385453</td>
<td>2292025</td>
<td>406947</td>
<td>373600</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>3</td>
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<td>685584</td>
<td>341941</td>
<td>2292025</td>
<td>545140</td>
<td>835745</td>
<td>1.51</td>
<td>0.70</td>
</tr>
<tr>
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<td>202.02</td>
<td>432713</td>
<td>117424</td>
<td>1158016</td>
<td>135939</td>
<td>336165</td>
<td>1.92</td>
<td>0.85</td>
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<tr>
<td>5</td>
<td>197.93</td>
<td>423124</td>
<td>189743</td>
<td>1187130</td>
<td>190178</td>
<td>605037</td>
<td>1.00</td>
<td>1.00</td>
</tr>
<tr>
<td>6</td>
<td>148.04</td>
<td>367012</td>
<td>97004</td>
<td>658910</td>
<td>86514</td>
<td>239760</td>
<td>2.04</td>
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<tr>
<td>7</td>
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<td>245542</td>
<td>91861</td>
<td>854188</td>
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<td>239360</td>
<td>1.58</td>
<td>0.70</td>
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<tr>
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<td>91710</td>
<td>606743</td>
<td>78357</td>
<td>208188</td>
<td>1.85</td>
<td>0.58</td>
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<tr>
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<td>295812</td>
<td>92409</td>
<td>736545</td>
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<td>49357</td>
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<td>46198</td>
<td>867467</td>
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<td>66640</td>
<td>319218</td>
<td>31726</td>
<td>169051</td>
<td>1.42</td>
<td>0.73</td>
</tr>
</tbody>
</table>

(3) Assume that $\alpha_o$ takes the form $\alpha_o = x_o + Wz$, where $W$ is a given vector with nonnegative components and $z$ is a parameter. In this case we propose the following LP instead of problem (VI):

$$\min z \quad \text{s.t. } (\theta_I(x_o + Wz), \beta_o) \in T.$$  

6. An application

Our method is illustrated here via an application with real-world data. The data is from Charnes et al. [17], where they evaluated the efficiency in the economic performance of 28 key Chinese cities during 1983 and 1984. The cities played a critical role in the government’s program of economic development. In total, 3 inputs and 3 outputs were employed. The outputs were Gross Industrial Output Value ($y_1$), Profits and Taxes ($y_2$), and Retail Sales ($y_3$). The inputs were Labor ($x_1$), Working Funds ($x_2$), and Investments ($x_3$). Data on the above factors for the 28 DMUs are reported in Table 4. We have produced output-oriented CCR and input-oriented BCC efficiency scores in the last two columns of Table 4.

Assume that DMU$_{17}$ is under consideration in the BCC model. As we see from Table 4, the efficiency index of the DMU is 0.64. Suppose the output vector is changed from $y_{17} = (482448, 67857, 158250)^t$ to $\beta_{17} = (434203, 74643, 142425)^t$. In order to estimate the input vector we solve the following MOLP:

$$\min \alpha = (\alpha_1, \alpha_2, \alpha_3) \quad \text{s.t. } (0.64\alpha, \beta_{17}) \in T_V.$$  

There are several methods that we can apply to solve this problem. Here we use weighted sum method (case 1 in Section 5) to solve this problem. Let $w = (1, 2, 3)$; hence instead of the above MOLP we solve the
following LP:
\[
\begin{align*}
\text{min } & \alpha_1 + 2\alpha_2 + 3\alpha_3 \\
\text{s.t. } & (0.64\alpha, \beta_{17}) \in T_V.
\end{align*}
\]
The optimal solution is \(\alpha_{17} = (106.86, 164656.33, 47501.27)^T, \lambda_8 = 0.027, \lambda_{26} = 0.973\) and the other variables are equal to zero. Therefore DMU_{17} should utilize input vector \(\alpha_{17} = (106.86, 164656.33, 47501.27)^T\) to produce output vector \(\beta_{17} = (434203, 74643, 142425)^T\).

Now consider DMU_{27} in the CCR model. Although this DMU is efficient in the BCC model, it is inefficient in the CCR model and its efficiency index in the CCR model is 1.85. Now suppose the input vector is changed from \(x_{27} = (20.09, 50717, 54650)^T\) to \(\alpha_{27} = (18.08, 45645, 60115)^T\). To estimate the output vector, consider the following problem:
\[
\begin{align*}
\text{max } & \beta = (\beta_1, \beta_2, \beta_3) \\
\text{s.t. } & (\alpha_{27}, 1.85\beta) \in T_C.
\end{align*}
\]
Assume that \(w = (2, 2, 1)\), then solve the following problem:
\[
\begin{align*}
\text{max } & 2\beta_1 + 2\beta_2 + \beta_3 \\
\text{s.t. } & (\alpha_{27}, 1.85\beta) \in T_C.
\end{align*}
\]
The optimal solution is \(\beta_{27} = (132133.12, 22533.86, 25172.93)^T, \lambda_1 = 0.019, \lambda_{24} = 0.134\) and the other variables are equal to zero. Thus the DMU should produce the output vector \(\beta_{27}\) by utilizing input vector \(\alpha_{27} = (18.08, 45645, 60115)^T\) to retain its efficiency index.

7. Conclusions

In this paper we discuss this problem: how should we control the changes in input/output levels of a given DMU such that the efficiency index of the DMU is preserved. To solve the problem we propose a MOLP model. The model is immediately applicable and easily implemented for solving practical problems. Our method in comparison with current methods has some advantages, as follows.

1. Other methods estimate inputs (outputs) for a given DMU when some or all outputs (inputs) are increased, while we consider the general case: the increase of some inputs and the decrease due to some of the others may be taken into account at the same time.

2. To estimate input/output levels [13] employs a MOLP for the inefficient DMUs and a linear programming when the DMU is weakly efficient, while we propose a MOLP which is able to estimate input/output levels, regardless of the efficiency or inefficiency of the DMU. The previous method may fail for input estimation, whereas our method overcomes this drawback.

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References