On Strongly Multiplicative Graphs

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Abstract — A graph G with p vertices and q edges is said to be strongly multiplicative if the vertices are assigned distinct integers 1, 2, 3, ..., p such that the labels induced on the edges by the product of the end vertices are distinct. We prove some of the following graphs are strongly multiplicative. Acharya, Germina, and Petersen graphs, ladders, and unicyclic graphs strongly multiplicative. We show that the labeling of edges within the edge sets and among the edge sets are distinct. If it is assumed in each case an induced subgraph of a strongly multiplicative graph. In this paper we study strongly multiplicative labeling for some special classes of graphs.

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I. INTRODUCTION

In this paper we deal with only finite, simple, connected and undirected graphs obtained through graph operations. A labeling of a graph G is an assignment of labels to vertices or edges or both following certain rules. Labeling of graphs plays an important role in application of graph theory in Neural Networks, Coding theory, Circuit Analysis etc. A useful survey on graph labeling by J.A. Gallian (2010) can be found in [5]. All graphs considered here are finite, simple and undirected. In most applications labels are positive (or nonnegative) integers, though in general real numbers could be used.

Beineke and Hegde [4] call a graph with p vertices strongly multiplicative if the vertices of G can be labeled with distinct integers 1, 2, 3, ..., p such that the labels induced on the edges by the product of the end vertices are distinct. They prove the following results are strongly multiplicative: trees; cycles; wheels; K_n if and only if n ≤ 5; K_{r,t} if and only if r ≤ 4; and P_n × P_n. Beineke and Hegde [4] obtain an upper bound for the maximum number of edges λ(n) for a given strongly multiplicative graph of order n. It was further improved by C. Adiga, H. N. Ramaswamy, and D. D. Somashekara [2] for greater values of n. It remains an open problem to find a nontrivial lower bound for λ(n). Seoud and Zid [7] prove the following results are strongly multiplicative: wheels; rK_n for all r and n at most 5; rK_n for r ≥ 2 and n ≥ 6 or 7; rK_n for r ≥ 3 and n = 8 or 9; K_{4,5} for all r; and the corona of P_n and K_n for all n and 2 ≤ m ≤ 8. Germina and Ajitha [6] prove that K_5 + K_n, quadrilateral snakes, Petersen graphs, ladders, and unicyclic graphs are strongly multiplicative. Acharya, Germina, and Ajitha [1] have shown that every graph can be embedded as

II. MAIN RESULTS

Theorem 1:

The graph C_n^+ has strongly multiplicative labeling.

Proof: Let V = \{v_1, v_2, ..., v_n, v_{n+1}, ..., v_{2n}\} be the vertex set and E = E_1 \cup E_2 \cup E_3 be the edge set where E_1 = \{v_i, v_{i+1}, 1 ≤ i ≤ n\}, E_2 = \{v_i, v_{n+i}, 1 ≤ i ≤ n\} and E_3 = \{v_n v_{2n}\} of the graph C_n^+. Define a bijection f : V → \{1, 2, ..., 2n\} such that

Case (i): When n is even

\[ f(v_i) = \begin{cases} 4i - 3 & 1 ≤ i ≤ \frac{n}{2} \\ 4(n-i) + 3 & \frac{n}{2} + 1 ≤ i ≤ n \end{cases} \]

Define an induced function g : E → N, such that

\[ g(v_i v_j) = f(v_i) f(v_j) \quad \forall v_i v_j ∈ E \quad v_i, v_j ∈ V \]

We show that the labeling of edges within the edge sets and among the edge sets are distinct. If it is assumed in each case...
that the induced label of the edges are same then we arrive at contradiction.

For the edges in \( E_1 \):

Sub Case (a) For \( i \neq j \) and \( 1 \leq i, j < \frac{n}{2} \)

If we assume that \( g(v_{ij}) = g(v_{ji}) \)
\[
\Rightarrow f(v_i)f(v_{ij}) = f(v_j)f(v_{ji})
\]
\[
(4i - 3)(4i + 1) = (4j - 3)(4j + 1)
\]
gives \( i = j \) a contradiction as \( i + j = \frac{1}{2} \) is not possible.

Sub Case (b) For \( i \neq j \) and \( \frac{n}{2} + 1 \leq i, j < n \)

If we assume that \( g(v_{ij}) = g(v_{ji}) \)
\[
\Rightarrow f(v_i)f(v_{ij}) = f(v_j)f(v_{ji})
\]
\[
(4n - i + 3)(4n - i - 1) = (4n - j + 3)(4n - j - 1)
\]
gives \( i = j \) a contradiction as \( i + j = \frac{4n - 1}{2} \) is not possible.

Sub Case (c) For \( 1 \leq i < \frac{n}{2} \) and \( \frac{n}{2} + 1 \leq i < n \) also \( i \neq j \) in \( E_1 \)

We have \( g(v_{ij}) = 16i^2 - 8i \) and
\[
g(v_{ji}) = 16j^2 - 8j + (16n^2 + 8n - 32n)
\]
Clearly \( 16i^2 - 8i \neq (16j^2 - 8j) + (16n^2 + 8n - 32n) \)
\( \therefore g(v_{ij}) \neq g(v_{ji}) \)

For the edges in \( E_2 \):

Sub Case (a) For \( i \neq j \) and \( 1 \leq i, j \leq \frac{n}{2} \)

If we assume that \( g(v_{ij}) = g(v_{ji}) \)
\[
\Rightarrow f(v_i)f(v_{ij}) = f(v_j)f(v_{ji})
\]
\[
(4i - 3)(4j - 2) = (4j - 3)(4i - 2)
\]
gives \( i = j \) a contradiction as \( i + j = \frac{5}{4} \) is not possible.

Sub Case (b) For \( i \neq j \) and \( \frac{n}{2} + 1 \leq i, j \leq n \)

If we assume that \( g(v_{ij}) = g(v_{ji}) \)
\[
\Rightarrow f(v_i)f(v_{ij}) = f(v_j)f(v_{ji})
\]
\[
(4n - i + 3)(4n - i + 4) = (4n - j + 3)(4n - j + 4)
\]
gives \( i = j \) a contradiction as \( i + j = \frac{8n + 7}{4} \) is not possible.

Sub Case (c) For \( 1 \leq i < \frac{n}{2} \) and \( \frac{n}{2} + 1 \leq j < n \) also \( i \neq j \) in \( E_2 \):

We see that \( g(v_{n+i}) = 16i^2 - 20i + 6 \)
\[
g(v_{n+j}) = 16j^2 - 20j + 6 + (16n^2 - 32nj + 28n - 8j + 12)
\]
\( \therefore g(v_{n+i}) \neq g(v_{n+j}) \)

Clearly the labeling of edges of \( E_1 \) and that of \( E_2 \) are all distinct as the labeling of edges of \( E_1 \) are all odd and those of \( E_2 \) are even. Also edges of \( E_1 \) & \( E_3 \) and \( E_2 \) & \( E_3 \) are also distinct as edge in \( E_3 \) is with the minimum odd label 3.

\( C_n^+ \) has strongly multiplicative labeling for \( n \) even.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure1.png}
\caption{Strongly multiplicative labeling for \( C_n^+ \)}
\end{figure}

Case (ii): When \( n \) is odd

The bijection \( f : V \rightarrow \{1, 2, 3, \ldots, 2n\} \) is defined as the following

\[
f(v_i) = \begin{cases}
4i - 3 & 1 \leq i \leq \frac{n + 1}{2} \\
4(n - i) + 3 & \frac{n + 1}{2} + 1 \leq i \leq n
\end{cases}
\]
and

\[
f(v_{n+i}) = \begin{cases}
4i - 2 & 1 \leq i \leq \frac{n + 1}{2} \\
4(n - i) + 4 & \frac{n + 1}{2} + 1 \leq i \leq n
\end{cases}
\]
The proof follows as above, replacing $1 \leq i \leq \frac{n}{2}$ by 

$$1 \leq i \leq \frac{n+1}{2} \text{ and } \frac{n+1}{2} + 1 \leq i \leq n.$$ 

Thus $C_n^+$ has strongly multiplicative labeling for $n$ odd. Hence $C_n^+$ has strongly multiplicative labeling for all $n$.

For the edges in $E_1$: 
For $i \neq j$, $1 \leq i, j \leq n - 3$
If we assume $g(v_i v_{n-1}) = g(v_j v_{n-1})$
$$f(v_i) f(v_{n-2}) = f(v_j) f(v_{n-2})$$
$$\implies i(n-2) = j(n-2)$$
$$\implies i = j \text{ a contradiction}$$

For the edges in $E_2$: 
For $i \neq j$, $1 \leq i, j \leq n - 3$
If we assume $g(v_i v_{n-1}) = g(v_j v_{n-1})$
$$f(v_i) f(v_{n-1}) = f(v_j) f(v_{n-1})$$
$$\implies i(n-1) = j(n-1)$$
$$\implies i = j \text{ a contradiction}$$

For the edges in $E_3$: 
For $i \neq j$, $1 \leq i, j \leq 3$
If we assume $g(v_i v_{n-1}) = g(v_j v_{n-1})$
$$f(v_i) f(v_{n-1}) = f(v_j) f(v_{n-1})$$
$$\implies n(n-i) = n(n-j)$$
$$\implies i = j \text{ a contradiction}$$

Now we have to show that the edges between different edge sets are distinct.

For the edges in $E_1$ and $E_2$: 
For $i \neq j$, $1 \leq i, j \leq n - 3$
If we assume $g(v_i v_{n-1}) = g(v_j v_{n-1})$
$$f(v_i) f(v_{n-2}) = f(v_j) f(v_{n-2})$$
$$\implies i(n-2) = j(n-1)$$
$$\implies i - j = \frac{2i-j}{n} \text{ a contradiction}$$

For the edges in $E_1$ and $E_3$: 
For $i \neq j$, $1 \leq i, j \leq n - 3$, $j = 1, 2, 3$
If we assume $g(v_i v_{n-1}) = g(v_j v_{n-1})$
$$f(v_i) f(v_{n-2}) = f(v_j) f(v_{n-1})$$
$$\implies n(n-i) = \frac{n-j}{i} \text{ a contradiction}$$

For the edges in $E_2$ and $E_3$: 
For $i \neq j$, $1 \leq i, j \leq n - 3$, $j = 1, 2, 3$
If we assume $g(v_i v_{n-1}) = g(v_j v_{n-1})$
$$f(v_i) f(v_{n-1}) = f(v_j) f(v_{n-1})$$
$$\implies i(n-1) = \frac{n-j}{i} \text{ a contradiction}$$
\[ \Rightarrow n - 1 = \frac{n - j}{i} \text{ a contradiction.} \]
This implies all the edge labeling are distinct. Hence the graph \((P_2 \cup mk_1) + N_2\) has strongly multiplicative labeling.

\[ \Rightarrow in = jn \]
\[ \Rightarrow i = j \text{ a contradiction} \]

Now we have to show that the edges between different edge sets are distinct.
For the edges in \(E_1\) and \(E_2\):
For \(i \neq j, \ 1 \leq i, j \leq n - 2\)
If we assume \(g(v_i, v_{n-1}) = g(v_j, v_{n-1})\)
\[ \Rightarrow f(v_i)f(v_{n-1}) = f(v_j)f(v_{n-1}) \]
\[ \Rightarrow i(n-1) = jn \]
\[ \Rightarrow n = \frac{i}{i-j} \text{ a contradiction} \]

For the edges in \(E_1\) and \(E_3\):
For \(1 \leq i \leq n - 2\)
If we assume \(g(v_i, v_n) = g(v_{n-1}, v_n)\)
\[ \Rightarrow f(v_i)f(v_n) = f(v_{n-1})f(v_n) \]
\[ \Rightarrow in = n(n-1) \]
\[ \Rightarrow i = n - 1 \text{ a contradiction} \]

This implies all the edge labeling are distinct. Hence the graph \(P_2 + mk_1\) has strongly multiplicative labeling.
Definition 4(3): Let $G = (V,E : R_1, R_2)$. The vertex set $V \rightarrow \{1,2,\ldots,n\}$ and the edge set is defined by the relations $R_1$ and $R_2$ such that:

$$R_1 : b = a + 1 \quad \forall \quad a,b \in V$$

$$R_2 : a + b = n + 1$$

If $n = 0 \mod 2$, we get cycle $C_n$ with $d = (n-2)/2$ non-intersecting chords and when $n = 1 \mod 2$ we get cycle $C_n$ with $d = (n-3)/2$ non-intersecting chords.

The graph obtained by this relation is $C_n^d$, $n \geq 5$.

Theorem 5: The graph $C_n^d$, $n \geq 5$ with non intersecting chords has strongly multiplicative labeling.

Proof: Let $V = \{v_1, v_2, v_3, \ldots, v_n\}$, be the vertex set and $E = E_1 \cup E_2 \cup E_3$ be the edge set of the graph $C_n^d$ with non intersecting chords, where $E_i = \{v_i, v_{i+1}, 1 \leq i < n\}$, $E_2 = \{v_{n-i}v_{n-i+1} : 1 \leq i < \frac{n}{2}-1\}$ and $E_3 = \{v_nv_1\}$. Define a bijection $f : V \rightarrow \{1,2,3,\ldots,n\}$ such that

Case (i): When $n$ is even

$$f(v_i) = \begin{cases} 2i - 1 & 1 \leq i \leq \frac{n}{2} \\ 2(n-i) + 2 & \frac{n}{2} + 1 \leq i \leq n \end{cases}$$

Define an induced function $g : E \rightarrow N$, such that $g(v_i, v_j) = f(v_i)f(v_j) \quad \forall \quad v_i, v_j \in E$ and $v_i, v_j \in V$. We show that the labeling of edges within the edge sets and among the edge sets are distinct.

If it is assumed in each case that the induced label of the edges are same then we arrive to a contradiction.

For the edges in $E_1$:

Sub Case (a) For $i \neq j$ and $1 \leq i, j \leq \frac{n}{2} - 1$

If we assume that $g(v_i, v_{i+1}) = g(v_j, v_{j+1})$

$$\Rightarrow f(v_i)f(v_{i+1}) = f(v_j)f(v_{j+1})$$

$$(2i-1)(2i) = (2j-1)(2j)$$

$$i = j$$

Sub Case (b) For $i \neq j$ and $\frac{n}{2} + 1 \leq i, j \leq n$

If we assume that $g(v_i, v_{i+1}) = g(v_j, v_{j+1})$

$$\Rightarrow f(v_i)f(v_{i+1}) = f(v_j)f(v_{j+1})$$

$$(2(n-i)+2)(2(n-i+1)+2) = (2(n-j)+2)(2(n-j-1)+2)$$

$$i = j$$

A contradiction as $i + j = 2n + 1$ is not possible.

Sub Case (c) Edges of $E_1$ for $1 \leq i < \frac{n}{2}$ are with odd labels and those of $\frac{n}{2} + 1 \leq j < n$ are with even labels. Thus the edge labels of these categories are distinct.

For the edges in $E_2$:

For $i \neq j$ and $1 \leq i, j \leq \frac{n}{2} - 1$

If we assume that $g(v_i, v_{i+1}) = g(v_j, v_{j+1})$

$$\Rightarrow f(v_i)f(v_{i+1}) = f(v_j)f(v_{j+1})$$

$$(2i+1)(2i+2)(2n-i+1)(2n-i+2) = (2j+1)(2j+2)(2n-j+1)(2n-j+2)$$

A contradiction as $i = j$ is not possible.

Now we have to show that the edges between different edge sets are distinct.

For the edges in $E_1$ and $E_2$:

The labels of edges of $E_2$ and those of $E_1$ for $1 \leq i < \frac{n}{2}$ are distinct as the labels of $E_2$ are with even numbers and those of $E_1$ for $1 \leq i < \frac{n}{2}$ are with odd numbers.

Also $i \neq j$ and for $\frac{n}{2} + 1 \leq i < n$ and $1 \leq j \leq \frac{n}{2} - 1$

$$g(v_i, v_{i+1}) = f(v_i)f(v_{i+1}) = 2i^2 + i + (2n^2 + 2n + 4ni - 3i)$$

$$g(v_j, v_{j+1}) = f(v_j)f(v_{j+1}) = 2j^2 + j$$

$$g(v_i, v_{i+1}) \neq g(v_j, v_{j+1})$$

Moreover $g(v_n, v_1) = f(v_n)f(v_1) = 2$

Thus all the edges of $C_n^d$, $n \geq 5$ with non intersecting chords are distinct. Hence $C_n^d$, $n \geq 5$ with non intersecting chords has strongly multiplicative labeling for $n$ even.
Fig. 5. $C_n^3$ with non intersecting chords

Case (ii): When $n$ is odd

We define the bijection $f : V \rightarrow \{1, 2, 3, \ldots, n\}$ such that

$$f(v_i) = \begin{cases} 2i - 1 & 1 \leq i \leq \frac{n+1}{2} \\ 2(n-i) + 2 & \frac{n+1}{2} + 1 \leq i \leq n \end{cases}$$

With the edge set being $E = E_1 \cup E_2 \cup E_3$ where

$E_1 = \{v_i v_{i+1}, 1 \leq i < n\}$, $E_2 = \{v_{i+1} v_{n-i+1}, 1 \leq i < \frac{n-1}{2} - 1\}$, and $E_3 = \{v_n v_1\}$.

The proof follows as above.

Thus $C_n^3$ with non intersecting chords has strongly multiplicative labeling for $n$ odd.

Hence $C_n^3$ with non intersecting chords has strongly multiplicative labeling for all $n$.

Fig. 6. $C_n^3$ with non intersecting chords

Theorem 6: The Union of two strongly multiplicative graphs is also a strongly multiplicative graph.

Proof: Let $G_1$ and $G_2$ be two strongly multiplicative graphs with number of vertices $n_1$ and $n_2$ respectively. The graph $G_1 \cup G_2$ will have $n_1 + n_2$ vertices. Since $G_1$ is strongly multiplicative the induced labeling of the edges are distinct. Relabel the vertices of $G_2$ as $n_1 + 1$ for the vertex with label 1, $n_1 + 2$ for the vertex of label 2 and so on. As vertices are relabeled, the induced edge labeling will be added by the quantity $n_1^2 + (i + j)n_1$ for all $1 \leq i, j \leq n_2$. The addition of the quantity $n_1^2 + (i + j)n_1$ with the labels of the edges of $G_2$ will still result in the label of edges distinct. As $G_2$ is also strongly multiplicative graph, $G_1 \cup G_2$ is strongly multiplicative graph.

III. REFERENCES: