On the Control of the Permanent Magnet Synchronous Motor: An Active Disturbance Rejection Control Approach

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Abstract—This brief presents an active disturbance rejection control scheme for the angular velocity trajectory tracking task on a substantially perturbed, uncertain, and permanent magnet synchronous motor. The presence of unknown, time varying, and load-torque inputs, unknown system parameters, and the lack of knowledge of the initial shaft’s angular position, prompts a high-gain generalized proportional integral (GPI) observer-based active disturbance rejection control (ADR) controller. This controller is synthesized on the basis of the differential flatness of the system and the direct measurability of the system’s flat outputs, constituted by the motor’s angular displacement and the d-axis current. As a departure from many previous treatments, the d-q-axes currents model is here computed on the basis of the measured displacement and not on the basis of the unknown position. The proposed high-gain GPI observer-based ADR controller is justified in terms of a singular perturbation approach. The validity and robustness of the scheme are verified by means of realistic computer simulations, using the MATLAB/SIMULINK-PSIM package.

Index Terms—Active disturbance rejection control (ADRC), differentially flatness systems, high-gain extended observers, multivariable control, permanent magnet synchronous motor (PMSM), robust control.

I. INTRODUCTION

ACTIVE disturbance rejection control (ADRC) is a well-founded control design methodology where numerous academic and industrial contributions add to a long history of practical developments and achievements (see [24]). The fundamental idea in the ADRC is to regard additive endogenous nonlinearities and exogenous disturbances, as lumped unknown signals devoid of any specific state variable structure thus simplifying the state and disturbance estimation tasks (see [11], [13]). The efforts are basically aimed at efficiently online estimating such disturbance signals and then proceed to cancel their influence in the closed-loop dynamics. Extended state observers are common tools to online achieve such challenging estimation and control tasks.

Several globally stable position and velocity feedback control schemes for the permanent magnet synchronous motor (PMSM) have been reported in the control literature. For thorough treatments and background on the control of the PMSM, the reader is referred to the excellent books [3], [17], and the articles [2], [5], [6], and [18]. A two-stage ADRC design scheme has already been proposed for the control of the PMSM systems in [21]. Their approach uses nonlinear controllers and nonlinear differentiation techniques. In this brief, linear controllers (modulo control input gain cancelation) and linear high gain extended Luenberger observers, here addressed as the generalized proportional integral (GPI) observers, are proposed for the simultaneous estimation of state and lumped disturbance effects. The setting is that of input–output representation of the differentially flat system, and the analysis is based on singular perturbation theory. For the background on flatness, the reader is referred to [10]. The developments in this brief are carried out in the context of robust PMSM flatness-based linear output feedback control for a joint desired current and desired angular velocity trajectory tracking tasks. The velocity tracking is performed solely on the basis of the angular displacement measurement.

II. BACKGROUND ON THE SYNCHRONOUS MOTOR MODEL

The synchronous motor is described by the following nonlinear multivariable system of the differential equations, written in, so-called, a-b coordinates:

\[
\begin{align*}
L_a \frac{di_a}{dt} &= -R_s i_a + K_m (n_p \theta_r) \omega + u_a \\
L_b \frac{di_b}{dt} &= -R_s i_b - K_m \cos(n_p \theta_r) \omega + u_b \\
L \frac{d\omega}{dt} &= K_m \left[ -i_a \sin(n_p \theta_r) + i_b \cos(n_p \theta_r) \right] - B \omega - \tau_L \\
L \frac{d\theta}{dt} &= \omega
\end{align*}
\]

(1)

where \(u_a\) and \(u_b\) are the control input voltages, \(i_a\) and \(i_b\) are the phase currents. \(\tau_L\) is the exogenous load torque, considered to be unknown, and possibly, time varying. The variables, \(\theta\) and \(\omega\), stand, respectively, for the measured angular displacement and the corresponding angular velocity. The variable \(\theta_r\) represents the absolute angular position of the rotor. A needed distinction between angular displacement and angular position must be emphasized. The angular displacement, denoted by \(\theta\), is measured from the unknown absolute initial position \(\theta_0\) onward. In other words

\[
\theta_r = \theta_0 + \theta.
\]

(2)

Clearly, the angular velocity, \(\omega\), is given by: \(\dot{\theta}_r = \dot{\theta} = \omega\).

A. d-q Model on the Basis of the Measured Displacement

The mathematical model of the system can also be conveniently expressed in the, so called, d-q coordinates.
Contrary to customary practice, the $d$-$q$ model here adopted is obtained by means of the following invertible state and input coordinate transformations, specifically based on the measured angular displacement, $\theta$, and not on the actual angular position: $\theta_t$. Define

$$
\begin{pmatrix}
i_d \\
i_q
\end{pmatrix} = \begin{pmatrix}
\cos(n_p\theta) & \sin(n_p\theta) \\
-\sin(n_p\theta) & \cos(n_p\theta)
\end{pmatrix}
\begin{pmatrix}
i_a \\
i_b
\end{pmatrix},
$$

$$
\begin{pmatrix}
v_d \\
v_q
\end{pmatrix} = T
\begin{pmatrix}
u_a \\
u_b
\end{pmatrix}.
$$

The $a$-$b$ model (1) is transformed into

$$
L_s \frac{di_d}{dt} = -Rsi_d + n_p\omega Lsi_q + K_m\omega \sin(n_p\theta_0) + v_d
$$

$$
L_s \frac{di_q}{dt} = -Rsi_q - n_p\omega Lsi_d - K_m\omega \cos(n_p\theta_0) + v_q
$$

$$
J \frac{d\omega}{dt} = K miq - B\omega - \tau_L
$$

$$
\frac{d\theta}{dt} = \omega. \quad (4)
$$

Notice that if $\theta_0 = 0$, i.e., when the angular position coincides with the angular displacement, one obtains the traditional $d$-$q$ model.

### B. Flatness of the $d$-$q$ Model

The load-torque perturbed, uncertain, $d$-$q$ model given by (4), is differentially flat with flat outputs given by $\theta$ and $i_d$.

Indeed, all perturbed system variables are differentially parameterizable in terms of the flat outputs, the unknown exogenous disturbance $\tau$ and the unknown initial position $\theta_0$

$$
\omega = \dot{\theta}
$$

$$
i_q = \left(\frac{J}{K_m}\right)\theta + \frac{B}{K_m}\dot{\theta} + \frac{\tau_L}{K_m}
$$

$$
v_q = \left(\frac{L_s}{K_m}\right)\ddot{\theta} + \frac{L_s B}{K_m} + \frac{JR_i}{K_m}\dot{\theta} + n_p Ls i_d
$$

$$
+ K_m \cos(n_p\theta_0)
$$

$$
v_d = L_s \frac{di_d}{dt} + Rsi_d - n_p\dot{\theta}L_s \left(\frac{J}{K_m}\theta + \frac{B}{K_m}\dot{\theta} + \frac{\tau_L}{K_m}\right)
$$

$$
- K_m \dot{\theta} \sin(n_p\theta_0). \quad (5)
$$

The proposed high-gain GPI observer-based linear feedback controller is based on rather particularly simplified visions of the above input-to-flat output system relations. Such simplifications are systematically advocated in the ADRC approach

$$
\frac{d^3\theta}{dt^3} = \left(\frac{K_m}{L_s}\right)v_q + \zeta_0, \quad \frac{di_d}{dt} = \left(\frac{1}{L_s}\right)v_d + \zeta_i
$$

where the (endogenous) state dependent and (exogenous) disturbance dependent functions $\zeta_0$ and $\zeta_i$ are of the form

$$
\zeta_0 = \zeta_0(R_s(t), B, \theta_0, i_d, \dot{\theta}, \dot{\tau}_L(t), \tau_L(t))
$$

$$
\zeta_i = \zeta_i(R_s(t), B, \theta_0, i_d, \dot{\theta}, \dot{\tau}_L(t)). \quad (7)
$$

These functions are to be considered as disturbances, without specific regard for their nonlinear state dependent, exogenous signal, and internal architecture. In addition to the ADRC approach, this viewpoint, which seems controversial in the light of the dominant influence of the differential equations field in the automatic control literature, is also at the heart of well established robust nonlinear control techniques, such as, disturbance accommodation control (see [14]) and sliding mode control [22]. More recently, although more generally, the philosophy has been used in model-free control [9], where the systematic use of low-order local ultramodels is advocated.

### III. Problem Formulation and Main Result

In this section is provided the problem formulation and main result, but before we assume the following.

#### A. Assumptions

1) The angular displacement, $\theta$, and the direct current, $i_d$, are the only measured variables. The initial position $\theta_0$ is unknown and, hence, the angular position $\theta_t$ is not available. It is also assumed that the initial shaft position is bounded within the interval $\theta_0 \in (-\pi/2n_p, \pi/2n_p)$.

2) The load torque, $\tau_L(t)$, is a time-varying signal, known only to be uniformly absolutely bounded. Otherwise, the load torque is completely unknown.

3) Only the system parameters: $(L_s, K_m, J, n_p)$ are perfectly known. The time varying resistance $R_s(t)$, the viscous friction coefficient, $B$, and the initial position, $\theta_0$, are unknown.

4) For positive integers, $m$ and $p$, the quantities, $\zeta_0$ and $\zeta_i$, respectively, exhibit uniformly absolutely bounded time derivatives of order, respectively, $m$ and $p$, i.e., $\zeta_0(m)(t)$ and $\zeta_i(p)(t)$ are $L_\infty$ scalar functions. Notice that if these two assumptions are not satisfied almost everywhere for any finite $m$ and $p$, then the quantities, $\zeta_0$ and $\zeta_i$, are themselves absolutely unbounded on open sets of the underlying time axis, and the system itself does not even have a solution for any finite set of bounded control inputs. The general result has been rigorously proven in [12].

**Remark 1:** Note that for a given angular velocity reference trajectory, $\omega^*(t)$, the relation $\omega^*(t) = \theta^*_t(t)$ defines, modulo an arbitrary constant, a corresponding angular displacement trajectory: $\theta^*_t(t)$, via straightforward integration [see (15)]. The tracking of the angular displacement, $\theta^*_t(t)$, guarantees the tracking of the desired angular velocity $\omega^*(t)$ regardless of the arbitrary constant. We shall not make further distinction between $\theta^*_t(t)$ and $\omega^*(t)$.

#### B. Problem Formulation

Given the uncertain PMSM model (4) with all the previous assumptions being valid, devise a multivariable output feedback controller that allows the arbitrarily close tracking of a given $d$-axis current reference trajectory, $i^*_d(t)$ and that of a given angular velocity reference trajectory: $\omega^*(t) = \theta^*_t(t)$. The angular velocity tracking is to be accomplished regardless of the values adopted by the time-varying torque, $\tau_L(t)$, by...
the windings resistance, \( R_s(t) \), and by the initial value of the angular shaft position, \( \dot{\theta}_0 \).

C. Main Results

Theorem 1: Given the uncertain PMSM model (4), then, within an arbitrarily small neighborhood of the origin of the estimation error phase spaces, the pair of high-gain GPI observers, (9) and (10), estimate both the lumped additive perturbation input functions, \( \tilde{z}_0 \) and \( \tilde{z}_1 \), given in (6), as well as the angular displacement phase variables, \( \omega = \dot{\theta} \) and \( \dot{\theta}_0 = \dot{\theta}_0 \). These estimations are accomplished, respectively, via the variables: 1) \( z_1 \); 2) \( z_1 \); and 3) \( \hat{\theta}_2 = \ddot{\theta} \) and \( \theta_3 = \dot{\theta}_0 \), provided the sets of observer design coefficients: \( \{\gamma_0, \gamma_{\theta(m+2)}\}, \{\gamma_0, \gamma_{\theta(m+2)}\} \), are chosen so that the roots of the characteristic polynomials: \( p_0(s) \) and \( p_1(s) \), in the complex variable, \( s \), are located sufficiently far from the left-half side of the complex plane

\[
p_0(s) = s^{m+3} + \gamma_{\theta(m+2)} s^{m+1} + \cdots + \gamma_0 \quad p_1(s) = s^{p+1} + \gamma_{ip} s^{p+3} + \cdots + \gamma_0 s + \gamma_0 \tag{8}
\]

with \( m \) and \( p \) being relatively low-order integers (typically, 3 or 4)

\[
\dot{\theta}_1 = \theta_2 + \gamma_{\theta(m+2)} (\theta - \theta_1) \\
\dot{\theta}_2 = \theta_3 + \gamma_{\theta(m+1)} (\theta - \theta_1) \\
\dot{\theta}_3 = \frac{K_m}{L_s} v_q + z_{10} + \gamma_0 (\theta - \theta_1) \\
\dot{z}_{10} = z_{20} + \gamma_{\theta(m-1)} (\theta - \theta_1) \\
\vdots \\
\dot{z}_{m0} = \gamma_0 (\theta - \theta_1) \\
\frac{d}{dt} \hat{\theta}_d = \left( \frac{1}{\kappa_{\theta}} \right) \nu_d + z_{11} + \gamma_{i(\nu_d - \hat{\theta}_d)} \\
\dot{z}_{11} = z_{21} + \gamma_{i(\nu_d - \hat{\theta}_d)} \\
\vdots \\
\dot{z}_{pi} = \gamma_{i(\nu_d - \hat{\theta}_d)}. \tag{9}
\]

Proof: See Appendix.

D. GPI Observer-Based Controller

Theorem 2: The linear, high-gain, GPI observer-based, and flat output feedback controllers, provided with ADR terms: \( z_{10} \) and \( z_{11} \)

\[
v_q = \frac{L_d}{K_m} \left[ (\theta^{(3)})^3(t) - z_{10} - \kappa_0 (\theta_3 - \dot{\theta}^s(t)) - \kappa_0 (\theta_2 - \dot{\theta}^s(t)) \right] - \kappa_0 (\theta - \dot{\theta}^s(t)) - \kappa_0 \int_0^t (\theta - \dot{\theta}^s(\sigma)) d\sigma \]

\[
v_d = L_s \left[ \frac{d\nu_d(t)}{dt} - z_{11} - \kappa_0 (\nu_d - \dot{\theta}_d) \right] \tag{11}
\]

asymptotically exponentially force the closed-loop tracking error trajectories of the permanent magnet (PM) synchronous motor toward small as desired neighborhoods of the origin of the corresponding trajectory tracking error phase spaces, provided the roots of the polynomials, in the complex variable \( s \)

\[
p_{\theta,c}(s) = s^4 + \kappa_0 s^3 + \kappa_1 s^2 + \kappa_0 s + \kappa_0 \]

\[
p_{ic}(s) = s + \kappa_0 \]

are located sufficiently far to the left of the imaginary axis in the complex plane.

Proof: See Appendix.

IV. SIMULATION RESULTS USING THE PSIM PACKAGE

Computer simulations of the GPI observer-based speed controller for the PMSM were performed using the MATLAB/SIMULINK-PSIM package. The PSIM is a simulation software specifically designed for power electronics and motor drives. The parameter values of the PMSM with R43H series number (Pacific Scientific) used in the PSIM environment are shown in the Table I. Fig. 1 shows the block diagram of the GPI observer-based controller. It is constituted by the following subblocks: 1) two GPI observers, respectively, for the estimated variables, \( z_{10} \) and \( z_{11} \); 2) two gain blocks for the GPI observers; 3) two controllers, respectively, synthesizing the control inputs, \( \nu_q \) and \( \nu_d \); 4) two gain blocks for both controllers; 5) a disruption block corresponding to the time-varying exogenous load torque [see (13)]; 6) an angular velocity trajectory block for the desired output reference in the controller \( \nu_d \); and 7) a plant block. Fig. 2 shows the block diagram of the plant and this diagram is constituted by the following elements: 1) a three-phase PM synchronous machine with sinusoidal back electromotive force; 2) a three-phase pulsewidth modulation (PWM) voltage source inverter; 3) three current and voltage sensors, and three random sources for the three current measurements; 4) two abc-to-dq transformation blocks for the phase voltages and phase currents; 5) a dqo-to-abc transformation block for the GPI observer-based controllers and a three-phase PWM block; 6) an angular displacement sensor (resolver), an externally controlled mechanical load block; and 7) a set of three-phase resistor branches for the sudden changes of the stator resistors.

The initial value of the resistor is \( R_1 \ (t \leq 5 \ s) = 5.5 \ \Omega \), whereas that the final value is \( R_2 \ (t \geq 5 \ s) = 1.25 \ \Omega \). By means of the simulation setup described above, the following objectives are achieved.

1) Rather accurate tracking of a desired reference trajectory under an unknown time-varying load torque.

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>( R_s )</td>
<td>1.25 ( \Omega )</td>
</tr>
<tr>
<td>( L_d = L_q = L_s )</td>
<td>6.65 mH</td>
</tr>
<tr>
<td>( V_{pe/krpm} )</td>
<td>63.639 V</td>
</tr>
<tr>
<td>No. of Poles ( P )</td>
<td>4</td>
</tr>
<tr>
<td>Moment of Inertia ( (J) )</td>
<td>0.22E-3 kg.m²</td>
</tr>
<tr>
<td>Mech. Time Constant ( (\tau_{mech}) )</td>
<td>2.0944 sec.</td>
</tr>
<tr>
<td>Friction coefficient ( (B = J/\tau_{mech}) )</td>
<td>1.05E-4 kg.m²/sec.</td>
</tr>
<tr>
<td>Typical Rated Speed/320 dc bus</td>
<td>380 rad/sec.</td>
</tr>
<tr>
<td>Typical Rated Torque/320 dc bus</td>
<td>3.6 Nm</td>
</tr>
</tbody>
</table>

TABLE I

MOTOR PARAMETERS AND WINDING DATA
i) trajectories, torque value of the PMSM data sheet. The desired reference trajectory tracking while stabilizing to zero the closed-loop angular position third-order dynamics and the corresponding \(d\)-axis current first-order dynamics. Two normalized integral square error (ISE) trajectory tracking parameters for the simulation: 1) start time: 0 s; 2) stop time: 10 s; 3) type: fixed step; 4) solver: ode1 (Euler); and 5) fixed-step size (fundamental sample time): 0.0001. The angular displacement controller parameters were chosen via a term-by-term identification of the fourth-order characteristic polynomial in \(s\) (because of the integral control term), with the known polynomial \((s^2 + 2\zeta\omega_n s + \omega_n^2)^2\) with \(\zeta = 1\) and \(\omega_n = 75\), appropriately, chosen according to classical recipes. Similarly, for the design parameters in the observers. Fig. 3 shows the performance of the proposed GPI observer-based linear output feedback controller. The double task is that of robustly achieving an accurate angular velocity reference trajectory tracking while stabilizing to zero the \(d\)-axis current in the presence of a time-varying load-torque input. Severe variations of the resistance parameter \(R_s\) were also implemented. In addition, we add to the current measurements \(i_a, i_b,\) and \(i_c\) a noise source, the amplitude of the noise source is near to 15\% of its final nominal value of the phase currents (see Fig. 2). Fig. 2 also shows the estimation of the lumped effects of the endogenous and exogenous disturbance signals hypothesized in the ultramodell. The accuracy of the redundant estimations of the angular velocity and of the \(d\)-axis current, as provided by the GPI observer, is observed to be quite high.

Remark 2: Other control objectives are also possible, such as nonzero reference trajectories for the current \(i_d^*(t)\). As a particular case, we mention the case of constant load torques and optimal steady-state behavior for the \(i_d\) and \(i_q\) currents in the presence of either voltage limitations or current limitations or both. Such strategies, known as field weakening strategies, deserve separate detailed treatment from the perspective of flatness and ADRC (see [3]).

A robustness assessment was carried out regarding the effects of a possible lack of precise knowledge of the constant control input gains: \(K_m/JL_s\) and \(1/L_s\) affecting, respectively, the closed-loop angular position third-order dynamics and the corresponding \(d\)-axis current first-order dynamics. Two
Fig. 3. Angular velocity tracking performance of the GPI observer-based ADRC controller with linear estimation of the aggregate effects of uncertain parameter \( R_s(t) \) and exogenous disturbance input torque \( \tau_L(t) \).

Fig. 4. Behavior of the ISE trajectories for different values of the (percent) uncertainty factor, \( \gamma \), affecting the control input gains knowledge.

Fig. 5. Behavior of the ISE trajectories for different values of the (percent) uncertainty factor, \( \alpha \), affecting the control input gains knowledge.

Performance indices were proposed as

\[
\text{ISE}_{\omega,\gamma}(t) = \frac{1}{N_{\theta}} \int_0^t (\omega(t) - \omega^*(t))^2 \, dt \\
\text{ISE}_{id,\alpha}(t) = \frac{1}{N_{id}} \int_0^\infty (i_{dc}(t) - i_{dc}^*(t))^2 \, dt
\]

with \( \omega(t) \) and \( i_{dc}(t) \) being, respectively, the closed-loop response trajectories obtained with the faulty input gains: \( \gamma \frac{K_m}{J_L} \) and \( \alpha 1/L_s \), acting, both, on the corresponding GPI observer and the corresponding linear controller. The parameters \( N_{\theta} \) and \( N_{id} \) are normalizing factors representing, respectively, the steady-state values of the functions: \( \text{ISE}_{\omega,\gamma}(t) \) and \( \text{ISE}_{id,\alpha}(t) \).
and ISE\(_{\text{i.o.a}}(t)\), when \(\gamma = 1\) and \(\alpha = 1\). The robustness tests for each gain variation were carried out separately with the factor \(\gamma\) and \(\alpha\). Figs. 4 and 5 show the trajectories for ISE\(_{\text{i.o.a}}(t)\) and ISE\(_{\text{i.d.a}}(t)\) in logarithmic scale for several values of \(\gamma\) and \(\alpha\). The constant steady state of these functions indicates that the corresponding output tracking errors are ultimately in nearly exact agreement with the desired reference trajectories. Notice that the lower bound represents a discrepancy factor of 20, with respect to the nominal value \(\gamma = 1\). The upper bound represents a 1500% variation above the nominal value. These wide variations in both gains indicates wide variations of \(L_s\) in the first case, and of \(J\) in the second case.

V. CONCLUSION

In this brief, an ADRC approach has been proposed for the flatness-based control of a PMSM in an angular velocity trajectory tracking task. The approach includes the use of a linear extended high-gain observer, known as the GPI observer. This observer is used to simultaneously estimate the phase variables associated with the flat outputs and the lumped effects of additive exogenous and endogenous perturbation inputs affecting the decoupled flat outputs dynamics. This on-line information is used by linear output feedback controllers in the effective, although approximate, cancelation of effects of additive exogenous and endogenous perturbation inputs, unknown initial shaft angular position, and the presence of uncertain parameters. The performance of the proposed GPI observer-based control scheme was illustrated by means of the realistic computer simulations using the MATLAB/SIMULINK-PSIM simulation package along with a rather objective control input gain robustness assessment.

APPENDIX

SINGULAR PERTURBATION-BASED PROOF OF THE MAIN RESULT

Consider the problem of tracking a given smooth trajectory \(y^*(t), t \in [0, \infty)\), by an appropriate feedback control action, \(u\), on the following \(n\)-dimensional smooth nonlinear system

\[
y^{(n)} = \phi(t, y)u + \psi(t, y, \dot{y}, \ldots, y^{(n-1)})
\]

where \(\phi(t, y)\) is known and uniformly bounded away from zero, i.e., \(\sup \|\phi(t, y(t))\| > \delta > 0\). The scalar function \(\psi(\cdot)\) is completely unknown except for the fact that it is uniformly absolutely bounded and a finite number of its time derivatives, say the first \(m\) of them \(\psi^{(i)}(\cdot), i = 1, 2, \ldots, m\), are also uniformly absolutely bounded \((\sup \|\psi^{(i)}(t, y(t), \dot{y}(t), \ldots, y^{(n-1)}(t))\| \leq K_i, i = 0, \ldots, m)\).\(^1\) The integer \(m\) is a design parameter chosen to be a low integer (typically \(m = 3\) or \(4\)). Let \({\kappa_0, \ldots, \kappa_{n-1}\})\) be the real coefficients chosen so that the following polynomial in the complex variable \(s\), \(p_{\text{cl}}(s)\), is Hurwitz:

\[
p_{\text{cl}}(s) = s^n + \kappa_{n-1}s^{n-1} + \cdots + \kappa_1 s + \kappa_0.
\]

Similarly, let \({\lambda_0, \ldots, \lambda_{m+n-1}\})\) be the real coefficients chosen so that the following polynomial in the complex variable \(s\), \(p_{\text{obs}}(s)\), is also Hurwitz:

\[
p_{\text{obs}}(s) = s^{n+m} + \lambda_{n+m-1}s^{n+m-1} + \cdots + \lambda_1 s + \lambda_0.
\]

By defining \(y_i = y^{(i)}\), \(i = 0, \ldots, n-1\), \(y_0 = y\), system (17) may be written in state-space representation as

\[
\begin{align*}
\dot{y}_0 &= y_1 \\
\vdots \\
\dot{y}_{n-1} &= \phi(t, y_0)u + \psi(t, y_0, \ldots, y_{n-1}) \\
\dot{y}_n &= \phi(t, y_0)u + \psi(t, y_0, \ldots, y_{n-1}).
\end{align*}
\]

Consider the following observer-based feedback linearizing controller, including a nonlinearity cancelation term, \(\psi:\)

\[
u = \frac{1}{\psi_n} \left[ -\hat{\psi} + [y^*(t)]^{(n)} - \sum_{i=0}^{n-1} k_i (\hat{y}_i - [y^*(t)]^{(i)}) \right]
\]

where \(\hat{y}_i, i = 0, 1, \ldots, n-1\), and, \(\hat{\psi} = z_1\), are variables obtained from the following extended (linear) Luenberger observer (also called the GPI observer):

\[
\begin{align*}
\dot{\hat{y}}_0 &= \hat{y}_1 + \lambda_{n+m-1}(y - \hat{y}_0) \\
\vdots \\
\dot{\hat{y}}_{n-2} &= \hat{y}_{n-1} + \lambda_{m+1}(y - \hat{y}_0) \\
\dot{\hat{y}}_{n-1} &= \phi(t, y)u + \lambda_{m}(y - \hat{y}_0) \\
\dot{z}_1 &= z_2 + \lambda_{m-1}(y - \hat{y}_0) \\
\vdots \\
\dot{z}_m &= \lambda_0(y - \hat{y}_0).
\end{align*}
\]

Define \(e_i = y_i - \hat{y}_i\) and \(i = 0, 1, \ldots, n-1\). One obtains the following estimation error vector representation:

\[
\begin{align*}
\dot{\hat{e}}_0 &= \hat{e}_1 - \lambda_{n+m-1}\hat{e}_0 \\
\dot{\hat{e}}_1 &= \hat{e}_2 - \lambda_{n+m-2}\hat{e}_0 \\
\vdots \\
\dot{\hat{e}}_{n-1} &= \psi(t, y_0, y_1, \ldots, y_{n-1}) - z_1 - \lambda_m\hat{e}_0 \\
\dot{z}_1 &= z_2 + \lambda_{m-1}\hat{e}_0 \\
\vdots \\
\dot{z}_m &= \lambda_0\hat{e}_0.
\end{align*}
\]

Letting \(\tilde{e}_0 = \hat{e}_0\), the output estimation error, \(\tilde{e}_0 = \hat{e} = y - \hat{y}_0\), satisfies the following perturbed \(n + m\)th-order differential equation:

\[
\begin{align*}
\dot{e}^{(n+m)} + \lambda_{n+m-1}e^{(n+m-1)} + \cdots + \lambda_1 \dot{e} + \lambda_0 e &= \psi^{(m)}(t, y, \dot{y}, \ldots, y^{(n-1)}) \\
&= \psi^{(m)}(t, e + y^*(t), \ldots, e^{(n-1)} + [y^*(t)]^{(n-1)})
\end{align*}
\]

where \(e = y - y^*(t)\) is defined to be the output trajectory tracking error. Substitute (20) in (17). After adding and subtracting \(y_i\) to the factor: \((\hat{y}_i - [y^*(t)]^{(i)})\) inside the sum in (21), then, the overall closed-loop system, and the output
estimation error (24), are observed to satisfy the following set of perturbed scalar high-order differential equations:

\[ \begin{align*}
  e^{(n)} + \kappa_n - 1 \epsilon^{(n-1)} + \cdots + \kappa_0 e &= \psi(t) + \left[ y^* (t) \right]^{(n-1)} - \tilde{\psi}, \\
  \tilde{e}^{(m)} + \lambda_{n+m} - 1 \epsilon^{(m-1)} + \cdots + \lambda_1 \tilde{e} + \lambda_0 \epsilon &= \psi(t) + y^* (t), \quad e^{(n-1)} + [y^* (t)]^{(n-1)}.
\end{align*} \]

(25)

Fact 1: From (23), it follows that if \( \tilde{e}_0 \) is identically zero, then \( \tilde{e}_0 = \tilde{e}_1 = \cdots = \tilde{e}_{n-1} = 0 \). In particular, from the differential equation for \( \tilde{e}_{n-1} \) in (23)

\[ \tilde{e}_{n-1} = \psi(t, y_0, \ldots, y_{n-1}) - z_1 - \lambda_m \epsilon_0. \]

(27)

It follows that if \( \tilde{e}_i, \ i = 0, 1, \ldots, n-1 \), are identically zero, then \( z_1 = \psi \) is an exact estimate of the nonlinear term: \( \psi(t, y, \ldots, y^{(n-1)}) \). In addition, \( z_j = \psi^{(j-1)}(\epsilon) \) for \( j = 2, \ldots, m \) and \( \tilde{z}_m = \psi^{(m)}(\epsilon) \). This fact will be used later on, in the context of a singular perturbation analysis of the closed-loop system (25), (26).

Fact 2: Similarly, from (22) itself, if all \( \tilde{e}_i \) were not identically zero, but ultimately uniformly arbitrarily close to zero, then \( z_1 = \psi \) would also be an ultimately arbitrarily close estimate of the nonlinear term: \( \psi(t, y, \ldots, y^{(n-1)}) \).

This justifies our choice of notation for \( z_1 \) as \( \psi \).

High Gain Observer Design: Let \( \epsilon \) and \( \lambda \) be strictly positive real parameters (with, say, \( \lambda > 1 \)). Choosing the observer gains \( \lambda_0, \ldots, \lambda_{n+m-1} \). In (22), as

\[ \begin{align*}
  \lambda_{n+m+i} &= \frac{(n+m)!}{i!(n+m-i)!} \left( \frac{\lambda}{\epsilon^2} \right)^i, \quad i = 1, 2, \ldots, n + m
\end{align*} \]

(28)

one obtains, from (26), the following perturbed observer error dynamics, written in polynomial differential operator form:

\[ (\frac{d}{dt} + \lambda) \tilde{e}^{n+m} = \epsilon \tilde{e}^{n+m} y^{(m)}(t, e + y^* (t), \dot{e} + y^* (t), \ldots, e^{(n-1)} + [y^* (t)]^{(n-1)}) \]

(29)

Multiplying out by \( \epsilon^{n+m} \), one obtains the equivalent system

\[ (\epsilon \frac{d}{dt} + \lambda) \tilde{e}^{n+m} \tilde{e} = \epsilon^{n+m} y^{(m)}(t, e + y^* (t), \dot{e} + y^* (t), \ldots, e^{(n-1)} + [y^* (t)]^{(n-1)}) \]

(30)

A Singular Perturbation Analysis: Summarizing, the closed-loop system and the injected dynamics of the high-gain observer satisfy the following set of differential equations, suitable for a singular perturbation analysis since they are easily reducible to normal form [16]:

\[ \begin{align*}
  e^{(n)} + \kappa_n - 1 \epsilon^{(n-1)} + \cdots + \kappa_0 e &= \psi(t, e + y^* (t), \dot{e} + y^* (t), \ldots, e^{(n-1)} + [y^* (t)]^{(n-1)}) \\
  - \dot{\tilde{\psi}} + \sum_{i=0}^{n-1} \kappa_i \tilde{e}_i \quad (31)
\end{align*} \]

\[ \left( \frac{d}{dt} + \lambda \right)^{n+m} \tilde{e} = \epsilon^{n+m} y^{(m)}(t, e + y^* (t), \dot{e} + y^* (t), \ldots, e^{(n-1)} + [y^* (t)]^{(n-1)}) \]

(32)

Following the standard procedure in singular perturbation analysis (see [15]), one first obtains the reduced order system of (31) and (32) by formally setting \( \epsilon = 0 \) in (32). This yields \( \tilde{\psi} \) identically zero, i.e., \( \tilde{e} = 0 \) is the only trivial equilibrium of the singularly perturbed observer error dynamics. One proceeds to replace this equilibrium solution, \( \tilde{e} = 0 \), in (31). Recalling from Fact 1 that whenever \( \tilde{e} \) and its time derivatives are zero, \( z_1 = \psi(\epsilon) = \psi(\epsilon) \), and, also \( \tilde{e}_0 = 0 \), \( i = 0, 1, \ldots, n-1 \), then, one obtains the following reduced order system for the tracking error dynamics:

\[ (\lambda \frac{d}{dt} + \lambda)^{n+m} \tilde{e} = 0. \]

(33)

By the initial assumptions on the controller parameters: \( \kappa_j, \ j = 0, \ldots, n-1 \), the reduced order system has the origin as an asymptotically exponentially stable equilibrium point. The corresponding boundary layer system, described in the stretched time scale, \( t = t/\epsilon \), and denoted by the correction variable \( e = \tilde{e} - \tilde{e} - \tilde{e} \), satisfies the following asymptotically stable linear homogenous differential equation:

\[ \left( \frac{d}{dt} + \lambda \right)^{n+m} \tilde{e} = 0. \]

(34)

as obtained by setting \( \epsilon = 0 \) in the right-hand side of (32) and performing the natural time rescaling \( \epsilon d/dt = d/dT \). The equilibrium \( \tilde{e} = 0 \) is asymptotically stable independently of the initial condition for the tracking error \( e(0) \). In other words, it is uniformly asymptotically stable in the initial condition \( e(0) \). In addition, any initial condition, \( e(0) \), for the boundary layer system (34) is, trivially, in its domain of attraction. The (repeated) eigenvalue, \( -\lambda \), of the trivial linearization around the origin of the boundary layer system (34) is strictly negative and it can be made smaller than a finite fixed negative real number \( -\epsilon \) by setting \( \lambda > \epsilon \).

It follows from the previous singular perturbation analysis, that all hypothesis of the well known Tikhonov’s (see [16]) are satisfied. Hence, for a sufficiently small \( \epsilon \), the observer error, \( \tilde{e} \), and its time derivatives, ultimately uniformly evolve, in a stable fashion, inside an arbitrarily small neighborhood of the origin of the observer error phase space \( (\epsilon, \tilde{e}^{(m+n-1)}) \) determined by \( \epsilon \). Indeed, setting \( d/dT \) to zero, in (32), one obtains the quasi-steady state equilibrium for the estimation error \( \tilde{e} \) as

\[ \tilde{e} = \left( \frac{\epsilon}{\epsilon^{n+m}} \right) \frac{d^{m+n}}{d\epsilon^{m+n}} \left[ \psi(t, e + y^* (t), \dot{e} + y^* (t), \ldots, e^{(n-1)} + [y^* (t)]^{(n-1)}) \right]. \]

(35)

An estimate of the radius of an ultimately uniformly bounding open neighborhood for \( \tilde{e} \) is just given by: \( (\epsilon/\lambda)^{m+n} K_m \), with \( K_m \) being the uniform absolute bound hypothesized on \( y^{(m)}(t, n, \ldots, y^{(n-1)}) \). Clearly, it is advisable that \( \lambda > 1 \). Under these circumstances, from Fact 2, the difference \( \psi - \psi^* \) is also arbitrarily small and the estimation of the nonlinearity can be made as approximate as desired.

Controller Design: Notice that from (23), the definition of \( \tilde{e} = y_0 - \tilde{y}_0 = y - \tilde{y}_0 \), it follows that:

\[ \tilde{y}_i = y^{(i)} - \sum_{j=0}^i \lambda_{n+m+j} \tilde{e}^{(i-j)} \quad i = 0, 1, 2, \ldots, n-1 \]

with \( \lambda_{n+m} = 1 \).
Then, the right-hand side of (25), after using (28), is given by
\[
e^{(n)} + \kappa_{n-1} e^{(n-1)} + \cdots + \kappa_0 e
= \left[ y(t) + y^*(t), \dot{e} + \dot{y}^*(t), \ldots, e^{(n-1)} + [y^*(t)]^{(n-1)} - \dot{\psi} \right]
+ \sum_{i=0}^{n-1} \kappa_i \left[ \sum_{j=0}^{i} \frac{(n+m)!}{j!(n+m-j)!} \frac{\psi}{e} (j-1) \right].
\]

The highest order power of \(\epsilon\), in a denominator of the right-hand side of (37), is \(n-1\). Multiplying out by \(\epsilon^n\) and time-scaling, as before, one obtains
\[
d^n e^n + \epsilon \kappa_{n-1} d^{n-1} e^{(n-1)} + \cdots + \epsilon^n \kappa_0 e
= \epsilon^n \left[ y(t) + y^*(t), \dot{e} + \dot{y}^*(t), \ldots, e^{(n-1)} + [y^*(t)]^{(n-1)} - \dot{\psi} \right]
+ \sum_{i=0}^{n-1} \kappa_i e^{n-i} \left[ \sum_{j=0}^{i} \frac{(n+m)!}{j!(n+m-j)!} \frac{\psi}{e} (j-1) \right].
\]

The natural choice for the set of coefficients \(\{\kappa_0, \ldots, \kappa_{n-1}\}\) is, for a finite positive real number \(\mu > 0\)
\[
\kappa_{n-j} = \left( \frac{\mu}{\epsilon} \right) \frac{n^j}{j!(n-j)!}, \quad j = 1, 2, \ldots, n.
\]

The perturbed tracking error dynamics is, then, of the form:
\[
(\epsilon \frac{d}{dt} + \mu)^n e = O(\epsilon).\]

The same singular perturbation analysis, carried out before, is applicable now. The tracking error, \(e\), ultimately uniformly evolves on an \(\epsilon\) neighborhood of zero.

REFERENCES