On the relationship between limit spaces, many valued topological spaces, and many valued preorders

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Abstract

Let \((L, *, 1)\) be a residuated complete lattice with the underlying lattice \(L\) a meet continuous lattice. This paper presents a systematic investigation of the interrelationship between the categories of limit spaces, \(L\)-topological spaces, and \(L\)-preorders. The results exhibit a close connection between these different mathematical structures.

Keywords: Category theory; Topology; Residuated lattice; Limit space; Many valued topology; Many valued preorder

1. Introduction

The interaction between topological structures and order structures is a stimulating topic in mathematics and computer science, e.g., the theory of locales [17] and the theory of domains [9]. This interaction also plays an important role in the theory of many valued topology, see, e.g., [5,8,13–16,19,20,22,29,30].

This paper presents, in the case that \(L\) is a meet continuous residuated lattice, a systematic investigation of the interrelationship between the categories of limit spaces, \(L\)-preorders, and \(L\)-topological spaces.

A residuated lattice [2] is a triple \((L, *, 1)\), where, \(L\) is a complete lattice with a top element 1 and a bottom element 0; * is a binary operation on \(L\) such that (i) \((L, *, 1)\) is a commutative monoid and (ii) * distributes over arbitrary joins.

A complete lattice \(L\) is meet continuous [9] if the binary meet operation \(\wedge\) distributes over directed joins. In particular, a residuated lattice \((L, *, 1)\) is called a meet continuous residuated lattice if \(L\) is meet continuous.

Let \((L, *, 1)\) be a meet continuous residuated lattice. This article is concerned with the interrelationship between the following categories:

- **Top**, the category of topological spaces and continuous functions;
- **Prord**, the category of preordered sets and order-preserving functions;
- **Lim**, the category of limit spaces and continuous functions;
- **TopLim**, the full subcategory of **Lim** consisting of topological limit spaces;

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2. Preliminaries

In this section, we recall some basic notions from lattice theory and category theory which shall be needed in the subsequent sections.

Given a residuated lattice \((L, *, 1)\), define a binary operation \(\rightarrow\) on \(L\) by

\[ b \rightarrow c = \bigvee \{a \in L|a * b \leq c\}. \]

The binary operation \(\rightarrow\) is called the residuation with respect to \(*\). The operations \(*\) and \(\rightarrow\) are interlocked with each other by the adjoint property: \(a * b \leq c \iff a \leq b \rightarrow c\). This adjoint property enables \(L\) to play the role of the set of truth-values in a many valued logic. \(1 \in L\) is interpreted as true, \(0 \in L\) as absurd, \(*\) as the logic connective conjunction, and \(\rightarrow\) as implication. The reader is referred to \([2,10–12,23]\) for more in this regard. So, the study concerning \(L\)-subsets (i.e., functions with codomain \((L, *, 1)\)) has a strong flavor of many valued logic. This is why \(L\)-topological spaces and \(L\)-preorders are also called many valued topological spaces and many valued preorders, respectively \([10,14,31]\).

Some basic properties of the operations \(*\) and \(\rightarrow\) in a residuated lattice are collected here for later use, they can be found in many places, e.g., \([11,12]\).

\[
\begin{align*}
(11) & \quad 0 * p = 0; \\
(12) & \quad p \rightarrow (q \rightarrow r) = (p * q) \rightarrow r; \\
(13) & \quad p \rightarrow q = 1 \iff p \leq q; \\
(14) & \quad (p \rightarrow q) * (q \rightarrow r) \leq (p \rightarrow r); \\
(15) & \quad \left(\bigvee_{j \in J} p_j\right) \rightarrow q = \bigwedge_{j \in J}(p_j \rightarrow q); \\
(16) & \quad p \rightarrow \left(\bigwedge_{j \in J} q_j\right) = \bigwedge_{j \in J}(p \rightarrow q_j).
\end{align*}
\]

For any set \(X\), the set \(L^X\) of mappings \(X \rightarrow L\) with the pointwise order is also a complete lattice. Elements of \(L^X\) are called \(L\)-subsets (or fuzzy subsets) of \(X\). For \(x, y \in L\) and \(x, y \in L\), we denote by \(x * y\) and \(x \rightarrow y\) the \(L\)-subsets defined by \((x * y)(x) = x * y(x), (x \rightarrow y)(x) = x \rightarrow y(x)\) for each \(x \in X\). Also, for each subset \(U \subseteq X, a \in L\), we denote by \(a \wedge 1_U\) the function \(X \rightarrow L\) defined by \(a \wedge 1_U(x) = a\) if \(x \in U\) and \(a \wedge 1_U(x) = 0\) if \(x \notin U\). When \(a = 1, a \wedge 1_U\) is simplified to \(1_U\).

Let \(L\) be a complete lattice and \(x, y \in L\). We say that \(x\) is way below \(y\) (in symbols, \(x \ll y\)) if for all directed subsets \(D \subseteq L, x \leq \sup D\) always implies that \(x \leq d\) for some \(d \in D\). A complete lattice \(L\) is said to be continuous \([9]\) if \(x = \sup \downarrow x\) for all \(x \in L\), where \(\downarrow x = \{y \in L : y \ll x\}\).

Finally, we recall some basic notions from category theory. Our reference to category theory is the monograph \([1]\).

Let \(A\) and \(B\) be categories. By an adjunction \([1]\) between \(A\) and \(B\) (in symbols, \(F: A \rightarrow B, G: B \rightarrow A\)) is meant a pair of functors \(F: A \rightarrow B, G: B \rightarrow A\) such that there is a natural isomorphism between the functors \(\hom_A(\cdot, G(\cdot))\)
and \( \text{hom}_B(F(-), -) \). In this case, \( F \) is called a left adjoint of \( G \) and \( G \) a right adjoint of \( F \). The symbol \( F \dashv G : A \longrightarrow B \) is often abbreviated to \( F \Rightarrow G \) if the categories \( A \) and \( B \) are evident.

In this paper, by a concrete category is meant a concrete category over \( \text{Set} \) (i.e., a construct in the terminologies of [1]). Precisely, a concrete category is a pair \( (A, U) \), where \( A \) is a category and \( U : A \longrightarrow \text{Set} \) is a faithful functor (called a forgetful functor). For each \( X \in A \), \( U(X) \) is called the underlying set of \( X \) (also denoted by \( |X| \) if no confusion would arise). So, every object in a concrete category is nothing but a set with a structure on it. We often write \( A \) for a concrete category \( (A, U) \) if the forgetful functor is evident.

A concrete functor \( F : (A, U) \longrightarrow (B, V) \) between concrete categories is a functor \( F : A \longrightarrow B \) such that \( U = V \circ F \). That means, \( F \) only changes the structures on the underlying sets, leaving the underlying sets and morphisms untouched.

Thus, in order to define a concrete functor is trivial for the functors involved in this paper, we often omit the verification of this step.

**Definition 2.1** (Adámek et al. [11]). Suppose that \( A \) and \( B \) are concrete categories; \( F : A \longrightarrow B \), \( G : B \longrightarrow A \) are concrete functors. The pair \( (F, G) \) is called a **Galois correspondence** if either of the following equivalent conditions hold:

1. \( \{ \text{id}_Y : F \circ G(Y) \longrightarrow Y | Y \in B \} \) is a natural transformation from the functor \( F \circ G \) to the identity functor on \( B \); and \( \{ \text{id}_X : X \longrightarrow G \circ F(X) | X \in A \} \) is a natural transformation from the identity functor on \( A \) to \( G \circ F \).

2. For each \( Y \in B \), \( \text{id}_Y : F \circ G(Y) \longrightarrow Y \) is a \( B \)-morphism; and for each \( X \in A \), \( \text{id}_X : X \longrightarrow G \circ F(X) \) is an \( A \)-morphism.

If \( (F, G) \) is a Galois correspondence, then \( F \) is a left adjoint of \( G \) [1].

### 3. Review of related results

This section reviews some basic notions and results about limit spaces, \( L \)-topological spaces, and \( L \)-preordered sets.

#### 3.1. The interrelationship between \( \text{Top} \), \( \text{Prord} \), and \( \text{Lim} \)

A preorder on a set \( X \) is a reflexive and transitive relation \( \preceq \) on \( X \). The pair \( (X, \preceq) \) is called a preordered set. A function \( f : X \longrightarrow Y \) between two preordered sets is order-preserving if \( x \preceq y \) implies \( f(x) \preceq f(y) \). The category of preordered sets and order-preserving functions is denoted by \( \text{Prord} \).

Given a preordered set \( (X, \preceq) \) and \( A \subseteq X \), let \( \uparrow A = \{ y \in X | x \preceq y \text{ for some } x \in A \} \). A subset \( A \) of \( X \) is an upper set if \( A = \uparrow A \). Dually, a subset \( B \) is a lower set if \( B = \downarrow B = \{ y \in X | y \preceq x \text{ for some } x \in B \} \). The family of all the upper sets of \( X \) is a topology on \( X \), called the Alexandrov topology on \( X \) and denoted by \( \mathcal{I}(\preceq) \). The correspondence \( (X, \preceq) \mapsto (\mathcal{O}(X, \mathcal{I}(\preceq))) \) defines a concrete functor \( \Gamma : \text{Prord} \longrightarrow \text{Top} \).

Given a topological space \( (X, T) \), define a binary relation \( \mathcal{O}(T) \) on \( X \) as follows: \( (x, y) \in \mathcal{O}(T) \) if for any open set \( U \), \( y \in U \) whenever \( x \in U \). Then \( \mathcal{O}(T) \) is a preorder on \( X \), called the specialization order [17] of \( (X, T) \). The correspondence \( (X, T) \mapsto (X, \mathcal{O}(T)) \) defines a concrete functor \( \Omega : \text{Top} \longrightarrow \text{Prord} \). \( (\mathcal{I}, \mathcal{O}) \) is a Galois correspondence [17].

For a set \( X \), let \( \mathcal{F}(X) \) denote the set of filters on \( X \). For each \( x \in X \), let \( \mathcal{I} \) denote the principal filter generated by \( x \), i.e., \( \mathcal{I} = \{ A \subseteq X | x \in A \} \).

A convergence structure on \( X \) is a subset \( T \subseteq \mathcal{F}(X) \times X \) such that

1. \( (x, x) \in T \) for all \( x \in X \),
2. \( (F, x) \in T \), \( \mathcal{F} \subseteq G \Rightarrow (G, x) \in T \).

The pair \( (X, T) \) is called a convergence space. If \( (\mathcal{F}, x) \in T \), we also write \( T \longrightarrow x \) (or simply, \( \mathcal{F} \longrightarrow x \), if no confusion would arise). A continuous function \( f : (X, T_X) \longrightarrow (Y, T_Y) \) between convergence spaces is a function \( f : X \longrightarrow Y \)
such that $\mathcal{F} \to x \Rightarrow f(\mathcal{F}) \to f(x)$, where $f(\mathcal{F})$ is the filter on $Y$ generated as a filterbase by \{ $f(A) | A \in \mathcal{F}$ \}. The category of convergence spaces and continuous functions is denoted by $\textbf{Con}$.

A convergence space $(X, T)$ is called a limit space if

(3) $\mathcal{F} \to x, \mathcal{G} \to x \Rightarrow \mathcal{F} \cap \mathcal{G} \to x$.

$\textbf{Lim}$ denotes the full subcategory of $\textbf{Con}$ consisting of limit spaces.

A convergence space $(X, T)$ is called pretopological if

(4) $\mathcal{F}_j \to x, \forall j \in J \Rightarrow \bigcap_{j \in J} \mathcal{F}_j \to x$.

Every pretopological convergence space is clearly a limit space. A limit space $(X, T)$ is called topological if it is pretopological and satisfies

(5) for all $x \in X$ and $U \in \bigcap\{\mathcal{F} | \mathcal{F} \to x\}$, there exists $V \in \bigcap\{\mathcal{F} | \mathcal{F} \to y\}$ such that $\forall y \in V$, $U \in \bigcap\{\mathcal{F} | \mathcal{F} \to y\}$.

The full subcategory of $\textbf{Lim}$ consisting of topological limit spaces is denoted by $\textbf{TopLim}$.

Let $(X, T)$ be a topological space. Then

\[ e(T) = \{(x, x), x \in (X, T) \} \]

is a limit structure on $X$. The correspondence $(X, T) \mapsto (X, e(T))$ defines a full and faithful (concrete) functor $e : \textbf{Top} \to \textbf{Lim}$. The image of $e$ is exactly the full subcategory $\textbf{TopLim}$ of $\textbf{Lim}$. Thus, if we write $\hat{e} : \textbf{Top} \to \textbf{TopLim}$ for the functor obtained by restricting the codomain of $e$ to $\textbf{TopLim}$, then $\hat{e}$ is an isomorphism [27]. Denote the inverse of $\hat{e}$ by $\hat{e}^{-1} : \textbf{TopLim} \to \textbf{Top}$.

The functor $e : \textbf{Top} \to \textbf{Lim}$ has a concrete left adjoint $\mathcal{R} : \textbf{Lim} \to \textbf{Top}$ given by $\mathcal{R}(X, T) = (X, \mathcal{R}(T))$, where

$\mathcal{R}(T) = \{U \subseteq X | \forall x \in U, \mathcal{F} \to x \Rightarrow U \in \mathcal{F} \}$.

Therefore, $\textbf{TopLim}$ (which is concretely isomorphic to $\textbf{Top}$) is concretely reflective, hence initially closed, in $\textbf{Lim}$ [1,27].

Given a limit structure $T$ on $X$, define a binary relation $\leq_T$ on $X$ as follows: $x \leq_T y$ if $\mathcal{F} \xrightarrow{T} x$ whenever $\mathcal{F} \xrightarrow{T} y$. Then $\leq_T$ is a preorder on $X$, called the specialization order of $(X, T)$ and denoted by $\Theta(T)$. We leave it to the reader to verify that if $(X, T)$ is topological, i.e., $(X, T) = e(X, T)$ for some topology $T$ on $X$, then the preorder $\Theta(T)$ coincides with preorder $\Omega(T)$.

It should be noted that the correspondence $(X, T) \mapsto (X, \Theta(T))$ is not functorial (this was pointed out to us by W. Yao in a personal communication). And it is not hard to check that for any topological limit space $(X, T)$, $\Theta(T) = \Omega \circ \hat{e}^{-1}(T)$.

The composition $\textbf{Lim} \xrightarrow{\mathcal{R}} \textbf{Top} \xrightarrow{\Omega} \textbf{Prord}$ defines another preorder on each limit space.

**Proposition 3.1.** Let $(X, T)$ be a limit space.

(1) $\Theta(T) \subseteq \Omega \circ \mathcal{R}(T)$.

(2) The topology $\mathcal{R}(T)$ is coarser than the topology $\Gamma \circ \Theta(T)$.

**Proof.** (1) Assume that $(x, y) \in \Theta(T)$, i.e., $x \leq_T y$. By definition of $\leq_T$, $x \leq_T y$ implies that $\hat{y} \to x$. Hence if $U$ is an open neighborhood of $x$ in $(X, \mathcal{R}(T))$, then $U \in \hat{y}$, i.e., $y \in U$. Thus, that $(x, y) \in \Omega(\mathcal{R}(T))$.

(2) It suffices to check that for every open set $U$ in $(X, \mathcal{R}(T))$, $U$ is an upper set w.r.t. the preorder $\leq_T$. In fact, if $x \in U$ and $x \leq_T y$, then $\hat{y} \xrightarrow{T} x$ by definition of $\leq_T$. Hence, $U \in \hat{y}$ by definition of $\mathcal{R}(T)$, whence $y \in U$.

**Example 3.2.** The converse statements of the above proposition are not true. Let $X = \{x, y, z\}$. $T$ is the limit structure on $X$ given by $(\mathcal{F}, x) \in T \iff \mathcal{F} \supseteq \breve{\hat{x}} \cap \breve{\hat{z}}$, $(\mathcal{F}, y) \in T \iff \mathcal{F} \supseteq \breve{\hat{y}}$, and $(\mathcal{F}, z) \in T \iff \mathcal{F} \supseteq \breve{\hat{y}} \cap \breve{\hat{z}}$. Then

$\mathcal{R}(T) = \{\{x, y, z\}, \emptyset, \{y, z\}, \{y, z\}\}$.

$\Theta(T) = \{(x, x), (y, y), (z, z), (z, y)\}$. 

But,
\[
\Theta \circ \Theta(T) = \{(x, y, z), \emptyset, \{x\}, \{y\}, \{x, y\}, \{y, z\}, \{x, y, z\}\}.
\]
\[
\Omega \circ \Phi(T) = \{(x, y), (x, z), (x, x), (y, y), (z, z), (z, y)\}.
\]

3.2. The adjunction \(\omega_L \vdash i_L : \text{Lim} \rightarrow \text{SL-Top}\)

\((L, \ast, 1)\) denotes a meet continuous residuated lattice in this subsection if not otherwise specified. An \(L\)-topology on a set \(X\) is a subset \(\tau\) of \(L^X\) closed with respect to finite meets and arbitrary joins. \((X, \tau)\) is called an \(L\)-topological space. An \(L\)-topology \(\tau\) on \(X\) is called stratified if it contains all the constant functions from \(X\) to \(L\). The category of (stratified) \(L\)-topological spaces is denoted by \((\text{SL-Top}) \rightarrow \text{SL-Top}\) respectively.

Limit spaces and \(L\)-topological spaces are extensions of topological spaces in quite different directions. However, they are closely related to each other via the Scott convergence on the complete lattice \(L\) [13–15,30].

An upper set \(U\) on a complete lattice \(L\) is called Scott open if for each directed set \(D \subseteq L\), \(\exists D \in U\) implies that \(D \cap U \neq \emptyset\). The Scott open sets on \(L\) form a topology \(\sigma L\) on \(L\), called the Scott topology. For a meet continuous lattice \(L\), it is routine to check that \(\mathcal{H}(L, \gamma L) = (L, \sigma L)\).

For every limit space \((X, T)\), let
\[
\omega_L(T) = \{\lambda : (X, T) \rightarrow (L, \gamma L)|\lambda\text{ is continuous}\}.
\]
Then \(\omega_L(T)\) is a stratified \(L\)-topology on \(X\). The correspondence \((X, T) \mapsto (X, \omega_L(T))\) defines a concrete functor \(\omega_L : \text{Lim} \rightarrow \text{SL-Top}[13,30]\).

Conversely, for an \(L\)-topological space \((X, \tau)\), let
\[
i_L(\tau) = \left\{ (F, x)|\lambda(x) \leq \bigvee_{A \in F} \bigwedge_{y \in A} \lambda(y) \text{ for all } \lambda \in \tau \right\}.
\]
Then \(i_L(\tau)\) is a limit structure on \(X\) and
\[
\{\lambda : (X, i_L(\tau)) \rightarrow (L, \gamma L)\}_{\lambda \in \tau}
\]
is an initial source in \(\text{Lim}\). The correspondence \((X, \tau) \mapsto (X, i_L(\tau))\) defines a concrete functor \(i_L : \text{SL-Top} \rightarrow \text{Lim}\) [13,30].

**Proposition 3.3** (Höhle [13], Zhang [30]). Let \(L\) be a meet continuous lattice. Then the pair \((\omega_L, i_L)\) is a Galois correspondence.

**Proposition 3.4.** Suppose that \(L\) is a continuous lattice.

1. For every stratified \(L\)-topological space \((X, \tau)\), \((X, i_L(\tau))\) is topological.
2. \(\omega_L = \omega_L \circ i \circ \Phi\).

**Proof.** (1) First, the limit space \((L, \gamma L)\) is topological since \(L\) is a continuous lattice. Second, because \(\text{TopLim}\) is initially closed in \(\text{Lim}\) and the source \(\{\lambda : (X, i_L(\tau)) \rightarrow (L, \gamma L)\}_{\lambda \in \tau}\) is initial in \(\text{Lim}\), the limit space \((X, i_L(\tau))\) is therefore topological.

(2) By (1), the correspondence \((X, \tau) \mapsto (X, i_L(\tau))\) actually defines a concrete functor \(i_L^c : \text{SL-Top} \rightarrow \text{TopLim}\) such that \(i_L = i \circ i_L^c\), where \(i\) is the embedding of \(\text{TopLim}\) in \(\text{Lim}\). Write \(\omega_L^c : \text{TopLim} \rightarrow \text{SL-Top}\) for the functor obtained by restricting the domain of \(\omega_L\) to \(\text{TopLim}\). Then, appealing to Proposition 3.3, we obtain an adjunction \(\omega_L^c \vdash i_L^c\).
Since $R+e$, $\omega_L \circ e_L$, and $e \circ e^{-1}$, where $e : \text{Top} \rightarrow \text{TopLim}$ is the isomorphism defined in the above subsection and $e^{-1}$ is the inverse of $e$, the composition

$$\text{Lim} \xrightarrow{\hat{\epsilon}} \text{Top} \xrightarrow{e} \text{TopLim} \xrightarrow{\omega_L} \text{L-STOP}$$

is a left adjoint of the composition

$$\text{SL-STOP} \xrightarrow{i_L} \text{TopLim} \xrightarrow{\hat{e}^{-1}} \text{Top} \xrightarrow{e} \text{Lim}.$$ 

Because $i_L = e \circ \hat{e}^{-1} \circ i_L$ and $\omega_L$ is a left adjoint of $i_L$, we obtain that $\omega_L = \omega_L \circ \hat{e} \circ R = \omega_L \circ e \circ R$, as desired. □

**Remark 3.5.** If $L = \{0, 1\}$, then $\text{SL-STOP} = \text{Top}$ and the adjunction $\omega_L \circ L$ is the adjunction $R+e$. So, $\omega_L : \text{Lim} \rightarrow \text{SL-STOP}$ is an extension of the functor $R : \text{Lim} \rightarrow \text{Top}$ to the many valued setting.

**Proposition 3.6** (Höhle and Kubiak [15], Lai and Zhang [21]). If $L$ is a meet continuous lattice, then the functor $\omega_L \circ e : \text{Top} \rightarrow \text{SL-STOP}$ has a left adjoint given by $\rho_L : \text{SL-STOP} \rightarrow \text{Top}$, $\rho_L(X, \tau) = (X, \rho_L(\tau))$, where, $\rho_L(\tau) = \{U \subseteq X | 1_U \in \tau\}$.

**3.3. The adjunction $\Gamma_L \circ \Omega_L : L-\text{Prord} \rightarrow \text{SL-STOP}**

In this subsection, $(L, \ast, 1)$ always denotes a residuated complete lattice.

**Definition 3.7** (Bělohlávek [2,3], Gottwald [10], Valverde [28]). An $L$-preorder (or, a fuzzy preorder) on a set $X$ is a function $R : X \times X \rightarrow L$ such that

1. $R(a, a) = 1$ for every $a \in X$ (reflexivity);
2. $R(a, b) \ast R(b, c) \leq R(a, c)$ for all $a, b, c \in X$ (transitivity).

The pair $(X, R)$ is called an $L$-preordered set.

An $L$-preorder $R$ is called an $L$-equivalence if it is symmetric in the sense that

3. $R(x, y) = R(y, x)$ for all $x, y \in X$.

In the sequel, we write simply $X$ for an $L$-preordered set $(X, R)$ and $X(x, y)$ for $R(x, y)$ if there would be no confusion with respect to the $L$-preorder $R$.

**Remark 3.8.** There is a more general notion of fuzzy order, fuzzy equivalence based preorders, introduced and studied in [4,6,7] under the name $M$-E-ordering, where $E$ is an $L$-equivalence. If $E$ is the identity relation on $X$, then an $M$-E-ordering is exactly an $L$-preorder on $X$ discussed in this paper.

An $L$-order-preserving function $f : (X, R) \rightarrow (Y, S)$ between $L$-preordered sets is a function $f : X \rightarrow Y$ such that $R(a, b) \leq S(f(a), f(b))$ for all $a, b \in X$. $f$ is said to be an isometry if $R(a, b) = S(f(a), f(b))$ for all $a, b \in X$. The class of all $L$-preordered sets and $L$-order-preserving functions constitute a category, denoted by $L-\text{Prord}$.

A classical preorder $\leq$ on a set $X$ defines an $L$-preorder on $X$ with range $\{0, 1\} \subseteq L$. Precisely, let $\phi_L(\leq)(x, y) = 1$ if $x \leq y$; otherwise, $\phi_L(\leq)(x, y) = 0$. The correspondence $(X, \leq) \mapsto (X, \phi_L(\leq))$ defines a concrete functor $\phi_L : \text{Prord} \rightarrow L-\text{Prord}$, which is indeed an embedding.

**Proposition 3.9** (Lai [22]). The embedding functor $\phi_L : \text{Prord} \rightarrow L-\text{Prord}$ has both a left and a right adjoint.

**Proof.** This conclusion was proved in [22] in the case $L$ is the unit interval. The proof therein is valid in the general case. We copy the proof here in order to fix notations for later use.

Let $R$ be an $L$-preorder on $X$. Define a binary relation $R^*$ on $X$ as follows: $x R^* y$ if $R(x, y) > 0$. $R^*$ is a reflexive relation. Let $\sigma_L(R) = \bigcup_{n=1}^{\infty} (R^*)^n$, where $(R^*)^1 = R^*$, $(R^*)^{n+1} = (R^*)^n \circ R^*$. Then $\sigma_L(R)$ is a preorder on $X$. The correspondence $(X, R) \mapsto (X, \sigma_L(R))$ defines a concrete functor $\sigma_L : L-\text{Prord} \rightarrow \text{Prord}$, which is a left adjoint of $\phi_L$. 
If we let \( \psi_L(R) = \{(x, y) \in X \times X | R(x, y) = 1\} \), then \( \psi_L(R) \) is a preorder on \( X \). The correspondence \( (X, R) \mapsto (X, \psi_L(R)) \) defines a concrete functor \( \psi_L : \text{L-Prord} \rightarrow \text{Prord} \), which is a right adjoint of \( \phi_L \). □

**Example 3.10.** In this example, we list some well-known examples of \( \text{L-preorders} \) and \( \text{L-order-preserving functions} \). The aim is to fix some notations for later use.

1. Suppose \((X, R)\) is an \( \text{L-preordered set} \). Let \( R^\text{op}(a, b) = R(b, a) \) for all \( a, b \in X \). Then \( R^\text{op} \) is an \( \text{L-preorder on } X \), called the **opposite** of \( R \).
2. Let \( R : L \times L \rightarrow L \) be defined by \( R(x, y) = x \rightarrow y \). Then \((L, R)\) is an \( \text{L-preordered set} \) because of (I3) and (I4), which shall be denoted by \((L, \rightarrow)\) in the sequel.
3. Let \((X, \leq)\) be a classical preordered set. Then \( \mu : (X, \phi_L(\leq)) \rightarrow (L, \rightarrow) \) is an \( \text{L-order-preserving function} \) if and only if \( \mu : (X, \leq) \rightarrow L \) is order-preserving.
4. Given \( \text{L-preordered sets } X, Y \), let \([X, Y]\) denote the set of \( \text{L-order-preserving functions} \) from \( X \) to \( Y \). For all \( f, g \in [X, Y] \), let

\[
[X, Y](f, g) = \bigwedge_{x \in X} Y(f(x), g(x)).
\]

Then \([X, Y]\) becomes an \( \text{L-preordered set} \).
5. For an \( \text{L-preordered set } X \), the function \([23]\)

\[
y : X \rightarrow [X^\text{op}, (L, \rightarrow)], \quad y(a)(x) = X(x, a)
\]

and the function

\[
y' : X \rightarrow [X, (L, \rightarrow)]^\text{op}, \quad y'(a)(x) = X(a, x)
\]

are both isometries.

Let \((X, R)\) be an \( \text{L-preordered set} \). Following the terminologies in [22], every \( \text{L-order-preserving function} \) \( \mu : (X, R) \rightarrow (L, \rightarrow) \) is called an upper \( \text{L-subset of } (X, R) \), and every \( \text{L-order-preserving function} \) \( \mu : (X, R^\text{op}) \rightarrow (L, \rightarrow) \) a lower \( \text{L-subset of } (X, R) \). For each \( a \in X \), \( y(a) \) is an upper \( \text{L-subset} \) (the principal upper \( \text{L-subset} \) generated by \( a \)); \( y(a) \) is a lower \( \text{L-subset} \) (the principal lower \( \text{L-subset} \) generated by \( a \)).

The following theorem asserts that the upper \( \text{L-subsets} \) in an \( \text{L-preordered set } (X, R) \) constitute a stratified \( \text{L-topology} \) on \( X \), called the Alexandrov \( \text{L-topology} \) on \( (X, R) \).

**Theorem 3.11** (Klavonv and Castro [18]). For a given \( \text{L-preorder } R \) on \( X \), the family \( \Gamma_L(R) \) of upper \( \text{L-subsets} \) satisfies the following properties: for all \( K \subseteq \Gamma_L(R) \), \( \mu \in \Gamma_L(R) \), and \( a \in L \),

1. Every constant \( \text{fuzzy set } X \rightarrow L \) belongs to \( \Gamma_L(R) \);
2. \( \forall K \in \Gamma_L(R); \bigvee K \in \Gamma_L(R) \);
3. \( a * \mu \in \Gamma_L(R); a \rightarrow \mu \in \Gamma_L(R) \).

If \( R \) is an \( \text{L-equivalence} \), then \( \Gamma_L(R) \) satisfies moreover

4. \( \mu \rightarrow a \in \Gamma_L(R) \).

Conversely, for a given set \( \mathcal{F} \subseteq L^X \) satisfying conditions (1)–(3), there is a unique \( \text{L-preorder } R \) such that \( \mathcal{F} = \Gamma_L(R) \). In this case, \( R(x, y) = \bigwedge_{\mu \in \mathcal{F}} \mu(x) \rightarrow \mu(y) \). Moreover, if \( \mathcal{F} \) satisfies moreover (4), then \( R \) is symmetric.

Suppose that \( f : (X, R) \rightarrow (Y, S) \) is an \( \text{L-order-preserving function} \). It is easy to verify that \( f : (X, \Gamma_L(R)) \rightarrow (Y, \Gamma_L(S)) \) is continuous. Therefore, the correspondence \( (X, R) \mapsto (X, \Gamma_L(R)) \) defines a concrete functor \( \Gamma_L : \text{L-Prord} \rightarrow \text{SL-Top} \).

**Proposition 3.12** (Valverde [28]). Let \( K \subseteq L^X \) be a family of functions from \( X \) to \( L \). Then \( \Omega_L(K)(x, y) = \bigwedge_{\mu \in K} \mu(x) \rightarrow \mu(y) \) is an \( \text{L-preorder on } X \), called the \( \text{L-preorder} \) determined by \( K \). Each \( \mu \in K \) is an upper \( \text{L-subset} \) in \((X, \Omega_L(K))\).
Given an $L$-topological space $(X, \tau)$, the $L$-preorder $\Omega_L(\tau)$ on $X$ is called the \textit{specialization order} of $(X, \tau)$ [22]. We leave it to the reader to check that the correspondence $(X, \tau) \mapsto (X, \Omega_L(\tau))$ defines a concrete functor $\Omega_L : \mathbf{SL}\text{-}\mathbf{Top} \rightarrow \mathbf{L\text{-}\mathbf{Prord}}$. It is easy to check that when $L = [0, 1]$, $\Omega_L$ is exactly the functor $\Omega : \mathbf{Top} \rightarrow \mathbf{Prord}$. So, $\Omega_L$ is an extension of the functor $\Omega$ to the many valued setting.

**Proposition 3.13** (Lai [22]). $(\Gamma_L, \Omega_L)$ is a Galois correspondence. Moreover, $\Omega_L$ is a left inverse of $\Gamma_L$.

An $L$-topological space $(X, \tau)$ is said to be pseudo-discrete [22] if $\lambda \rightarrow a \in \tau$ for all $\lambda \in \tau$ and $a \in L$. Then part of Theorem 3.11 can be restated as follows.

**Proposition 3.14.** Suppose that $R$ is an $L$-preorder on $X$. Then $R$ is an $L$-equivalence if and only if $\Gamma_L(R)$ is pseudo-discrete.

For a non-empty subset $K$ of $L^X$,

$$\tau_K = \bigcap \{ \tau | \tau \text{ is a stratified } L\text{-}topology on } X \text{ with } K \subseteq \tau \}$$

is a stratified $L$-topology on $X$, called the stratified $L$-topology generated by $K$.

**Proposition 3.15** (Lai [22]). Suppose that $\tau$ is an $L$-topology on $X$ generated by $K \subseteq L^X$. Then for all $x, y \in X, \Omega_L(\tau)(x, y) = \bigwedge_{\mu \in K} \mu(x) \rightarrow \mu(y)$.

### 4. The main results

This section presents the main results in this paper about the connection between limit spaces, $L$-topological spaces, and $L$-preorders. Throughout this section, $(L, \ast, 1)$ is assumed to be a meet continuous residuated lattice if not otherwise specified.

We begin with two lemmas.

**Lemma 4.1.** Let $(X, T)$ be a limit space. Every continuous function $\lambda : (X, T) \rightarrow (L, \gamma L)$ is order-preserving with respect to the specialization order $\preceq_T$ on $X$.

**Proof.** If $x \preceq_T y$ then $\lambda(y) \rightarrow \lambda(x)$ since $\lambda$ is continuous. This means that $\lambda(\lambda(x)) \preceq \bigwedge_{a \in A} \lambda(a) = \lambda(y)$ by definition of the Scott convergence structure $\gamma L$. $\square$

**Lemma 4.2.** Let $(X, T)$ be a topological space and $L$ a meet continuous lattice. Then $f : (X, e(T)) \rightarrow (L, \gamma L)$ is continuous if and only if there is a family of open sets $\{U_s|s \in S\}$ and a subset $\{a_s|s \in S\}$ of $L$ such that $f = \bigwedge_{s \in S} a_s \land 1_{U_s}$. Therefore, the set $\{a \land 1_{U}|a \in L, U \in T\}$ is a base for the stratified $L$-topology $\omega_L \circ e(T)$.

**Proof.** Let $(X, T)$ be a topological space. First of all, we note that if a filter $G$ on $X$ converges to $x$ in $(X, e(T))$ if and only if $\hat{G} \succeq N^\circ(x)$, where $N^\circ(x) = \{U \in T | x \in U\}$ is the set of open neighborhoods of $x$ in $(X, T)$. For each $x \in X$, let $\hat{N}(x) = \{A \subseteq X | \exists U \in N^\circ(x), U \subseteq A\}$. Then $\hat{N}(x)$ is the smallest filter converging to $x$. Therefore, $f : (X, e(T)) \rightarrow (L, \gamma L)$ is continuous if and only if $f(\hat{N}(x)) \rightarrow f(x)$ for every $x \in X$ if and only if

$$f(x) = \bigwedge_{A \in \hat{N}(x)} \bigwedge_{y \in A} f(y) = \bigwedge_{U \in N^\circ(x)} \bigwedge_{y \in U} f(y)$$

for every $x \in X$.

**Sufficiency:** Suppose that $f = \bigwedge_{s \in S} a_s \land 1_{U_s}$ for some family of open sets $\{U_s|s \in S\}$ and some subset $\{a_s|s \in S\}$ of $L$. For each $x \in X$, let $M_x = \{s \in S | x \in U_s\}$. Then

$$f(x) = \bigvee_{s \in M_x} a_s \leq \bigvee_{s \in M_x, y \in U_s} f(y) \leq \bigwedge_{U \in N^\circ(x)} \bigwedge_{y \in U} f(y),$$

where the first inequality holds because $f(y) \succeq a_s$ for any $y \in U_s$. 


Proof. Suppose that \( f : (X, e(T)) \to (L, \gamma L) \) is continuous. Then for each \( x \in X \), \( f(x) \leq \bigvee_{U \in N'(x)} \bigwedge_{y \in U} f(y) \). The converse inequality \( f(x) \geq \bigvee_{U \in N'(x)} \bigwedge_{y \in U} f(y) \) is obvious. Thus, \( f(x) = \bigvee_{U \in N'(x)} \bigwedge_{y \in U} f(y) \).

For each \( U \in \mathcal{T} \), let \( a_U = \bigwedge_{y \in U} f(y) \). Then for all \( x \in X \),

\[
f(x) = \bigvee_{U \in N'(x)} \bigwedge_{y \in U} f(y) = \bigvee_{U \in N'(x)} a_U \wedge 1_U(x) = \left( \bigvee_{U \in \mathcal{T}} a_U \wedge 1_U \right)(x),
\]

where the last equality holds because \( a_U \wedge 1_U(x) = 0 \) if \( U \notin N'(x) \). Therefore, \( f = \bigvee_{U \in \mathcal{T}} a_U \wedge 1_U \).

Finally, since each function of the form \( a_U \wedge 1_U \) with \( U \) an open set is clearly a member of \( \omega_L \circ e(\mathcal{T}) \), it follows that \( \{a \wedge 1_U | a \in L, U \in \mathcal{T}\} \) is a base for \( \omega_L \circ e(\mathcal{T}) \). \( \square \)

**Theorem 4.3.** \( \Gamma_L \circ \phi_L = \omega_L \circ e \circ \Gamma \). That is, the following diagram commutes.

\[
\begin{array}{ccc}
\text{Prord} & \xrightarrow{\Gamma} & \text{Top} & \xrightarrow{e} & \text{Lim} \\
\phi_L \downarrow & & \downarrow \omega_L & & \\
L\text{-Prord} & \xrightarrow{\Gamma_L} & SL\text{-Top} \\
\end{array}
\]

**Proof.** Let \((X, \leq)\) be a preorder set. By Example 3.10(3), \( \lambda \in \Gamma_L \circ \phi_L(\leq) \) if and only if \( \lambda : (X, \leq) \to L \) is order-preserving.

For any open set \( U \) in \((X, \Gamma(\leq))\) and \( a \in L \), \( a \wedge 1_U \) is an order-preserving function since \( U \) is an upper set in \((X, \leq)\). Since every element in \( \omega_L \circ e \circ \Gamma(\leq) \) can be written as a supremum of a family of functions of the form \( a_U \wedge 1_U \) with \( U \in \mathcal{T} \) by the above lemma, we obtain that every element in \( \omega_L \circ e \circ \Gamma(\leq) \) is order-preserving. Therefore, \( \omega_L \circ e \circ \Gamma(\leq) \subseteq \Gamma_L \circ \phi_L(\leq) \).

On the other hand, suppose that \( \lambda : (X, \leq) \to L \) is order-preserving. If \( F \xrightarrow{e \circ \Gamma} x \), then \( \uparrow x \in F \) because \( \uparrow x \) is an open neighborhood (indeed, the smallest one) of \( x \) in \((X, \Gamma(\leq))\). Thus,

\[
\bigvee_{A \in F} \bigwedge_{y \in A} \lambda(y) \geq \bigwedge_{y \in \uparrow x} \lambda(y) = \lambda(x).
\]

This means that \( \lambda : (X, e \circ \gamma(\leq)) \to (L, \gamma L) \) is continuous. Therefore, \( \Gamma_L \circ \phi_L(\leq) \subseteq \omega_L \circ e \circ \Gamma(\leq) \). \( \square \)

**Proposition 4.4.** For every limit space \((X, T)\), \( \phi_L \circ \Theta(X, T) \leq \Omega_L \circ \omega_L(X, T) \).

**Proof.** For all \( x, y \in X \), we distinguish two cases. If \( \phi_L \circ \Theta(T)(x, y) = 0 \), the conclusion holds trivially. If \( \phi_L \circ \Theta(T)(x, y) = 1 \) (i.e., \( x \leq_T y \)), then for all \( \lambda \in \omega_L(T) \), \( \lambda(x) \leq \lambda(y) \) by Lemma 4.1. Therefore,

\[\Omega_L \circ \omega_L(T)(x, y) = \bigwedge_{\lambda \in \omega_L(T)} \lambda(x) = 1. \square\]

**Example 4.5.** The inequality \( \phi_L \circ \Theta(X, T) \leq \Omega_L \circ \omega_L(X, T) \) can be strict. Let \( L = \{0, 1\} \). Then \( \phi_L \) is the identity functor, \( \Omega_L = \Omega \) and, as observed in Remark 3.5, \( \omega_L = \mathcal{P} \). Let \((X, T)\) be the limit space in Example 3.2. The conclusion in Example 3.2 just asserts that \( \phi_L \circ \Theta(X, T) \neq \Omega_L \circ \omega_L(X, T) \).

**Theorem 4.6.** \( \Omega_L \circ \omega_L \circ e = \phi_L \circ \Omega \). That is, the following diagram commutes.

\[
\begin{array}{ccc}
\text{Top} & \xrightarrow{e} & \text{Lim} & \xrightarrow{\omega_L} & SL\text{-Top} \\
\Omega \downarrow & & \downarrow \Omega_L & & \\
\text{Prord} & \xrightarrow{\phi_L} & L\text{-Prord} \\
\end{array}
\]

**Proof.** Let \((X, T)\) be a topological space and \( \leq \) be the specialization order \( \Omega(T) \) on \( X \). By Lemma 4.2, the set \( \{a \wedge 1_U | a \in L, U \in \mathcal{T}\} \) is a base of \( \omega_L \circ e(T) \). Thus, \( \omega_L \circ e(T) \) is generated by \( \{1_U | U \in \mathcal{T}\} \).
For all $x, y \in X$, appealing to Proposition 3.15, we have that

$$\Omega_L \circ \omega_L \circ e(T)(x, y) = \bigwedge_{U \in T} 1_U(x) \to 1_U(y) = \begin{cases} 1, & x \leq y, \\ 0, & x \leq y. \end{cases}$$

Therefore, $\Omega_L \circ \omega_L \circ e(T)(x, y) = \phi_L \circ \Omega(T)(x, y)$. □

**Corollary 4.7.** $\Gamma \circ \sigma_L = \rho_L \circ \Gamma_L$. That is, the following diagram commutes.

$$\begin{array}{ccc}
L\text{-Prord} & \xrightarrow{\Gamma_L} & SL\text{-Top} \\
\sigma_L \downarrow & & \downarrow \rho_L \\
\text{Prord} & \xrightarrow{\Gamma} & \text{Top}
\end{array}$$

**Proof.** Since $\sigma_L \circ \phi_L \colon \Gamma \circ \Omega \colon \Gamma_L \circ \Omega_L$ and $\rho_L \circ \omega_L \circ e$ (Proposition 3.6), we obtain that $\Gamma \circ \sigma_L \circ \phi_L \circ \Omega$ and $\rho_L \circ \Gamma_L \circ \Omega_L \circ \omega_L \circ e$. By Theorem 4.6, $\Omega_L \circ \omega_L \circ e = \phi_L \circ \Omega$. Thus, $\Gamma \circ \sigma_L = \rho_L \circ \Gamma_L$. □

**Theorem 4.8.** For every stratified $L$-topological space $(X, \tau)$, $\Theta \circ i_L(\tau) = \psi_L \circ \Omega_L(\tau)$.

**Proof.** Let $(X, \tau)$ be a stratified $L$-topological space. For all $(x, y) \in X \times X$, we distinguish two cases.

*Case 1: $(x, y) \in \Theta \circ i_L(\tau)$. By definition of $\Theta(i_L(\tau))$, we have $\hat{y} \xrightarrow{i_L(\tau)} x$. Thus, for all $\lambda \in \tau$,*

$$\lambda(x) \leq \bigvee_{A \in \mathcal{F}_0} \bigwedge_{a \in A} \lambda(a) = \lambda(y).$$

Hence

$$\Omega_L(\tau)(x, y) = \bigwedge_{\lambda \in \tau} \lambda(x) \to \lambda(y) = 1.$$ Therefore, $(x, y) \in \psi_L \circ \Omega_L(\tau)$.

*Case 2: $(x, y) \notin \Theta \circ i_L(\tau)$. By definition of $\Theta(i_L(\tau))$, there is some $(\mathcal{F}_0, y) \in i_L(\tau)$ such that $(\mathcal{F}_0, x) \notin i_L(\tau)$.*

We claim that there is some $\lambda_0 \in \tau$ such that $\lambda_0(x) \neq \lambda_0(y)$. Otherwise, for all $\lambda \in \tau$, it holds that

$$\lambda(x) \leq \lambda(y) \leq \bigvee_{A \in \mathcal{F}_0} \bigwedge_{a \in A} \lambda(a).$$

Then, $(\mathcal{F}_0, x) \in i_L(\tau)$ by definition of $i_L(\tau)$, contradictory to the assumption. Thus,

$$\Omega_L(\tau)(x, y) = \bigwedge_{\lambda \in \tau} \lambda(x) \to \lambda(y) \leq \lambda_0(x) \to \lambda_0(y) < 1.$$ Therefore, $(x, y) \notin \psi_L \circ \Omega_L(\tau)$.

A combination of the two cases yields that $\Theta \circ i_L(\tau) = \psi_L \circ \Omega_L(\tau)$. □

**Proposition 4.9.** (1) $e \circ \Gamma \circ \psi_L \subseteq i_L \circ \Gamma_L$; (2) $\sigma_L \circ \Omega_L \leq \Omega \circ \rho_L$.

**Proof.** (1) Let $R$ be an $L$-preorder on $X$. It is easy to check that if $R(x, y) = 1$ then $\lambda(x) \leq \lambda(y)$ for all $\lambda \in \Gamma_L(R)$.

For any $x \in X$, the principal upper set generated by $x$ w.r.t the preorder $\psi_L(R)$ is given by $\uparrow x = \{y \mid R(x, y) = 1\}$. Thus, if $(\mathcal{F}, x) \in e \circ \Gamma \circ \psi_L(R)$, then $\uparrow x \in \mathcal{F}$ by definition of $\Gamma \circ \psi_L(R)$. Therefore, for any $\lambda \in \Gamma_L(R)$,

$$\bigvee_{A \in \mathcal{F}} \bigwedge_{y \in \uparrow x} \lambda(y) \geq \bigwedge_{y \in \uparrow x} \lambda(y) \geq \lambda(x).$$

Then we obtain that $(\mathcal{F}, x) \in i_L \circ \Gamma_L(R)$ by definition.
The inequality can be strict since the right side is a limit space in general, but the left side is always a topological limit space.

(2) Let \( \tau \) be a stratified \( L \)-topology on \( X \). Denote the preorders \( \Omega \circ \rho_L(\tau) \) and \( \sigma_L \circ \Omega_L(\tau) \) by \( \leq \) and \( \leq' \), respectively. First, note that for any \( x, y \in X \), if \( \Omega_L(\tau)(x, y) = \bigwedge_{\lambda \in \mathcal{L}} (\hat{\lambda}(x) \rightarrow \hat{\lambda}(y)) > 0 \), then for any \( U \in \rho_L(\tau), x \in U \) implies that \( y \in U \). Therefore, \( \Omega_L(\tau)(x, y) > 0 \) implies that \( x \leq y \).

Suppose that \( x \leq y \). By definition of \( \sigma_L(\Omega_L(\tau)) \) (Proposition 3.9), there exist \( x_0, \ldots, x_n \in X \) such that

\[
x_0 = x, \quad x_n = y, \quad \Omega_L(\tau)(x_i, x_{i+1}) > 0, \quad i = 0, 1, \ldots, n - 1.
\]

Then \( x_i \leq x_{i+1} \) for each \( i \leq n - 1 \), whence \( x \leq y \) by transitivity.

The inequality can be strict as shown in [22], Example 4.21(1). \( \square \)

In the following we present another description of the limit structure \( \iota_L \circ \Gamma_L(R) \) for an \( L \)-preordered set \( (X, R) \). Let \( T(R) \) be the limit structure on \( X \) such that the source

\[
\{ y'(a) : (X, T(R)) \rightarrow (L, \gamma_L) \}_{a \in X}
\]

is initial in \( \text{Lim} \), where \( \gamma_L \) is the Scott convergence structure on \( L \) and \( y'(a) : (X, R) \rightarrow (L, \rightarrow) \) is the \( L \)-order-preserving function given by \( y'(a)(x) = R(a, x) \) (also see Example 3.10(5)).

**Proposition 4.10.** For every \( L \)-preordered set \( (X, R) \), \( T(R) = \iota_L \circ \Gamma_L(R) \). Therefore, the correspondence \( (X, R) \mapsto (X, T(R)) \) defines a concrete functor \( \iota_L : \text{L-Prord} \rightarrow \text{Lim} \), which is equal to the composition \( \iota_L \circ \Gamma_L \).

**Proof.** For each \( a \in X \), \( y'(a) : (X, \iota_L \circ \Gamma_L(R)) \rightarrow (L, \gamma_L) \) is continuous. Therefore, \( (\mathcal{F}, x) \in \iota_L(\Gamma_L(R)) \) implies that \( (\mathcal{F}, x) \in T(R) \), whence, \( \iota_L(\Gamma_L(R)) \subseteq T(R) \).

On the other hand, if \( (\mathcal{F}, x) \in T(R) \), then for every \( a \in X \),

\[
R(a, x) = y'(a)(x) = \bigvee_{Z \in \mathcal{F}} \bigwedge_{z \in Z} y'(a)(z) = \bigvee_{Z \in \mathcal{F}} \bigwedge_{z \in Z} R(a, z).
\]

Letting \( a = x \) we obtain that

\[
1 = R(x, x) = \bigvee_{Z \in \mathcal{F}} \bigwedge_{z \in Z} R(x, z).
\]

Then, for any \( \lambda \in \Gamma_L(R) \),

\[
\lambda(x) \rightarrow \bigvee_{Z \in \mathcal{F}} \bigwedge_{z \in Z} \lambda(z) \supseteq \bigvee_{Z \in \mathcal{F}} \bigwedge_{z \in Z} \lambda(x) \supseteq \bigvee_{Z \in \mathcal{F}} \bigwedge_{z \in Z} R(x, z) \supseteq 1,
\]

whence

\[
\lambda(x) \leq \bigvee_{Z \in \mathcal{F}} \bigwedge_{z \in Z} \lambda(z).
\]

which implies that \( (\mathcal{F}, x) \in \iota_L(\Gamma_L(R)) \). \( \square \)

**Example 4.11.** If \( \leq \) is a classical preorder on \( X \), then \( T(\phi_L(\leq)) \) is equal to \( e \circ \Gamma(\leq) \) (hence, topological). In fact, by definition, \( (\mathcal{F}, x) \in T(\phi_L(\leq)) \) if and only if for all \( a \in X \),

\[
\phi_L(\leq)(a, x) = y'(a)(x) = \bigvee_{Z \in \mathcal{F}} \bigwedge_{z \in Z} y'(a)(z) = \bigvee_{Z \in \mathcal{F}} \bigwedge_{z \in Z} \phi_L(\leq)(a, z).
\]

Since

\[
\phi_L(\leq)(a, x) = \begin{cases} 0, & a \not\leq x, \\ 1, & a \leq x, \end{cases}
\]


we obtain that
\[(\mathcal{F}, x) \in T(\phi_L(\leq)) \Leftrightarrow \forall a \leq x, 1 = \phi_L(\leq)(a, x) \leq \bigvee_{Z \in \mathcal{F}} \bigwedge_{z \in Z} \phi_L(\leq)(a, z)\]
\[\Leftrightarrow \forall a \leq x, \exists A \in \mathcal{F} \text{ such that } A \subseteq a \]
\[\Leftrightarrow \uparrow x \in \mathcal{F} \]
\[\Leftrightarrow (\mathcal{F}, x) \in e \circ I(\leq),\]
where the last equivalence holds because \(\uparrow x\) is the smallest neighborhood of \(x\) in \((X, e \circ I(\leq))\).

If \(L\) is a continuous lattice, by Proposition 3.4, \((X, T(R))\) is a topological limit space, that is \(T(R) = e(T)\) for some topology \(T\) on \(X\). In the following, we present a simple description of this topology \(T\). First, we recall some properties of the way below relation \(\ll\) on a continuous lattice [9]: (i) \(x \ll y \Rightarrow x \leq y\); (ii) \(x \ll y, y \ll z \Rightarrow x \ll z\); and (iii) for a directed set \(D\), \(x \ll z \leq \sup D\) implies \(x \ll d\) for some \(d \in D\).

**Proposition 4.12.** Suppose that \(L\) is a continuous lattice and that \(R\) is an \(L\)-preorder on \(X\). For any \(x \in X, r \in L \setminus \{0\}\), let \(B(x, r) = \{y | r \ll R(x, y)\}\). Then \([B(x, r)|x \in X, r \in L \setminus \{0\}]\) is a base for some topology \(T\) on \(X\) with \(T(R) = e(T)\).

**Proof.** That \([B(x, r) : x \in X, r \in L \setminus \{0\}]\) is a base for some topology \(T\) on \(X\) was proved in [22] in the case \(L\) is the unit interval \([0, 1]\). The proof therein is valid for a continuous lattice by replacing the strictly smaller-than relation \(<\) on \([0, 1]\) by the way below relation \(\ll\) on \(L\). So, we need only prove that \(T(R) = e(T)\).

We note firstly that for any \(x \in X, N(x) = \{B(a, r)|x \in B(a, r)\}\) is a neighborhood base of \(x\) w.r.t. the topology \(T\), hence, \(\mathcal{F}\) converges to \(x\) with respect to \(e(T)\) if and only if \(N(x) \subseteq \mathcal{F}\).

Assume that \((\mathcal{F}, x) \in T(R)\) and \((B(a, r) \in N(x)\). Because \(r \ll R(a, x)\) and
\[y'(a)(x) = R(a, x) \leq \bigvee_{Z \in \mathcal{F}} \bigwedge_{z \in Z} R(a, z)\]
by definition of \(T(R)\), there is some \(A \in \mathcal{F}\) such that \(r \ll \bigwedge_{z \in A} R(a, z)\). Therefore \(A \subseteq B(a, r)\), whence \(B(a, r) \in \mathcal{F}\) and \(N(x) \subseteq \mathcal{F}\). This proves that \(T(R) \subseteq e(T)\).

What remains is to show that \(e(T) \subseteq T(R)\). It suffices to show that if \((\mathcal{F}, x) \in e(T)\), then for all \(a \in X\) it holds that
\[R(a, x) = y'(a)(x) \leq \bigvee_{Z \in \mathcal{F}} \bigwedge_{z \in Z} R(a, z) = y'(a)(z) \leq \bigvee_{Z \in \mathcal{F}} \bigwedge_{z \in Z} R(a, z)\]
This inequality holds trivially if \(R(a, x) = 0\). Suppose that \(R(a, x) = r > 0\). Then, \(x \in B(a, s)\) for all \(s \ll r\). Thus, \(B(a, s) \in N(x) \subseteq \mathcal{F}\). Therefore,
\[\bigvee_{Z \in \mathcal{F}} \bigwedge_{z \in Z} R(a, z) \geq \bigvee_{s \ll r} \bigwedge_{z \in B(a, s)} R(a, z) \geq \bigvee_{s \ll r} s = r = R(a, x),\]
as desired. \(\square\)

**Definition 4.13 (Lowen [24]).** A limit space \((X, T)\) is called symmetric if for any \(x, y \in X, x\) converges to \(y\) implies \(x\) and \(y\) have the same convergent filters.

The following proposition shows that the functor \(T = t_L \circ I_L\) preserves symmetry in the sense that if \((X, R)\) is an \(L\)-equivalence, then \((X, T(R))\) is a symmetric limit space.

**Proposition 4.14.** Let \(R\) be an \(L\)-preorder on \(X\). Then \((X, T(R))\) is symmetric if and only if the kernel of \(R\), \(\ker(R) = \{(x, y) \in X \times X | R(x, y) = 1\}\), is an equivalence relation on \(X\).

**Proof.** Necessity: Suppose that \(\hat{x} \rightarrow y\). By definition of \(T(R)\), for all \(a \in X\),
\[y'(a)(y) = R(a, y) \leq \bigvee_{A \in \hat{x} \subseteq \mathcal{A}} \bigwedge_{z \in Z} y'(a)(z) = R(a, x),\]
Letting \( a = y \) we obtain that \( R(y, x) = 1 \). Hence \( R(x, y) = 1 \) by assumption. Consequently, \( R(a, x) = R(a, y) \) for every \( a \in X \) by transitivity of \( R \). If \( \mathcal{F} \xrightarrow{T(R)} x \), then

\[
R(a, y) = R(a, x) \leq \bigvee_{A \in \mathcal{F}} \bigwedge_{z \in A} y'(a)(z),
\]

for all \( a \in X \). Thus, \( \mathcal{F} \xrightarrow{T(R)} y \). Similarly, one can check that \( \mathcal{F} \xrightarrow{T(R)} y \) implies that \( \mathcal{F} \xrightarrow{T(R)} x \). Therefore, \( x \) and \( y \) have the same convergent filters.

**Sufficiency:** From the proof of the necessity part, we see that \( x \xrightarrow{T(R)} y \) if and only if \( R(y, x) = 1 \). If \( T(R) \) is symmetric, then \( R(x, y) = 1 \) if and only if \( R(y, x) = 1 \) if and only if \( \text{ker}(R) \) is symmetric. Since \( \text{ker}(R) \) is clearly reflexive and transitive, it is therefore an equivalence relation. \( \square \)

**Example 4.15.** Suppose that \( L \) has at least three elements. Take \( a \in L \setminus \{0, 1\} \). Let \( X = \{x, y\} \), \( R(x, y) = a \), \( R(y, x) = 0 \), \( R(x, x) = R(y, y) = 1 \). Then \( R \) is an \( L \)-preorder on \( X \) which is not symmetric, but \( T(R) \) is symmetric by the above proposition. Thus, the functor \( T \) does not reflect symmetry in general.

To summarize the results in this paper, for a meet continuous residuated lattice \((L, \ast, 1)\), consider the following diagrams:

\[
\begin{array}{ccc}
\text{Lim} & \xrightarrow{e \circ \Gamma} & \text{Prord} \\
\omega_L & \xrightarrow{\Theta} & \phi_L \\
\downarrow{L\text{-Top}} & & \downarrow{L\text{-Prord}} \\
\Omega_L & \xrightarrow{\theta} & \psi_L
\end{array}
\quad
\begin{array}{ccc}
\text{Top} & \xrightarrow{\Gamma} & \text{Prord} \\
\omega_L \circ e & \xrightarrow{\rho_L} & \phi_L \\
\downarrow{L\text{-Top}} & & \downarrow{L\text{-Prord}} \\
\Omega_L & \xrightarrow{\sigma_L} & \psi_L
\end{array}
\]

- The arrow \( \Theta \) is not a functor; all the other arrows are functors.
- Adjunctions: \( \omega_L + \downarrow L \); \( \Gamma_L + \downarrow \Omega_L \); \( \phi_L + \downarrow \psi_L \); \( \rho_L + \downarrow \omega_L \circ e \); \( \sigma_L + \downarrow \phi_L \); and \( \Gamma + \downarrow \Omega \).
- If \( L = \{0, 1\} \), then the adjunctions \( \Gamma_L + \downarrow \Omega_L \) and \( \omega_L + \downarrow L \) are exactly the adjunctions \( \Gamma + \downarrow \Omega \) and \( \rho_L + \downarrow e \), respectively.
- For the left square: \( \Gamma_L \circ \phi_L = \omega_L \circ e \circ \Gamma \); \( \Theta \circ \downarrow L = \psi_L \circ \downarrow \Omega_L \); \( \phi_L \circ \Theta \leq \downarrow \omega_L \circ e \); \( e \circ \Gamma \circ \psi_L \leq \downarrow \Omega_L \circ \downarrow L \).
- For the right square: \( \Gamma_L \circ \phi_L = \omega_L \circ e \circ \Gamma \); \( \Gamma \circ \sigma_L = \rho_L \circ \Gamma_L \); \( \phi_L \circ \Omega = \Omega_L \circ \omega_L \circ e \); and \( \Omega \circ \rho_L \geq \sigma_L \circ \Omega_L \).

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**References**