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s_β-compact and s_β-closed spaces

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Abstract - The objective of this paper is to obtain the properties of s_β -compact and s_β –closed spaces by using nets, filterbase and s_β -complete accumulation points.

Index Terms— s_β -open sets, semi open sets, s_β -compact spaces, s_β –closed spaces, s_β -complete accumulation points.

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1 INTRODUCTION AND PRELIMINARIES

It is well- Known that the effects of the investigation of properties of closed bounded intervals of real numbers, spaces of continuous functions and solution to differential equations are the possible motivations for the formation of the notion of, compactness. Compactness is now one of the most important, useful, and fundamental notions of not only general topology but also of other advanced branches of mathematics. Recently Khalaf A.B. et al. [13] introduced and investigated the concepts of s_β-space and S_β-continuity. A semi open subset A of a topological space (X,τ) is said to be S_β-open if for each $x \in X$ there exists a β-closed set F such that $x \in F \subseteq A$. The aim of this paper to give some characterizations of S_β-compact spaces in terms of nets and filter-bases. We also introduce the notion of S_β-complete accumulation points by which we give some characterizations of S_β-compact spaces. Throughout the present paper, (X,τ) and (Y,υ) or simply X and Y denote topological space . In [14] Levine initiated semi open sets and their properties.

Mathematicians gives in several papers interesting and different new types of sets. In [1] Abd-El-Moonsef defined the class of β-open set. In [18] Shareef introduced a new class of semi-open sets called S_P- open sets. We recall the following definitions and characterizations. The closure (resp., interior) of a subset A of X is denoted by clA (resp., intA). A subset A of X is said to be semi-open [14] (resp., pre-open [15], α-open [16], β-open [1] regular open [19] and regular β-open [22]) set if $A \subseteq cl\,\text{int}A$, (resp., $A \subseteq \text{int}clA$, $A \subseteq \text{int}cl\,\text{int}A$, $A \subseteq cl\,\text{int}clA$, $A = \text{int}\,clA$ and $A = \beta\,\text{int}\,\beta clA$). The complement of S_β-open (resp., semi-open, pre-open, α-open , β-open, regular open, regular β-open) set is said to be S_β-closed (resp., semi-closed, pre- closed, α-closed , β-closed, regular closed, regular β-closed). The intersection of all S_β-closed (resp., semi-closed , pre- closed, β-closed) sets of X containing a subset A is called the S_β-closure (resp., semi-closure, pre-closure β-closure) of A and denoted by $S_\beta$clA (resp., sclA, pclA , βclA). The union of all S_β-open (resp., semi-open ,pre-open, β-open) set of X contained in A is called the S_β-interior (resp., semi-interior, pre-interior, β-interior) of A and denoted by $S_\beta$intA (resp., sintA , pintA , βintA). The family of all S_β-open (resp., semi-open, pre-open, α-open β-open, regular β-open, regular open, S_β-closed, semi-closed , pre- closed, α-closed , β-closed , regular β-closed , and regular closed) subset of a topological space X is denoted by S_βO(X) (resp., SO(X), PO(X), αO(X), βO(X), RβO(X), RO(X), S_βC(X), SC(X), PC(X), αC(X), βC(X) , RβC(X) and RC(X)). A subset A of X is called δ-open [21] if for each $x \in A$, there exists an open set B such that $x \in B \subseteq \text{int}clB \subseteq A$. A subset A of a space X is called θ-semi-open [12] (resp., semi-θ-open [7] if for each $x \in A$, there exists a semi-open set B such that $x \in B \subseteq clB \subseteq A$ (resp., $x \in B \subseteq SclB \subseteq A$.

Definition 1.1 [15]. A topological space (X,τ) is said to be :

1- Extremely disconnected if $clV \in \tau$ for every $V \in \tau$.

2- Locally indiscrete, if every open subset of X is closed.

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3- Hyper-connected if every non-empty open subset of X is dense.

Theorem 1.2 [14] Let A be any subset of a space X. Then $A \in SC(X)$ if and only if $clA = cl\,\text{int}\,A$.

The following results can be found in [13].

Proposition 1.3: If a space X is T_1, then $SO(X) = S_\beta O(X)$.

Proposition 1.4: If a topological space X is locally indiscrete, then every semi-open set is $S_\beta -$ open set.

Proposition1.5: A space X is hyper-connected if and only if $S_\beta O(X) = \{X, \varphi\}$.

Corollary 1.6: For any space X, $S_\beta O(X,\tau) = S_\beta O(X,\tau_\alpha)$

Proposition 1.7: If a topological space (X,τ) is β-regular, then $\tau \subseteq S_\beta O(X)$

Proposition 1.8: 1-Every S_p-open set is S_β-open.

2- An S_β-open set is regular β-open.

3- A regular closed set is S_β-open.

4- Every Regular open set is S_β-closed.

Definition 1.9: A subset A of a space X is said to be S_β-open [18] (resp., S_C [4]) if for each $x \in A \in SO(X)$ there exists a pre-closed (resp., closed) set F such that $x \in F \subseteq A$.

Definition 1.10: A filter base $\Im$ in a space X is s-converges [20] (resp., rc-converges [11]) to a point x if for every semi-open (resp., regular closed) subset U of X containing x there exist $B \in \Im$ such that $B \subseteq clU$ (resp., $B \subseteq U$).

Definition 1.11: A filter base $\Im$ in a space X is s-accumulates [20] (resp., rc- accumulates [11])to a point x if for every semi-open (resp., regular closed) subset U of X containing x there exist $B \in \Im$ such that $B \cap clU \neq \varphi$ (resp., $B \cap U \neq \varphi$).

2 S_β-compact and S_β-closed spaces

In this section, we introduce new classes of topological space called S_β-compact and S_β- closed spaces.

Definition 2.1: A filter base $\Im$ is S_β-convergent (resp., S_β-θ convergent) to a point $x \in X$, if for any S_β-open set V containing x, there exists $F \in \Im$ such that $F \subset V$ (resp., $F \subseteq S_\beta clV$).

Definition 2.2: A filter base $\Im$ is S_β-accumulates (resp., S_β-θ accumulates) to a point $x \in X$, if $F \cap V \neq \varphi$ (resp., $F \cap S_\beta clV \neq \varphi$) for any S_β-open set V containing x and every $F \in \Im$.

It is clear from the definition above, that S_β-converges (resp., S_β-accumulates) of a filter bases in a topological spaces implies S_β-θ - converges (resp., S_β-θ - accumulates) but the converses are not true in general as shown in the following examples

Example 2.3: Let $X = \{a,b,c,d\}$ and $\tau = \{\varphi, X, \{a,b\}, \{a,b,c\}\}$ we get $S_\beta O(X) = \{\varphi, X, \{a,b\}, \{a,b,c\}, \{a,b,d\}\}$ and let $\Im = \{X, \{c,d\}, \{\{b,c,d\}\}$, then $\Im$ S_β-θ -converges to a point a, but $\Im$ does not S_β-converges to a because the set $\{a,b\} \in S_\beta O(X)$ contains a, but there exists no $B \in \Im$ such that $B \subseteq \{a,b\}$. Also $\Im$ S_β-θ-accumulates to a point b, but $\Im$ does not S_β-accumulates to b, because the set $\{a,b\} \in S_\beta O(X)$ contains b, but there exist $\{c,d\} \subseteq \Im$ such that $\{a,b\} \cap \{c,d\} = \varphi$.

Theorem 2.4: Let (X,τ) be a topological space and let $\Im$ be a filter base on X. Then the following statements are equivalent:

1- There exists a filter base finer than $\{U_x\}$, where $\{U_x\}$ is the family of S_β-open sets of X containing x.

2- There exists a filter base $\Im_1$ finer than $\Im$ and S_β-converges to x.

Proof: Let $\Im_1$ be a filter base which is finer than both $\Im$ and $\{U_x\}$. Then $\Im$ S_β- converges to x since it contains $\{U_x\}$.

Conversely, let $\Im_1$ be the filter base which is finer than $\Im$ and which converges to x. Then $\Im$ must contain $\{U_x\}$ by definition.

Corollary 2.5: If $\Im$ is a maximal filter base in a topological space (X,τ), then $\Im$ S_β-converges (resp., S_β-$\theta-$ converges) to a point $x \in X$ if and only if $\Im$ S_β -accumulates (resp., S_β-θ -accumulates) to a point x.

Proof: Let $\Im$ be a maximal filter base in X and S_β -accumulates to a point $x \in X$, and then by Theorem 2.4, there exists a filter base $\Im_1$ finer than $\Im$ and S_β-converges to x. But $\Im$ is maximal filter base. Thus it is S_β-convergent to x.

Theorem 2.6: Let $\Im$ be a filter base in a topological space (X,τ). If $\Im$ S_β- converges (resp., S_β- accumulates) to a point $x \in X$, then $\Im$ rc-converges (resp., rc-accumulates) at a point $x \in X$.

Proof: Suppose that $\Im$ be a filter base S_β- converges to a point $x \in X$. Let V be any regular closed set containing x, by Theorem 2.1.15, then V is S_β-open set containing x. since $\Im$ S_β- converges (resp., S_β- accumulates) to a point $x \in X$, There exists $F \in \Im$ such that $F \subset V$ (resp., $F \cap V \neq \varphi$). This shows that $\Im$ rc-converges (resp., rc-accumulates) to a point $x \in X$.

The following example Show that the converse of Theorem 2.6 is not true in general.

Example 2.7: Let $X = \{a,b,c,d\}$ and $\tau = \{\varphi, X, \{a,c\}\}$, then the family $RC(X) = \{\varphi, X\}$ and $S_\beta O(X) = \{\varphi, X, \{a,c\}, \{a,b,c\}, \{a,c,d\}\}$ and $\Im = \{\{b\}, \{a,b,c\}, X\}$. Then $\Im$ is rc-converges (resp., rc-accumulates) to a point $c \in X$, but not S_β- converges (resp., S_β- accumulates) to a point $c \in X$.

Theorem 2.8: Let $\Im$ be a filter base in a topological space (X,τ). If $\Im$ S_β- converges (resp., S_β- accumulates) to a point $x \in X$, then $\Im$ s- converges (resp., s- accumulates) at a point $x \in X$.

Proof: Suppose that $\Im$ is S_β- converges to a point $x \in X$. Let V be any semi-open set containing x, then by Theorem 1.2 $clV = cl\,\mathrm{int}\,V$, so clV is regular closed , by Theorem Proposion 1.8 clV is S_β-open set containing x. Since $\Im$ is S_β- converges (resp., S_β- accumulates) to a point $x \in X$, then there exists $B \in \Im$ such that $B \subseteq clV$ (resp., $B \cap clV \neq \varphi$). This implies that $\Im$ is s- converges (resp., s- accumulates) at a point $x \in X$.

The converse of Theorem 2.8 is not true in general as shown in the following example:

Example 2.9: In Example 2.7 the family $SO(X) = \{\varphi, X, \{a,b\}, \{a,b,c\}, \{a,b,d\}\} = S_\beta O(X)$, considering the family $\Im = \{\{c\}, \{a,b,c\}, X\}$. Then $\Im$ is filter base s-converges (resp., s-accumulates) to a point $b \in X$, but not S_β- converges (resp., S_β- accumulates) to a point $b \in X$.

Theorem 2.10: Let $\Im$ be a filter base in a topological space (X,τ) and E is any β-closed set containing x. If there exists $F \in \Im$ such that $F \subseteq E$ (resp., $F \subseteq S_\beta clE$). Then $\Im$ S_β-converges (resp., S_β-θ converges) to a point $x \in X$.

Proof: Let V be any S_β-open set containing x. then V is semi-open and for each $x \in V$, there exist a β-closed set E such that $x \in E \subseteq V$. By hypothesis, there exists $F \in \Im$ such that $F \subseteq E \subseteq V$ (resp., $F \subseteq S_\beta clE \subseteq S_\beta clV$). Which implies that $F \subseteq V$ (resp., $F \subseteq S_\beta clV$). Hence $\Im$ S_β-converges (resp., S_β-θ converges) to a point $x \in X$.

Theorem 2.11 : Let $\Im$ be a filter base in a topological space (X,τ) and E is any β-closed set containing x. If there exists $F \in \Im$ such that $F \cap E \neq \varphi$ (resp., $F \cap \beta_S clE \neq \varphi$), then $\Im$ S_β- accumulates (resp., S_β θ -accumulates) at a point $x \in X$.

Proof: Similar to proof of Theorem 2.10.

Definition 2.12: A topological space (X,τ) is said to be S_β-compact (resp., S_β-closed) if for every cover $\{V_\alpha : \alpha \in \nabla\}$ of X, by S_β-open sets, there exists a finite subset ∇_0 of ∇ such that $X = \cup\{V_\alpha : \alpha \in \nabla_0\}$ (resp., $X = \cup\{S_\beta clV_\alpha : \alpha \in \nabla_0\}$)

It is clear that S_β-compact is S_β-closed, but not conversely as shown in the following examples

Example 2.13: Consider an countable space X with Co-countable topology. Since X is T_1, then by Proposition 1.3 the family of open, semi-open set and S_β- open sets are identical. Hence X is an S_β- closed because every open set in X is dense , not S_β- compact [19] p-194.

Theorem 2.14: If every β-closed cover of a space has a finite subcover, then X is S_β-compact.

Proof: Let $\{V_\alpha : \alpha \in \nabla\}$ be any S_β- open cover of X, and $x \in X$, then for each $x \in V_{\alpha(x)}$ and $\alpha \in \nabla$, there exists a β-closed sets $x \in X$ such that $x \in F_{\alpha(x)} \subseteq V_{\alpha(x)}$, so the family $\{F_{\alpha(x)} : \alpha \in \nabla\}$ is a β-closed cover of X, then by hypothesis, this family has a finite subcover such that $X = \cup\{F_{\alpha(xi)}; i = 1,2,...n\} \subseteq \cup\{V_{\alpha(xi)}; i = 1,2,...n\}$. Therefore $X = \cup\{V_{\alpha(xi)} : i = 1,2,...n\}$. Hence X is S_β-compact.

The following theorem shows the relation between S_β-compact and some other compactness

Theorem 2.15: Every semi-compact space, is S_β-compact spaces.

Proof: Let $\{V_\alpha : \alpha \in \nabla\}$ be any S_β- open cover of X. Then $\{V_\alpha : \alpha \in \nabla\}$ is a semi--open cover of X. Since X is semi-compact, there exists a finite subset ∇_0 of ∇ such that $X = \cup\{V_\alpha : \alpha \in \nabla_0\}$. Hence X is S_β-compact.

The following example shows that the converse of Theorem 2.15 is not true in general.

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Example 2.16: Let X= R with the topology $\tau = \{X, \phi, \{0\}\}$. Then (X,τ) is not semi-compact, since the space X is hyperconnected, then by Proposion 1.5 $S_\beta O(X) = \{\phi, X\}$. Then (X,τ) is S_β-compact.

Theorem 2.17: If a topological space (X,τ) is T_1 and S_β-compact space , then it is semi-compact.

Proof: Suppose that X is T_1 and S_β-compact space. Let $\{V_\alpha : \alpha \in \nabla\}$ be any semi-open cover of X. Then for every $x \in X$, there exist $\alpha(x) \in \nabla$ such that x$\in V_{\alpha(x)}$. Since X is T_1 by proposition 1.3, the family $\{V_\alpha : \alpha \in \nabla\}$ is S_β-open cover of X. Since X is S_β-compact, there exists a finite subset ∇_0 of ∇ in X such that $X = \cup\{V_\alpha : \alpha \in \nabla_0\}$. Hence X is semi-compact.

Theorem 2.18: If a topological space (X,τ) is locally indiscrete. Then S_β-compact space X is semi-compact .

Proof: Follows from Theorem 1.4.

In general S_β-compact spaces and compact spaces are not comparable as shown in the following examples.

Example 2.19: The one-point compactification of any discrete spaces is not S-closed [10] corollary 4, therefore the space is not S_β-compact, but it is compact.

Example 2.20: Let X=(0,1) with the topology $\tau = \{X, \varphi, G = (0, 1-\frac{1}{n}), n = 2,3,...\}$ then (X,τ) is not compact [19], p. 76, but it is S_β-compact since the only S_β-open subset of (X,τ) are X and ϕ.

Theorem 2.21: Let (X,τ) be a topological space. If X is β-regular and S_β-compact , then X is compact.

Proof: Let $\{V_\alpha : \alpha \in \nabla\}$ be any open cover of X. Since X is β-regular By Proposition 1.7, $\{V_\alpha : \alpha \in \nabla\}$ forms an S_β-open cover of X. Since X is S_β-compact, there exists a finite subset ∇_0 of ∇ such that $S_\beta \operatorname{int} F_{\alpha 0} \cap S_\beta cl(X \setminus F_{\alpha 0}) = \varphi$, hence X is compact.

Theorem 2.22: If X is an s- regular S_β-closed $T_1 -$space, then X is compact.

Proof: Let $\{V_\alpha : \alpha \in \nabla\}$ be any open cover of an s-regular and S_β-closed T_1 space X, then for each $\alpha \in \nabla$ and for each $x \in X$ there exists $\alpha(x) \in \nabla$ such that x$\in V_{\alpha(x)}$. Since X is s-regular T_1 space there exist $G_x \in SO(X)$ and $x \in G_x \subset SclG_x \subset V_{\alpha(x)}$. Then by Proposition 1.3, the family $\{G_x : x \in X\}$ is an S_β-open cover of X. Since X is S_β-closed space, then there exists a subfamily $\{G_{xi} : i = 1,2,...n\}$ such that $X = \bigcup_{i=1}^{n} S_\beta clG_{xi} \subseteq \bigcup_{i=1}^{n} V_{\alpha(xi)}$. Thus X is compact.

The following example shows that the condition of s-regularity in Theorem 2.22 can not be dropped

Example 2.23: In Example 2.20 (X,τ) is neither s-regular nor compact but it is S_β-closed because the only non-empty S_β-open set in (X,τ) is X itself.

Theorem 2.24: Let X be an almost- regular space if X is S_β-compact, then it is nearly compact.

Proof: Let $\{V_\alpha : \alpha \in \nabla\}$ be any regular open cover of X, Since X is almost- regular space then for each $x \in X$ and each regular open set $V_{\alpha(x)}$, there exist an open set G_x such that $x \in G_x \subset clG_x \subset V_{\alpha(x)}$. But clG_x is regular closed for each $x \in X$. Therefore $X = \bigcup_{x\in X} clG_x = \bigcup_{\alpha(x)\in\nabla} V_{\alpha(x)}$ This implies that the family $\{clG_x : x \in X\}$ is an S_β-open cover of X. Since X is S_β-compact, then there exists a subfamily $\{clG_{xi} : i = 1,2,...n\}$ such that $X = \bigcup_{i=1}^{n} clG_{xi} \subset \bigcup_{i=1}^{n} V_{\alpha(xi)}$. Thus X is nearly compact.

Theorem 2.25: Let X be semi- regular s-closed space, then it is S_β-compact.

Proof: Let $\{V_\alpha : \alpha \in \nabla\}$ be any S_β- open cover of X, Then V_α is semi-open for each $\alpha \in \nabla$, since X is semi-regular for each $x \in X$ and $V_{\alpha(x)}$, there exists a semi-open set G_x such that $x \in G_x \subseteq SclG_x \subseteq V_{\alpha(x)}$. Then the family $\{G_x : x \in X\}$ is semi-open cover of X. Since X is s-closed space, then there exists a subfamily $\{G_{xi} : i = 1,2,...n\}$ such that $X = \bigcup_{i=1}^{n} SclG_{xi} \subseteq \bigcup_{i=1}^{n} V_{\alpha(xi)}$. Thus X is S_β-compact.

Theorem 2.26: For any topological space (X,τ). The following statements are equivalent:

1- (X,τ) is S_β-compact spaces (resp., S_β-closed)

2- For any S_β-open cover $\{V_\alpha : \alpha \in \nabla\}$ of X, there exists a finite subset ∇_0 of ∇ such that $X = \cup\{V_\alpha : \alpha \in \nabla_0\}$ (resp.,

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$X = \bigcup\{S_\beta clV_\alpha : \alpha \in \nabla_0\}$)

3- Every maximal filter base $\Im$ in X S_β- converges (resp. S_β-θ- converges) to some point $x \in X$.

4- Every filter base $\Im$ in X S_β- accumulates (resp., S_β-θ-accumulates) to some point $x \in X$.

5- For every family $\{F_\alpha : \alpha \in \nabla\}$ of S_β-closed subsets of X such that $\bigcap\{F_\alpha : \alpha \in \nabla\} = \varphi$ there exists finite subset ∇_0 of ∇ such that $\bigcap\{F_\alpha : \alpha \in \nabla_0\} = \varphi$ (resp., $\bigcap\{S_\beta \text{int} F_\alpha : \alpha \in \nabla_0 = \varphi\}$)

proof: $1 \rightarrow 2$. Straightforward.

$2 \rightarrow 3$ Suppose that for every S_β-open cover $\{V_\alpha : \alpha \in \nabla\}$ of X , there exist a finite subset ∇_0 of ∇ such that $X = \bigcup\{V_\alpha : \alpha \in \nabla_0\}$ (resp., $X = \bigcup\{S_\beta clV_\alpha : \alpha \in \nabla_0\}$ and let $\Im = \{F_\alpha : \alpha \in \nabla\}$ be a maximal filter base. Suppose that $\Im$ does not S_β-converges (resp., S_β-θ- converges) to any point of X. Since $\Im$ is maximal, by Corollary 5.1.4 $\Im$ dose not S_β-accumulates (resp., S_β-θ-accumulates) to any point of X. This implies that for every $x \in X$ there exists S_β-open set V_x and $F_{\alpha(x)} \in \Im$ such that $F_{\alpha(x)} \cap V_x = \varphi$ (resp., $F_{\alpha(x)} \cap S_\beta clV_x = \varphi$). The family $\{V_x : x \in X\}$ is an S_β-pen cover of X and by hypothesis, there exists a finite number of points $x_1, x_2, ..., x_n$ of X such that $X = \bigcup\{V_{(xi)} : i = 1,2,...n\}$ (resp., $X = \bigcup\{S_\beta clV_{(xi)} : i = 1,2,...n\}$). Since $\Im$ is a filter base on X, there exists a $F_0 \in \Im$ such that $F_0 \subseteq \bigcap\{F_{\alpha(xi)} : i = 1,2,...,n\}$ **.** Hence $F_0 \cap V_{\alpha(xi)} = \varphi$ **(**resp**.,** $F_0 \cap S_\beta clV_{\alpha(xi)} = \varphi$) for $i = 1,2,...n$ which implies that $F_0 \cap (\bigcup\{V_{(xi)} : i = 1,2,...n\})$ (resp., $F_0 \cap (\bigcup\{S_\beta clV_{(xi)} : i = 1,2,...n) = F_0 \cap X = \varphi$ Therefore, we obtain $F_0 = \phi$. Which is contradict the fact that $\Im \neq \varphi$, thus $\Im$ is S_β- converges to some point $x \in X$

$3 \rightarrow 4$. Let $\Im$ be any filter base on X. Then, there exists a maximal filter base $\Im_0$ such that $\Im \subseteq \Im_0$. By hypothesis $\Im_0$ is S_β-converges (resp., S_β-θ- converges) to some point $x \in X$. For every $F \in \Im$ and every S_β-open set V containing x, there exists $F_0 \in \Im_0$ such that $F_0 \subseteq V$ (resp., $F_0 \subseteq S_\beta clV$), hence $\varphi \neq F_0 \cap F \subseteq V \cap F$ (resp., $S_\beta clV \cap F$. This shows that $\Im$ S_β-accumulates at x (resp., S_β-θ-accumulates).

$4 \rightarrow 5$. Let $\{F_\alpha : \alpha \in \nabla\}$ be a family of S_β-closed subsets of X such that $\bigcap\{F_\alpha : \alpha \in \nabla\} = \varphi$. If possible suppose that every finite subfamily $\bigcap\{F_{\alpha i} : i = 1,2,...,n\} \neq \varphi$. Therefore $\Im = A \subseteq Y \subseteq X$ form a filter base on X. By hypothesis, $\Im$ S_β- accumulates (resp., S_β-θ-accumulates) to some point $x \in X$. This implies that for every S_β-open set V containing x, $F_\alpha \cap V \neq \varphi$ (resp., $F_\alpha \cap S_\beta clV \neq \varphi$), for every $F_\alpha \in \Im$ and every $\alpha \in \nabla$ .Since $x \notin \bigcap F_\alpha$,there exist $\alpha_0 \in \nabla$ such that $x \notin F_{\alpha 0}$. Hence $X \setminus F_{\alpha 0}$ is S_β-open set containing x and $F_{\alpha 0} \cap X \setminus F_{\alpha 0} = \varphi$ (resp., $S_\beta \text{int} F_{\alpha 0} \cap S_\beta cl(X \setminus F_{\alpha 0}) = \varphi$). Which contracting the fact that $\Im$ S_β- accumulates to x (resp., S_β-θ-accumulates). So the assertion in (5) is true.

$5 \rightarrow 1$. Let $\{V_\alpha : \alpha \in \nabla\}$ be S_β-open cover of X.

Then $\{X \setminus V_\alpha : \alpha \in \nabla\}$ is a family of S_β-closed subsets of X such that $\bigcap\{X \setminus V_\alpha : \alpha \in \nabla\} = \varphi$. By hypothesis, there exists a finite subset ∇_0 of ∇ such that $\bigcap\{X \setminus V_\alpha : \alpha \in \nabla_0\} = \varphi$ (resp., $\bigcap\{S_\beta \text{int}(X \setminus V_\alpha) : \alpha \in \nabla_0\} = \varphi$. Hence $X = \bigcup\{V_\alpha : \alpha \in \nabla_0\}$ (resp., $X = \bigcup\{S_\beta clV_\alpha : \alpha \in \nabla_0\}$. This shows that X is S_β-compact (resp., S_β-closed) .

Theorem 2.27: If a topological space (X, τ) is S_β-closed and T_1-space then it is nearly compact.

Proof: Let $\{V_\alpha : \alpha \in \nabla\}$ be any regular open cover of X. Then $\{V_\alpha : \alpha \in \nabla\}$ is a S_β-open cover of X. Since X is S_β-closed , there exists a finite subset ∇_0 of ∇ such that $X = \bigcup\{V_\alpha : \alpha \in \nabla_0\}$. Hence X is nearly compact.

3 Characterization of S_β – compact spaces

Definition 3.1: A point x in X is said to be S_β-complete accumulation point of a subset A of X if Card $(A \cap U) =$ Card(A) for each $U \in S_\beta(X, x)$. Where Card(A) denotes the cardinality of A .

Definition3.2: In a topological space X, a point x is said to be an S_β-adherent point of a filter $\Im$ on X if it lies in the S_β-closure of all sets of $\Im$.

Theorem3.3 : A space X is S_β-compact spaces if and only if each infinite subset of X has S_β-complete accumulation point.

Proof: Let the space X be S_β-compact and S an infinite subset of X. Let K be the set of points x in X which are not S_β-complete accumulation points of S. Now it is obvious that for each point x in K , we are able to find $U_{(x)} \in S_\beta O(X, x)$ such that $Card(S \cap U_{(x)}) \neq Card(S)$. If K is the Whole space , then

$\Re=\{U_{(x)}:x\in X\}$ is S_β-cover of X. By hypothesis X is S_β-compact, so there exists a finite subcover $\Psi=\{U_{(xi)};i=1,2,...,n\}$ such that $S\subseteq\bigcup\{U_{(xi)}\cap S;i=1,2,...,n\}$, then $Card(S)=\max\{Card(U_{(xi)}\cap S);i=1,2,...n\}$, which does not agree with what we assumed. This implies that S has an S_β-complete accumulation. Now assume that X is not S_β-compact and that every infinite subset S of X has an S_β-complete accumulation point in X. it follows that there exists an cover Θ with no finite subcover. Set $\delta=\min\{Card(\Xi):\Xi\subset\Theta, wher\ \Xi\ is an S_\beta-cover\ of\ X\}$. Fix $\Psi\subseteq\Theta$, for which $Card(\Psi)=\delta$ and $\bigcup\{U:U\in\Psi\}=X$. Let N denote the set of natural numbers, then by hypothesis $\delta\geq Card(N)$ [By well-ordering of Ψ] . By some minimal well-ordering "~" , suppose that U is any member of Ψ. By minimal well-ordering "~", we have $Card(\{V:V\in\Psi,V\sim U\})<Card(\{V:V\in\psi\})$. Since Ψ can not have any subcover with cardinality less that δ, then for each $U\in\Psi$ we have $X\neq\bigcup\{V:V\in\Psi,V\sim U]$. For each $U\in\Psi$ choose a point $x(U)\in X-\bigcup\{V\bigcup\{x(V)\}:V\in\Psi,V\sim U\}$. We are always able to do this, if not, one can choose a cover of smaller cardinality from Ψ. If $H=\{x(U):U\in\Psi\}$, then to finish the proof we will show that H has no S_β-complete accumulation point in X. Suppose that z is a point of the space X. Since Ψ is S_β-cover of X, then z is a point of some set W in Ψ. By the fact that U~V we have $x(U)\in W$. It follows that $T=\{U:U\in\Psi\ and\ x(U)\in W\}\subset\{V:V\in\Psi,V\sim W\}$. But $Card(T)<\delta$. Therefore $Card(H\cap W)<\delta$. But $Card(H)=\delta\geq Card(N)$, since for two distinct points U and W in Ψ, we have $x(U)\neq x(W)$, this means that H has no S_β-complete accumulation point in X which contradicts our assumptions. Therefore X is S_β-compact.

Theorem3.4: For a topological space the following are equivalent:

i- X is S_β-compact.

ii- Every net in X with well-ordered directed set as its domain accumulates to some point of X..

Proof:(i)$\Rightarrow$(ii): Suppose that X is S_β-compact and $\xi=\{x_\alpha:\alpha\in\nabla\}$ a net with a well-ordered set ∇ as domain. . Assume that ξ has no S_β- adherent point in X. Then for each point x in X there exists $V_{(x)}\in S_\beta O(X,x)$ and an $\alpha(x)\in\nabla$ such that $V_{(x)}\cap\{x_\alpha:\alpha\geq\alpha(x)\}=\varphi$. This implies that $\{x_\alpha:\alpha\geq\alpha(x)\}$ is a subset of $X-V_{(x)}$. Then the collection $\omega=\{V_{(x)}:x\in X\}$ is S_β- cover of X. By hypothesis of theorem, X is S_β-compact and so ω has a finite subfamily $\{V_{(xi)}:i=1,2,...n\}$ such that $X=\bigcup\{V_{(xi)}:i=1,2,...n\}$ .Suppose that the corresponding elements of ∇ be $\{\alpha(xi)\}$ where i=1,2,…,n, since ∇ is well-ordered and $\{\alpha(xi)\}$ where i=1,2,…,n is finite. The largest elements of $\{\alpha(xi)\}$ exists. Suppose it is $\{\alpha(xi)\}$. Then for $\gamma\geq\{\alpha(xi)\}$. We have $\{x_\delta:\delta\geq\gamma\}\subset\bigcap_{i=1}(X-V_{(xi)})=X-\bigcup_{i=1}V_{(xi)}=\varphi$. Which is impossible. This shows that ξ has at least one S_β-adherent point in X .

(ii) $\Rightarrow$ (i): Now it is enough to prove that each infinite subset has an S_β-complete accumulation point by utilizing above theorem. Suppose that $S\subseteq X$ is an infinite subset of X. According to Zorns Lemma, the infinite set S can be well-ordered. This means that we can assume S to be a net with a domain which is a well ordered index set. It follows that S has S_β- adherent point z. Therefore is an S_β-complete accumulation point of S this shows that X is S_β-compact.

Theorem3.5 : A space X is S_β-compact if and only if each family of S_β-closed subsets of X with the finite intersection property has a non-empty intersection .

Proof: Given a collection ω of subsets of X. let $\upsilon=\{X-w:w\in\omega\}$ be the collection of their complements . Then the following statements hold.

i- ω is the collection of S_β-open sets if and only if υ is a collection of S_β-closed sets.

ii- the collection ω covers of X if and only if the intersection $\bigcap_{v\in\upsilon}\upsilon$ of all the elements of υ is non empty

iii- The finite sub collection $\{w_n,...w_n\}$ of ω covers X if and only if the intersection of the corresponding elements $v_i=X-w_i$ of υ is empty.

The statement (i) is trivial . While the statement (ii) and (iii) follows from De-Morgan Law $X-\bigcup_{\alpha\in j}v_\alpha=\bigcap_{\alpha\in j}(X-v_\alpha)$. The proof of theorem now proceeds in two steps. Taking the contra positive of the theorem and the complement .

The statement X is S_β-compact is equivalent to: Given any collection ω of S_β-open subsets of X, if ω covers X, then some finite sub collection of ω covers X. This statement is equivalent to its contra positive, Which is the following.

Given any collection ω of S_β-open sets, if no finite sub collection of ω covers X, then ω does not cover X . Letting υ be as earlier, the collection $\{X - w : w \in \omega\}$, and applying (i) to (iii), we see that this statement is in turn equivalent to the following.

Given any collection υ of S_β-closed sets , if every finite intersection of elements of υ is non empty. This is just the condition of our theorem.

Theorem3.6: A space X is S_β-compact if and only if each filter base in X has at least one S_β-adherent point.

Proof: Suppose that X is S_β-compact and $\Im = \{F_\alpha : \alpha \in \nabla\}$ is a filter base in it. Since all finite intersections of F_α's are nonempty . It follows that all finite intersections of $S_\beta cl(F_\alpha)$'s are also nonempty. Now it follows from Theorem [F.I.P] that $\bigcap_{\alpha\in\nabla} S_\beta clF_\alpha$ is nonempty. This means that $\Im$ has at least one S_β-adherent point. Now suppose that $\Im$ is any family of S_β-closed sets. Let each finite intersection be nonempty the set F_α with their finite intersection establish the filter base $\Im$. Therefore $\Im$ S_β-accumulates to some point z in X. It follows that $z \in \bigcap_{\alpha\in\nabla} F_\alpha$. Now we have by Theorem 3.5, that X is S_β-compact.

Theorem3.7: A space X is S_β-compact if and only if each filter base on X , with at most one S_β-adherent point, is S_β-convergent.

Proof: Suppose that X is S_β-compact, x is a point of X, and $\Im$ is a filter base on X. The S_β-adherent of $\Im$ is a subset of $\{x\}$. Then the S_β-adherent of $\Im$ is equal to $\{x\}$, by Theorem 3.7 . Assume that there exists a $V \in S_\beta O(X,x)$ such that for all $F \in \Im$, $F \bigcap (X - V)$ is nonempty . Then $\Psi = \{F - V : F \in \Im\}$ is a filter base on X . It follows that the S_β-adherence of Ψ is nonempty. However $\bigcap_{F\in\Im} S_\beta cl(F - V) \subseteq (\bigcap_{F\in\Im} S_\beta clF) \bigcap (X - V) = \{x\} \bigcap (X - V) = \varphi$. But this is a contradiction . Hence, for each $V \in S_\beta O(X,x)$ there exist $F \in \Im$ with $F \subseteq V$. This shows s that S_β-converges to x.

To prove the converse , It suffices to show that each filter base in X has at least one S_β-accumulation point. Assume that $\Im$ is a filter base on X with no S_β-adherent point. By hypothesis $\Im$ S_β-converges to some point z in X. Suppose F_α is an arbitrary element of $\Im$. Then for each $V \in S_\beta O(X,z)$, there exists an $F_\beta \in \Im$ such that $F_\beta \subseteq V$. Since $\Im$ is a filter base there exists a γ such that $F_\gamma \subseteq F_\alpha \bigcap F_\beta \subseteq F_\alpha \bigcap V$ where F_γ is a nonempty. This means that $F_\alpha \bigcap V$ is nonempty for every $V \in S_\beta O(X,z)$ and correspondingly for each α , z is a point of $S_\beta clF_\alpha$. It follows that $z \in \bigcap_\alpha S_\beta clF_\alpha$ Therefore, z is S_β – adherent point of $\Im$. Which is contradiction . This shows that X is S_β compact.

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