Truncated Predictor Feedback Control for Exponentially Unstable Linear Systems with Time-Varying Input Delay

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Abstract—The stabilization of exponentially unstable linear systems with time-varying input delay is considered in this paper. We extend the truncated predictor feedback (TPF) design method, which was recently developed for systems with all poles on the closed left-half plane, to be applicable to exponentially unstable linear systems. Assuming that the time-varying delay is known and bounded, the design approach of a time-varying state feedback controller is developed based on the solution of a parametric Lyapunov equation. An explicit condition is derived for which the stability of the closed-loop system with the proposed controller is guaranteed. It is shown that, for the stability of the closed-loop system, the maximum allowable time-delay in the input is inversely proportional to the sum of the unstable poles in the plant. The effectiveness of the proposed method is demonstrated through numerical examples.

I. INTRODUCTION

The control of most dynamic systems in the real world is affected by time-delays, which degrade the closed-loop performance and stability characteristics. Modern digital controller implementations broaden the reach of control theory to many industrial applications and allow the development of remotely controlled and complex networked systems. However, with the added complexity and the time required to complete the digital computations and communications in the control loop, many applications need to deal with substantial time delays. A straightforward approach to dealing with delays in control systems is to treat it as a stability robustness problem. In such an approach, the information of the delay is typically not used in the design of the controller. However, the difficult problem remains of establishing the conditions for stability and the corresponding bound on the allowable delays. The difficulty of this problem is more evident for multiple input multiple output systems, where the concept of gain/phase margin becomes indefinite. As a result, methods such as the predictor feedback, which explicitly use the delay information to design the stabilizing controller, have been widely explored in recent years.

The control of linear and nonlinear systems with time-delays have been a topic of extensive research, where [1]–[3] and all other references cited in this section are only a small sample of the available literature on this topic. The stabilization of a linear oscillator system was explored in [4] and [5]. The stabilization of a delayed chain of integrators was discussed in [6] and [7]. One extensively explored method that has proven to be efficient in dealing with time delays is the predictor feedback. Although much of the work in this field has focused on systems with constant time-delays as found in [8] and [9], research on the stabilization of systems with time-varying delays has been very active since the work of Artstein in [10]. A small sample of the work found in the literature on the stabilization of linear and nonlinear systems with time-varying delays can be found in [11]–[14]. When the time-delays are unknown, adaptive prediction feedback methods can be developed as in [15] and [16], for linear and nonlinear systems, respectively.

Lin and Fang in [13] developed a low gain feedback approach to the stabilization of a class of linear systems with constant input delays. By using an eigenstructure assignment based low gain feedback design [17], the authors of [13] show that a stabilizable and detectable linear system with an arbitrarily large time delay in the input can be asymptotically stabilized either by linear state feedback or by linear output feedback as long as the open loop system is not exponentially unstable. A salient feature of this low gain design is that it takes the structure of a predictor feedback control law but with the distributed portion of the predictor feedback control dropped, and hence the resulting feedback law is of a finite dimension. A simple example was also constructed in [13] to show that such a result would not be true if the open loop system is exponentially unstable. Another byproduct of this low gain feedback design is that, with no additional conditions, the resulting linear feedback laws would also semi-globally asymptotically stabilize such systems when they are also subject to input saturation.

The low gain feedback design approach proposed in [13] was further developed in [14], where a parametric Lyapunov equation based low gain feedback design was developed and the design method is termed “truncated predictor feedback (TPF).” In addition, time-varying delays are allowed.

In this paper we revisit the problem of asymptotically stabilizing a linear system with time-varying bounded input delay through the use of the Lyapunov equation based low gain feedback method. By allowing the system to have poles in the open right-half plane, here we extend the TPF control approach, which was developed in [14] for systems whose open loop poles are all in the closed left-half plane, to the general linear systems with time-varying input delays. An explicit condition is established that guarantees the global asymptotic stability of the closed-loop system. We will see that for the special case where the system poles are contained in the closed left-half plane, the developed feedback law and the stability condition reduce to the same results presented in [14], and the upper bound on the delay function can
be arbitrarily large. On the other hand, if the considered system is exponentially unstable, the upper bound of the delay function is inversely proportional to the sum of the unstable poles.

The extension of the results in [14] to exponentially unstable systems is not straightforward, and many of the simplifying assumptions employed in the above mentioned papers do not apply for systems with eigenvalues in the open right-half plane. However, by manipulating the structure and utilizing the intricate properties of the state space matrices, we were able to demonstrate that exponentially unstable systems with time varying input delays can be stabilized using a similar controller design procedure as in [14].

The remaining of this paper is organized as follows. The control problem to be studied in this paper is defined in Section II. Preliminary results necessary for presenting our main results are included in Section III. Section IV contains the main results of this paper for the state feedback case, and Section V demonstrates the case of output feedback. Numerical examples are included in Section VI to verify the theoretical derivation. Finally, Section VII draws the conclusion to this paper.

II. PROBLEM DEFINITION

Consider a linear time-invariant system with input delay,

\[ \dot{x}(t) = Ax(t) + Bu(\phi(t)), \]

where \( x(t) \in \mathbb{R}^n \) is the state vector and \( u(t) \in \mathbb{R}^m \) is the input vector, and the pair \((A,B)\) is assumed to be controllable. The time-varying delay function \( \phi(t) : \mathbb{R}^+ \rightarrow \mathbb{R} \) is assumed to be exactly known, continuously differentiable and invertible, with \( \frac{d}{dt}\phi(t) > 0 \) for all \( t > 0 \) ([11], [14]). Here, we define the delay function to have the standard form

\[ \phi(t) = t - D(t), \]

for a bounded function \( D(t) : \mathbb{R}^+ \rightarrow \mathbb{R}^+ \), where \( 0 \leq D(t) \leq \overline{D} \).

Without loss of generality, we also assume that the state matrix in (1) is structured as,

\[ A = \blkdiag\{A_1, A_2, \ldots, A_l\} = \blkdiag\{A_-, A_+\}, \quad (3) \]

where each block \( A_i \) for \( i = 1 \) to \( l \) contains the eigenvalues of \( A \) with an equal real part. The existence of a coordinate transformation to obtain the system realization in (3) was demonstrated in [17] and [18], which include discussions on how to obtain the corresponding transformation matrices. We further assume that the blocks are ordered such that \( \text{Re}(\lambda(A_1)) \leq \text{Re}(\lambda(A_2)) \leq \ldots \leq \text{Re}(\lambda(A_l)) \).

Therefore, matrix \( A \) can be divided into the block \( A_- \in \mathbb{R}^{\overline{p} \times \overline{p}} \) with all the eigenvalues in the open left-half plane, \textit{i.e.}, \( \max\{\text{Re}(\lambda(A_-))\} < 0 \), and \( A_+ \in \mathbb{R}^{\overline{q} \times \overline{q}} \) block with all the eigenvalues in the closed right-half plane, \textit{i.e.}, \( \min\{\text{Re}(\lambda(A_+))\} \geq 0 \).

The predictor feedback is an approach used in the stabilization of a delayed system, where the delay in the system is compensated by predicting the future trajectory of the states from the system equations and initial conditions. In what is known as the “truncated predictor feedback” (TPF) approach, the state prediction is simplified by eliminating the input dependent term from the computation of the stabilizing control law. As a result, the controller equation for the feedback gain \( K \) simplifies to \( u(t) = K e^{A(\phi^{-1}(t) - t)}x(t), \) for all \( t \geq 0 \).

The structure of the TPF type controllers proposed in this paper is given as

\[ u(t) = -B^TP(\gamma)e^{A(\phi^{-1}(t) - t)}x(t), \quad \forall t \geq 0. \]

The semi-positive definite matrix \( P(\gamma) \) is the solution to the parametric algebraic Riccati equation (ARE)

\[ A^TP + PA - PBB^TP = -\gamma P, \]

where \( \gamma > 0 \). Differently from the derivation in [14], the solution to (5) may not be strictly positive definite because \( A \) is allowed to have eigenvalues in the open right-half plane. The parameter \( \gamma \) is related to the minimum rate of decay of the closed-loop system. The role of \( \gamma \) and the condition for \( P > 0 \) are discussed in detail in [19].

In order to simplify the notation, we define the following matrices. Let \( \tilde{A} = (A + \gamma/2I) \). Because of the assumed structure of matrix \( A \) in (3), \( \tilde{A} \) is also a block diagonal matrix,

\[ \tilde{A} = \blkdiag\{\tilde{A}_-, \tilde{A}_+\}. \]

As before, the first block \( \tilde{A}_- \in \mathbb{R}^{\overline{p} \times \overline{p}} \) contains all the stable eigenvalues in the open left-half plane \( \max\{\text{Re}(\lambda(\tilde{A}_-))\} < 0 \), and \( \tilde{A}_+ \in \mathbb{R}^{\overline{q} \times \overline{q}} \) has all eigenvalues in the closed right-half plane \( \min\{\text{Re}(\lambda(\tilde{A}_+))\} \geq 0 \). The diagonal blocks \( A'_- \in \mathbb{R}^{\overline{p} \times \overline{p}} \) and \( A'_+ \in \mathbb{R}^{\overline{q} \times \overline{q}} \) of the state matrix \( A \) are defined, such that

\[ \tilde{A}_- = A'_- + \frac{\gamma}{2}I, \quad \text{and} \quad \tilde{A}_+ = A'_+ + \frac{\gamma}{2}I. \]

It is important to notice here that \( A'_- \) may not be equal to \( A_- \) for \( \gamma > 0 \), but the eigenvalues of \( A'_- \) are always in the open left-half plane and \( \bar{p} \leq p \). The input matrix \( B \) in (1) is partitioned accordingly as

\[ B = [B_- \ B_+]^T, \]

where \( B_- \in \mathbb{R}^{\overline{p} \times m} \) and \( B_+ \in \mathbb{R}^{\overline{q} \times m} \). With the matrices as defined in (6) and (8), the ARE in (5) can be rewritten as

\[ \tilde{A}^TP + P\tilde{A} - PBB^TP = 0. \]

Furthermore, for the special case where \( P \) is positive definite, the Riccati equation in (9) can be transformed into the Lyapunov equation [19]. The advantage of the Lyapunov equation over (9) is that the matrix equation becomes linear with respect to the unknown positive definite matrix.

III. PRELIMINARY RESULTS

In this section we present some properties of the solution to the ARE (9), as well as some basic theories for time-delay systems that will be valuable in establishing our main results. The first two lemmas we will introduce are extensions of the results presented in [14], [20] and [21] on a system with all poles on the imaginary axis to a general time-invariant linear system (1).
Lemma 1: Given matrices $\tilde{A}$ and $B$ as defined in (6) and (8), the parametric ARE in (9) has a positive semidefinite solution $P \geq 0$ in the form of
\[
P = \text{blkdiag} \{0, P_+\},
\]
where $P_+ > 0$ is the unique positive definite solution to
\[
\tilde{A}_+^T P_+ + P_+ \tilde{A}_+ - P_+ B_+ B_+^T P_+ = 0.
\]
Additionally, it follows that
\[
\text{tr}(B^T PB) = 2 \text{tr}(\tilde{A}_+),
\]
where “tr” represents the trace of a matrix, and
\[
PBB^T P \leq 2 \text{tr}(\tilde{A}_+) P.
\]
Proof: The first part of the proof is straightforward and involves substituting the $P$ in (10) into (9), which gives the same expression as in (11). The existence and uniqueness of positive definite $P_+$ have been established in [19].

The second part of the proof is obtained from the ARE in (11) after multiplying both sides of the equality to the right by the inverse of $P_+$. Then, by the properties of the trace operation, it follows that
\[
\text{tr}(B^T P B_+) = 2 \text{tr}(\tilde{A}_+),
\]
and it follows from [14] that
\[
P_+ B_+ B_+^T P_+ \leq 2 \text{tr}(\tilde{A}_+) P_+.
\]
Finally, it follows from the structure of $P$ that (12) and (13) must hold.

Lemma 2: Assume that we are given the state space system as defined in (1), where the pair $(A, B)$ is controllable, and $P$ is the solution (10) to the parametric ARE in (9). Then, it holds that
\[
e^{A^T t} P e^{A t} \leq e^{\omega t} P,
\]
for an arbitrary $t \geq 0$, and a positive scalar $\omega$ such that
\[
\omega \geq 2 \frac{\text{tr}(\tilde{A}_+)}{\gamma} - 1.
\]
Proof: Let $\omega$ be a positive scalar, $P$ be the solution to the parametric ARE in (9), and the matrix $Q(\omega)$ be defined as
\[
Q(\omega) = \gamma P + \omega \gamma P - PBB^T P.
\]
Then, it was shown in [22] that
\[
e^{A^T t} P e^{A t} - e^{\omega t} P = -e^{\omega t} \int_0^t e^{-\omega s} e^{A^T s} Q(\omega) e^{A s} ds
\]
is true for all $t \geq 0$. Additionally, the right-hand side of the equality is greater than or equal to zero if
\[
Q(\omega) \geq \left((\omega + 1)\gamma - 2 \text{tr}(\tilde{A}_+)\right) P \geq 0.
\]
Since $P \geq 0$, the above inequality is satisfied if (17) is true. Thus, the inequality in (16) holds if (17) is satisfied.

Lemma 3: [23] For any positive definite matrix $P > 0$, any scalars $\gamma_1$ and $\gamma_2$ such that $\gamma_2 \geq \gamma_1$, and a vector valued function $x : [\gamma_1, \gamma_2] \rightarrow \mathbb{R}^n$ such that the integrals in the following are well defined,
\[
\left(\int_{\gamma_1}^{\gamma_2} x^T(s) ds\right) P \left(\int_{\gamma_1}^{\gamma_2} x(s) ds\right) \leq (\gamma_2 - \gamma_1) \int_{\gamma_1}^{\gamma_2} x^T(s) P x(s) ds.
\]

IV. State Feedback Control

The linear system with time delay (1) can be written in subsystems
\[
x_-(t) = A_- x_-(t) + B_- u(\phi(t)),
\]
\[
x_+(t) = A_+ x_+(t) + B_+ u(\phi(t)),
\]
where the eigenvalues of $A_-$ are all in the open left-half plane, and some eigenvalues of $A_+$ can be in the open right-half plane. Additionally, due to the assumed structures of the matrices $P$, $A$, and $B$, the TPF control input only depends on the trajectory of $x_+(t)$,
\[
u(t) = B_+ P_x e^{A_+ (\phi^{-1}(t)-t)} x_+(t).
\]

Based on the TPF control input given above and how the linear system is structured in (21) and (22), we observe that the subsystem in (21) is asymptotically stable. The size of the matrix $A'_-$ may vary with the value of $\gamma$ as defined in (7), but its eigenvalues are always contained in the open left-half plane. The input to the subsystem (21) is a feedback law of the states of (22). Therefore, the input $u(t)$ is an external signal to (21), which is bounded and converging when (22) is stabilized. This implies that the full system will be asymptotically stable if we establish the asymptotic stability of (22).

Theorem 1: Consider the linear system (1) with the corresponding subsystems in (21) and (22). If there exist $\delta > 0$ and $\bar{\delta} > 0$ such that
\[
\delta (1 - n \bar{\delta} \delta (e^{\delta}) - 1) > 2 \text{tr}(A_+) \bar{D}
\]
holds for $\delta \in (\delta, \bar{\delta})$, then there exist $\gamma > 0$ and $\bar{\gamma} > 0$ such that
\[
\text{tr}(\tilde{A}_+(\gamma)) = \frac{\delta}{2D} , \quad \text{tr}(\tilde{A}_+(\bar{\gamma})) = \frac{\bar{\delta}}{2D}.
\]
Furthermore, for any $\gamma \in (\gamma, \bar{\gamma})$ the TPF control in (4) asymptotically stabilizes the delayed system (1).

Proof: The proof of this theorem is similar to the results presented in [14], except for the modifications introduced in Lemmas 1 and 2. Because of the similarities, we will only present the steps in the proof that are critical to arriving at our results. To simplify the notation, we will denote $K = K(\gamma) = -B^T P(\gamma) = -B^T P$.

Given the linear time-invariant system (1) with an arbitrary initial condition $x(t) = \varphi(t)$, $t \in [0(\phi(0)), 0]$, and the TPF control in (4), it was demonstrated in [14] that the closed-loop system states are bounded for all $t \in [0, \phi^{-1}(\phi^{-1}(\phi^{-1}(0)))].$ Thus, the stability can be established by considering $t \geq \phi^{-1}(\phi^{-1}(0))$.

The state trajectory at time $t$ of the time-delay system in (1), under the TPF control (4) and with initial condition...
\( x(\phi(t)) \), can be found explicitly from the system equation. Then, the closed-loop state equation becomes
\[
\dot{x}(t) = (A + BK) x(t) - B K (A x(\phi(t))) + \gamma x(\phi(t)),
\]
where
\[
\lambda(t) = \int_{t-}^{t_\phi} e^{A(t-s)} BK e^{A(t-\phi(s))} x(\phi(s)) \, ds.
\]

As mentioned earlier, the subsystem (21) is asymptotically stable. Consider the following Lyapunov function for the subsystem (22)
\[
V(x_+(t)) = \lambda^T P \lambda(t),
\]
where the properties derived in Lemma 1 were used.

Next, we define the scalar parameter \( \omega = 2 \text{tr}(A_+)/\gamma \). Referring to Lemma 3, we can simplify the term \( \lambda^T(t) P \lambda(t) \) in (27) as,
\[
\lambda^T(t) P \lambda(t) \leq (t - \phi(t)) \int_{\phi(t)}^{t} x^T(\phi(s)) e^{A(t-s-\phi(s))} P BB^T e^{A(t-s-\phi(s))} x(\phi(s)) \, ds.
\]
The above inequality can be further simplified by employing the results of Lemmas 1 and 2, and by making use of the upper bound information of \( D \),
\[
\lambda^T(t) P \lambda(t) \leq 4 \text{tr}(A_+) \int_{t-}^{t_\phi} e^{\gamma(t-s)} V(x_+(\phi(s))) \, ds.
\]
As noted in [14], \( \phi(t - \bar{D}) \geq t - 2 \bar{D} \). Thus, under the condition that
\[
V(x_+(t + \theta)) < \eta V(x_+(t)), \quad \forall \theta \in [-2\bar{D}, 0],
\]
for \( t \geq \phi^{-1}(\phi^{-1}(0)) \) and some \( \eta > 1 \), we can simplify the expression in (29) as
\[
\lambda^T(t) P \lambda(t) \leq 4 \text{tr}(A_+) \int_{t-}^{t_\phi} e^{\gamma(t-s)} V(x_+(\phi(s))) \, ds.
\]
Substituting the inequality in (31) into (27) results in
\[
V(x_+(t)) \leq -\left( \gamma - 8 \text{tr}(A_+) \right) \int_{t-}^{t_\phi} e^{\gamma(t-s)} ds V(x_+(\phi(s))).
\]
We define a new variable
\[
\delta = \omega_\gamma \bar{D} = 2 \text{tr}(A_+) \bar{D}.
\]
Then, there are sufficiently small values of \( \eta > 1 \) and \( \epsilon > 0 \), such that
\[
V(x_+(t)) \leq -\epsilon V(x_+(\phi(s))),
\]
if it holds that
\[
\gamma - 2 \text{tr}(A_+) \delta e^\delta (e^\delta - 1) > 0.
\]
It is also observed that \( \gamma \geq 2 \text{tr}(A_+)/n - 2 \text{tr}(A_+)/n \), and the inequality in (34) will be satisfied if
\[
\delta (1 - n \delta e^\delta (e^\delta - 1)) > 2 \text{tr}(A_+) \bar{D}.
\]

From the above expression, we can deduce that the left-hand side of the inequality in (35) is a concave function of \( \delta \). Therefore (33) holds if there exist \( \delta \) and \( \delta \) such that (35) is true for all \( \delta \in [\delta, \delta] \), and the delayed input system is asymptotically stable by the Razumikhin Stability Theorem [24] and the assumption in (30).

Next we demonstrate that if (35) is satisfied for some \( \delta \), then it is always possible to find a \( \gamma > 0 \), such that
\[
\delta = 2 \text{tr}(A_+) \bar{D} = 2 \text{tr}(A_+ + \gamma \bar{I}) \bar{D}.
\]
Consider the case where (35) is satisfied. In this case, there exists a \( \delta \) such that
\[
\delta (1 - n \delta e^\delta (e^\delta - 1)) = 2 \text{tr}(A_+) \bar{D}.
\]
We assume that this \( \delta \) is outside the range of \( \gamma > 0 \) given by (36). Since \( \text{tr}(A_+) \) is a continuous and nondecreasing function of \( \gamma \) with \( \lim_{\gamma \to \infty} \text{tr}(A_+) = \infty \), the only possibility is that
\[
\delta < 2 \text{tr}(A_+) \bar{D} = 2 \text{tr}(A_+) \bar{D}.
\]
Then, it follows that
\[
\delta < \delta (1 - n \delta e^\delta (e^\delta - 1)).
\]
Simplifying the above inequality, we conclude that the following must hold,
\[
n \delta e^\delta (e^\delta - 1) < 0,
\]
which is a contradiction since the left-hand side of the above inequality is positive for all \( \delta > 0 \). Therefore, there is a \( \gamma > 0 \) such that \( 2 \text{tr}(A_+) \bar{D} = \delta \).

**Remark 1:** The existence of a stabilizing controller for (1) depends on the upper bound of the delay function \( D \), and the trace of the block \( A_+ \). We observe from the right-hand side of the stability condition in (35) that the bound on the input delay for stability is inversely proportional to the sum of the unstable poles of the plant. Thus, the stability of the closed-loop system results from the trade-off between the magnitude of the unstable poles in the plant and the maximum delay in the input signal. If the eigenvalues of the matrix \( A \) in (1) are all in the closed left-half plane, then (35) becomes the same as the condition developed in [14], where \( \omega = n - 1 \).

**Remark 2:** The relationship between \( \gamma \) and \( \bar{D} \) in the stability conditions (35) and (36) can be inverted numerically to obtain the maximum delay for a given gain \( \gamma \). Given a gain \( \gamma \), the maximum \( \bar{D} \) can be found from (24) by solving,
\[
e^{\gamma \bar{D}} \left( e^{\gamma \bar{D}} - 1 \right) > \frac{1}{\eta \gamma \omega} \left( 2 \text{tr}(A_+) - 1 \right).
\]
This can be helpful in combining the TPF control with other design methods for satisfying additional stability and performance objectives.
V. Output Feedback Control

In this section, we extend the state feedback results to the case where the TPF controller is based on the output signal of the time-delay system. Let the output of the time-delay system in (1) be defined as

\[ y(t) = Cx(t), \quad C \in \mathbb{R}^{r \times n}. \]  

(41)

In addition to the controllability of the pair \((A, B)\) and the matrix structures described in (3), (6) and (8), we further assume that the pair \((A, C)\) is detectable.

The truncated predictor output feedback control law is constructed as

\[
\begin{align*}
\dot{x}(t) &= A\hat{x}(t) + Bu(t) + L(y(t) - C\hat{x}(t)), \\
u(t) &= -B^{T}\hat{P}e^{A(t-t)L}\hat{x}(t), \quad \forall t \geq 0,
\end{align*}
\]

where \(\hat{x}\) is the estimate of the state vector, the positive semi-definite matrix \(P\) is the solution of the ARE in (5), and the observer gain \(L \in \mathbb{R}^{r \times r}\) is selected such that all the eigenvalues of \((A - LC)\) are in the open left-half plane. The condition for stability of the delayed system with the above output feedback TPF control can be readily derived using the results in Theorem 1.

**Theorem 2:** Consider the time-delay system in (1). Assume that the pair \((A, B)\) is controllable and the pair \((A, C)\) is detectable. If \((A - LC)\) is Hurwitz, and there exist \(\delta\) and \(\delta\) such that (24) is satisfied for \(\delta \in (\tilde{\delta}, \bar{\delta})\), then there exist \(\bar{\gamma} > 0\) and \(\gamma > 0\) such that the closed-loop system under the output feedback TPF control in (42) is asymptotically stable for all \(\gamma \in (\bar{\gamma}, \gamma)\).

**Proof:** Define the observer error vector as

\[ e(t) = x(t) - \hat{x}(t). \]

(43)

We can rewrite the time-delay system in (1) with the output feedback TPF control law as,

\[
\begin{align*}
\dot{e}(t) &= (A\hat{x}(t) + B\psi(t)x(\phi(t))) - B\psi(t)e(\phi(t)), \\
e(t) &= (A - LC)e(t),
\end{align*}
\]

(44)

where \(K = -B^{T}P\) and \(\psi(t) = Ke^{A(t-t)}\).

We observe that the subsystem (44) is the closed-loop system of (1) under the state feedback TPF control (4) and in the presence of an external input signal as a function of the error \(e(t)\). Furthermore, this external input is bounded and converges to zero exponentially since \(A - LC\) is Hurwitz. Therefore, the delayed system under the output feedback TPF control is asymptotically stable if (44) is asymptotically stable in the absence of \(e(t)\).

Since \(\delta\) and \(\delta\) are such that (24) is satisfied for \(\delta \in (\tilde{\delta}, \bar{\delta})\). Then, by Theorem 1, there exist \(\bar{\gamma} > 0\) and \(\gamma > 0\) such that the subsystem (44) is asymptotically stable for all \(\gamma \in (\bar{\gamma}, \gamma)\).

VI. Numerical Examples

We consider the case of a double oscillator system, as considered in [14], with a positive real pole added to the plant equation. The resulting state space matrices \(A\) and \(B\) as in (1) are given by

\[
A = \begin{bmatrix}
p & 1 & 0 & 0 & 0 \\
0 & 0 & \omega & 0 & 0 \\
0 & -\omega & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & \omega \\
0 & 0 & 0 & -\omega & 0
\end{bmatrix}, \quad B = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
1
\end{bmatrix}.
\]

The scalar \(p = 0.1\) represents the location of the unstable real pole, and \(\omega = 1\) locates the resonance frequency of the double oscillator. We assume that the initial conditions are given by \(x(\theta) = [-1, 2, 2, -1, 2]^{T}\) for all \(\theta \in [-D, 0]\).

A delay function \(\phi(t)\) similar to the example presented in [12] and [14] is considered here. The upper bounds of the delay function was reduced, in order to accommodate the additional constrains introduced by the open right-half plane pole in the plant.

A. Sinusoidal Delay Function

In this example case, the inverse of the delay function is specified as, \(\phi^{-1}(t) = \rho(t) = t + 0.3(1 + 0.5 \cos(t))\). The corresponding delay signal \(\phi(t)\) in this case is oscillatory, with an upper bound of \(D = 0.45\). We find the range of \(\gamma\) such that the stability condition in (35) is satisfied for the given \(D\). We observe that the stability condition is satisfied if \(0.002 < \gamma < 0.092\). In the simulation, we select \(\gamma = 0.09\).

The time response of the closed-loop states is shown in Fig. 1. The amplitude of the state oscillation increases initially, but the states are later brought back to zero asymptotically. The closed-loop control signal \(u(t)\) and the delayed input to the plant \(u(\phi(t))\) are shown in Fig. 2.

B. Improvement in Delay Compensation with TPF

In this subsection, the TPF controller in (4) is compared to the state feedback control.
The feedback gain \( K = -B^TP \) is taken to be the same as in the previous example for both the TPF control and (46). The purpose of this comparison is to examine the contribution of the exponential factor in the TPF control for the stabilization of the delayed system. The phase margin of the system with the control input \( u(t) = Kx(t) \) is found to be 68.1 degrees, which corresponds to a delay margin of 0.809 s.

Figure 3 demonstrates the state responses of the closed-loop system under the TPF controller (4) and under the static feedback (46), both with a constant delay of 1 s. As expected from the stability margin information, the system under the feedback (46) is unstable. On the other hand, the simulation results with the TPF controller show a stable response.

VII. CONCLUSIONS

The stabilization of general linear systems with time-varying input delays was examined in this paper. The truncated predictor feedback (TPF) method, which had recently been developed for systems with poles in the closed left-half plane and is based on low gain feedback, was extended to be applicable to exponentially unstable systems. An explicit design procedure of the control law was presented along with a stability condition for the closed-loop system.

In the special case where the system poles are all in the closed left-half plane, it was observed that the stability condition obtained here reduces to the result presented in [14], and a stabilizing controller can be found for an arbitrarily large delay. On the other hand, if the system has exponentially unstable poles, then the closed-loop stability condition reveals a clear trade-off between the sum of the unstable poles and the maximum allowable delay in the system input. Numerical examples validated the mathematical derivation in this paper.

The results presented in this paper are built upon the existence of \( \phi(t) \) and the assumption that \( \phi(t) \) is continuously differentiable. Such assumptions are satisfied in many applications, such as the case of systems with constant delay. However, the assumptions may not be appropriate in some applications such as control over networks, where the delay function is generally not continuous. As discussed in the introduction, some methods exist for the stabilization of delayed system with minimum information on the delay, but those methods mostly focus on constant delays. The limitations that come from the assumptions on \( \phi(t) \) will need to be addressed in future research.

REFERENCES